1 Lecture 9: overview

- discussion: monitoring
  deterministic vs. random
- moral hazard: optimal UI
- lack of commitment:
  - one sided / partial equilibrium
  - two sided / GE (2 agent/2 shock case)

2 optimal UI

- Utility:

$$U = E\sum_{t=0}^{\infty}\beta^t\left[u\left(c_t\right) - a_t\right]$$

- a = effort
- $p\left(a\right)$ = prob of finding a job
- $c_t = w$ if employed $c_t = \tau_t$ otherwise
- "planner" is risk neutral and evaluates cost

$$C \equiv E\sum_{t=0}^{\infty}\beta^t\tau_t$$

- Pareto Problem: frontier between C and U:
  $\min C$ s.t. $U \geq V$
  or $\max V$ s.t. $C \leq \bar{C}$

2.1 First Best: Effort observable

- sequence: tedious notation but possible

- recursive:

$$C(V) = \min\{c + \beta(1 - p(a))C(V^u)\}$$
$$V = u(c) - a + \beta(1 - p(a))V^u + \beta p(a)V^e$$

- focs:

$$1 - \theta = 0$$
$$\beta(1 - p(a))C'(V^u) - \theta\beta(1 - p(a)) = 0$$
$$-\beta p'(a)C(V^u) + \theta\beta p'(a)V^u - \theta\beta p'(a)V^e + \theta = 0$$

- simplifying

$$\theta = \frac{1}{u'(c)}$$
$$C'(V^u) = \theta$$
$$C(V^u) = \left[V^u - V^e + \frac{1}{\beta p'(a)}\right]\theta$$

- result: V_t^u is constant $\Rightarrow c_t^u$ is constant

2.2 Second Best: Effort Unobservable

- sequence problem is not very tractable (Shavell and Weiss perform a variational argument for their results)
- recursive formulation

$$C(V) = \min\{c + \beta(1 - p(a))C(V^u)\}$$
$$V = u(c) - a + \beta(1 - p(a))V^u + \beta p(a)V^e$$
$$\beta p'(a)[V^e - V^u] = 1$$

- focs

$$\begin{aligned} 1 - \theta u'(c) &= 0 \\ \beta(1 - p(a))C'(V^u) - \theta\beta(1 - p(a)) + \eta\beta p'(a) &= 0 \\ -\beta p'(a)C(V^u) + \underbrace{\theta\beta p'(a)V^u - \theta\beta p'(a)V^e + \theta}_{=0 \text{ from agent's foc}} + \eta\beta p''(a)[V^e - V^u] &= 0 \end{aligned}$$

- simplifying:

$$\theta = \frac{1}{u'(c)}$$
$$C'(V^u) = \theta - \eta \frac{p'(a)}{1-p'(a)}$$

envelope condition implies

$$C'(V) = \theta$$

- results:
V_t^u is decreasing
$\Rightarrow$ (using envelope and f.o.c. w.r.t. c) c_t^u decreasing

Remarks:

- relaxing $c_t^e = w$ (Hopenhayn and Nicolini, 1997)
- agents save/borrow
- welfare gains of optimal program

3 Lack of commitment: One Sided / Partial Equilibrium

Source: LS Chapter 15 section 401-409

$$P(w) = \min \sum_{s \in S} \pi_s \left[c_s - y_s + \beta P(w_s')\right]$$
$$u(c(s)) + \beta w_s \geq u(y_s) + \beta V_{aut}$$
$$\sum \pi_s \left[u(c(s)) + \beta w_s\right] \geq w$$

$$w \in [V_{aut}, V_{\max}]$$
$$c\,[c_{\min}, c_{\max}]$$

foc

$$\begin{aligned}(\lambda_s + \mu\pi_s)\, u'(c_s) &= \pi_s \\ \lambda_s + \mu\pi_s &= -\pi_s P'(w_s)\end{aligned}$$

combing:

$$u'(c_s) = \frac{-1}{P'(w_s)}$$

since P is decreasing and concave then cs is increasing in ws.

- $ws = w$ if constraint not binding $\mu = 0$
- otherwise $ws > w$
- dynamics:

  eventually converge to a high enough w so that participation constraint is not binding (this result is more general: see Debraj Ray's Econometrica paper)

4 Two Sided / GE

Sources:

- LS Chapter 15 section 413-418: good treatment of Kocherlakota (iid shocks) but no formal analysis of long-run distribution
- we follow: simplified version of Alvarez-Jermann (2000) "Quantitative Asset Pricing Implications of Endogenous Solvency Constraints". only section 4 and sub-section 5.1 (section 3 introduces the notation).
  This version has individual persistence of income (not necessarily iid), only 2 shocks, no aggregate shocks (in our version). We are able to study the whole dynamics.

4.1 Dynamics

- environment:
  - symmetric, two agents $i = 1, 2$; equal population

- $y^1 > y^2$, $y_t = y^1 + y^2 \equiv e$ (no aggregate uncertainty)
- aggregate state $s = 1, 2$ denotes realization of income for type 1 (equivalently: s denotes who gets high shock)
- p is probability of transition from $s = 1$ to $s = 2$ and from $s = 2$ to $s' = 1$

- Problem (recursive version)

$$V(w, s) = \max_{c^1, c^2, w'(\cdot)} \left[u\left(c^1\right) + \beta \sum_{s'} \pi\left[s'|s\right] V\left(w'\left(s'\right), s'\right) \right]$$

$$\begin{aligned} c^1 + c^2 &= e \\ u\left(c^2\right) + \beta \sum \pi\left(s'|s\right) w'\left(s'\right) &\geq w \\ w'\left(s'\right) &\geq U_{aut}^2\left(s'\right) \\ V\left(w'\left(s'\right), s'\right) &\geq U_{aut}^1\left(s'\right) \end{aligned}$$

- last two constraints equivalent to

$$w'\left(s'\right) \in \left[L\left(s'\right), H\left(s'\right)\right]$$

  for some $L(s')$ and $H(s')$

- we take as given that we have V, L and H

  properties: V is decreasing, differentiable and concave in w

  we then derive some properties of the allocation

- graphical analysis: two shock case

- first order conditions:

$$\begin{aligned} u'\left(c^1\right) &= \lambda \\ \theta u'\left(c^2\right) &= \lambda \\ V_1\left(w'\left(s'\right), s'\right) &\lesseqgtr -\theta \end{aligned}$$

  with equality if $w'(s') \in (L(s'), H(s'))$, with $\leq$ if $w'(s') = L(s')$ and $\geq$ if $w'(s') = H(s')$

- Envelope condition:
$$V_1(w,s) = -\theta$$

- result 1: $c^2(w,s)$ is increasing in w

  since V is concave V_1 is decreasing thus $-V_1$ is increasing in w:
$$\frac{u'(e-c^2)}{u'(c^2)} = \theta = -V_1(w,s)$$
  which requires c^2 to increase with w

- result 2: if $s = s'$ then $w(s') = w$. FOC:
$$V_1(w'(s'), s') \lesseqgtr -\theta = V_1(w,s)$$
  is satisfied with equality if $(w'(s'), s') = (w,s)$. This satisfies the constraint since $w \in [L(s), H(s)]$ by assumption.

- result 3: in the 2 shock case if $s \neq s'$
$$V_1(w'(s'), s') \lesseqgtr V_1(w,s)$$

- collecting results:

  - $c^2(w,s)$ is increasing in w
  - if $s' = s \to w'(s') = w$ (constraint not binding)
  - if $s \neq s'$ if binding then go to closest value possible
  - show graph of policy
  - convergence (main result): **stationary distribution is history independent and symmetric** (we turn to studying this in more detail next)

4.2 Stationary Distributions

Given our previous result we now look for stationary symmetric distributions:

- given (c^1, c^2) let $V^1(c^1, c^2)$ and $V^2(c^1, c^2)$ be the unique solutions to:
$$\begin{aligned} V^1 &= u\left(c^1\right) + \beta\left[pV^1 + (1-p)V^2\right] \\ V^2 &= u\left(c^2\right) + \beta\left[pV^2 + (1-p)V^1\right] \end{aligned}$$
  clearly: $V^2(y,x) = V^1(x,y)$

- grinding out:

$$V^1\left(c^1,c^2\right) = \frac{1}{1-\beta}\left\{\omega u\left(c^1\right)+(1-\omega)\,u\left(c^2\right)\right\}$$
$$\text{where } \omega = \frac{1-\beta p}{1+\beta-2p\beta} > \frac{1}{2}$$

- stationary symmetric feasible allocations satisfies:

$$c^1+c^2=e \tag{1}$$

$$V^1\left(c^1,c^2\right) \geq V^1\left(y^1,y^2\right) \equiv V^1_{aut} \tag{2}$$
$$V^2\left(c^1,c^2\right) \geq V^2\left(y^1,y^2\right) \equiv V^2_{aut} \tag{3}$$

i.e. resource constraint and participation constraints.

- substituting

$$\omega u\left(c^1\right)+(1-\omega)\,u\left(c^2\right) \geq \omega u\left(y^1\right)+(1-\omega)\,u\left(y^2\right)$$
$$\omega u\left(c^2\right)+(1-\omega)\,u\left(c^1\right) \geq \omega u\left(y^2\right)+(1-\omega)\,u\left(y^1\right)$$

rearranging

$$\omega\left[u\left(c^1\right)-u\left(y^1\right)\right]+(1-\omega)\left[u\left(c^2\right)-u\left(y^2\right)\right] \geq 0 \tag{4}$$
$$(1-\omega)\left[u\left(c^1\right)-u\left(y^1\right)\right]+\omega\left[u\left(c^2\right)-u\left(y^2\right)\right] \geq 0 \tag{5}$$

if $c^1 \leq y^1$ then (4) implies (5) $\Rightarrow$ participation constraint for type 2 never binds

- full risk sharing is attainable iff

$$u\left(e/2\right) \geq \omega^1 u\left(y^1\right)+\left(1-\omega^1\right)u\left(y^2\right)$$

- otherwise, look for allocations with:

$$c^1+c^2=e \tag{6}$$

$$\omega\left[u\left(c^1\right)-u\left(y^1\right)\right]+(1-\omega)\left[u\left(c^2\right)-u\left(y^2\right)\right]=0$$

and $y^2 \leq c^2 \leq c^1 \leq y^1$ (i.e. with less variability than autarky).

- example: $u(c) = c^{1-\sigma}/(1-\sigma)$ then

$$\omega c^{1-\sigma} + (1-\omega)(e-c)^{1-\sigma} = \omega \left(y^1\right)^{1-\sigma} + (1-\omega)\left(y^2\right)^{1-\sigma}$$

- gives us $c(\omega)$ is decreasing in ω :

$$c^{1-\sigma} - \left(y^1\right)^{1-\sigma} - \left[(e-c)^{1-\sigma} - \left(y^2\right)^{1-\sigma}\right] + \left[\omega c^{-\sigma} - (1-\omega)(e-c)^{-\sigma}\right] \frac{\partial c}{\partial \omega} = 0$$

  and we have $\omega(\beta, p)$ (increasing in β and p)

- implications risk sharing
  - decreasing in p
  - decreasing in β
  - increasing in risk aversion

5 Trash

5.1 Grinding $V^i(\cdot,\cdot)$ formula

From

$$\begin{aligned} V^1 &= \frac{u(c^1)}{1-\beta p} + \beta \frac{1-p}{1-\beta p} V^2 \\ V^2 &= \frac{u(c^2)}{1-\beta p} + \beta \frac{1-p}{1-\beta p} V^1 = \frac{u(c^2)}{1-\beta p} + \beta \frac{1-p}{1-\beta p}\left[\frac{u(c^1)}{1-\beta p} + \beta \frac{1-p}{1-\beta p} V^2\right] \end{aligned}$$

we get that

$$\begin{aligned} V^2 &= \frac{u(c^2)}{1-\beta p} + \beta \frac{1-p}{(1-\beta p)^2} u\left(c^1\right) + \beta^2 \frac{(1-p)^2}{(1-\beta p)^2} V^2 \\ &= \frac{1}{(1-\beta p)^2 - \beta^2 (1-p)^2}\left[(1-\beta p)\, u\left(c^2\right) + \beta (1-p)\, u\left(c^1\right)\right] \\ &= \frac{1}{1-\beta}\left\{\frac{1-\beta p}{1+\beta-2p\beta} u\left(c^2\right) + \frac{\beta(1-p)}{1+\beta-2p\beta} u\left(c^1\right)\right\} \end{aligned}$$

so $(1-\beta)V^2$ is a weighted average of $u(c^1)$ and $u(c^2)$.