1 Lecture 4

1.1 Overview

- cross-sectional and cohort data:
  - show serious flaws in the simples life-cycle model...
  - ...not necessarily in a richer version (uncertainty, borrowing constraints, high-discounting, demographics, labor supply, etc...)
  - much debate over relative roles of these extensions
- precautionary savings
  - 2 periods: role of $u''' > 0$
  - exponential utility example
  - quantify aggregate effects
- income-fluctuations problem:
  - sequence representation:
    Euler equation
  - recursive representation:
    Euler equation
    policy functions

2 Precautionary Savings

- two period savings problem:

$$\max u(c_0) + \beta EU(\tilde{c}_1)$$

$$c_1 = \tilde{y}_1 + (1+r)k_1$$

$$k_1 + c_0 = (1+r)k_0 + y_0$$

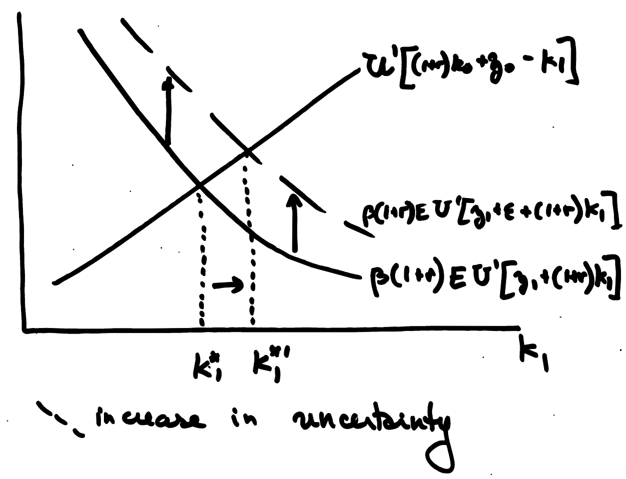
- subsituting:

$$\max_{k_1} u\left(\left(1+r\right)k_0 + y_0 - k_1\right) + \beta EU\left(y_1 + \left(1+r\right)k_1\right)$$

  f.o.c. is the Euler equation

$$u'\left(\left(1+r\right)k_0 + y_0 - k_1\right) = \left(1+r\right)\beta EU'\left(y_1 + A_1\right)$$

- graph LHS and RHS as functions of k_1, optimum is at the unique intersection k_1^* in the figure



- mean preserving spread
  i.e. replace the random variable y_1 with $y_1' = y_1 + \varepsilon$ with $E\left(\varepsilon|y\right) = 0$.
  $\Rightarrow$three possibilities:
  - $U'\left(\cdot\right)$ is linear: no effect
  - $U'\left(\cdot\right)$ is convex: RHS rises $\Rightarrow k_1^*$ increases
  - $U'\left(\cdot\right)$ is concave: RHS falls $\Rightarrow k_1^*$ decreases
- introspection: 2nd case most relevant

  standard parameterization implies u' is convex
  if $u'\left(c\right)$ is positive and $c \geq 0$ then u' is convex near 0 and ∞
- precautionary savings with more than 2 periods later

3 Income Fluctuation Problem

3.1 Sequence Formulation

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$A_{t+1} = (1+r)(A_t + y_t - c_t)$$

or equivalently:

$$c_t + k_{t+1} = (1+r) k_t + y_t$$

where $k_t \equiv A_t / (1+r)$; or,

$$x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

where $x_t \equiv A_t + y_t$.

- explicit about uncertainty:
  - let $s_t \in S$ be the state at time t (assume S is finite)
  - determines output $y_t = y(s_t)$ (one example: $s_t = y_t$)
  - history: $s^t = (s_0, s_1, ..., s_t) \in S^t \subset \mathbb{R}^{t+1}$
  - probability of history s^t having occured at time t is denoted $\Pr(s^t)$
    this is very general
    example: s_t is i.i.d. $\Pr(s^t) = p(s_0) p(s_1) \cdots p(s_t)$
  - information: at time t, s^t is known

- finite life income fluctuation problem:

$$\max_{\{c_t(s^t), A_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \sum_{s^t \in S^t} \beta^t \Pr(s^t) u(c_t(s^t))$$

  subject to,

$$A_{t+1}(s^t) = (1+r)\left(A_t(s^{t-1}) + y(s_t) - c_t(s^t)\right),$$

  for all t and $s^t \in S^t$ and

$$A_T(s^T) \geq 0$$

  for all s^T.

- Lagrangian: multiplier on budget constraint $\beta^t \Pr(s^t)\lambda(s^t)$

$$L \equiv \sum_{t=0}^{T-1} \sum_{s^t \in S^t} \beta^t \Pr\left(s^t\right) \left\{u\left(c_t\left(s^t\right)\right) + \lambda\left(s^t\right)\left[R\left(A_t\left(s^{t-1}\right) + y\left(s_t\right) - c_t\left(s^t\right)\right) - A_{t+1}\left(s^t\right)\right]\right\}$$

- first order conditions:

$$u'\left(c_t\left(s^t\right)\right) = \lambda\left(s^t\right)$$

$$\beta^t \Pr\left(s^t\right)\lambda\left(s^t\right) = \beta^{t+1}(1+r)\sum_{s_{t+1}} \Pr\left(s^{t+1}\right)\lambda\left(s^t, s_{t+1}\right)$$

  rearranging:

$$\lambda\left(s^t\right) = \beta^t (1+r)\sum_{s_{t+1}} \Pr\left(s_{t+1}|s^t\right)\lambda\left(s^t, s_{t+1}\right)$$

- combining (Euler):

$$u'\left(c_t\left(s^t\right)\right) = \beta(1+r)\sum_{s_{t+1}} u'\left(c_{t+1}\left(s^t, s_{t+1}\right)\right)\Pr\left(s_{t+1}|s^t\right)$$

  or simply,

$$u'(c_t) = \beta(1+r)E_t\left[u'(c_{t+1})\right].$$

3.2 Recursive Formulation

- sequence problem is not natural
- recursive formulation of finite period

$$v_t(x) = \max_c \left\{u(c) + \beta E v_{t+1}((1+r)(x-c)+y)\right\}$$

  and $v_{T-1}(x_{T-1}) = u(x_{T-1})$.

- infinite period problem:

$$v(x) = \max_c \left\{u(c) + \beta E v((1+r)(x-c)+y)\right\}$$

- (interior) first order condition:

$$u'(c) = \beta(1+r) Ev'((1+r)(x-c)+y) = \beta(1+r) Ev'(x')$$

$\Rightarrow$defines $c(x)$

- envelope condition:

$$v'(x) = u'(c(x))$$

thus $v'(x') = u'(c(x'))$ or informally $v'(x) = u'(c')$

- foc+envelope $\Rightarrow$ Euler equation

$$u'(c) = \beta(1+r) Eu'(c')$$

- consumption function is back

4 Precuationary Savings: $T > 2$

- first order condition

$$u'(c_t) = \beta(1+r) Ev'_{t+1}((1+r)(x-c)+y)$$

- if $v'''_{t+1} > 0$ then there are precautionary savings during period t but do we know anything about v'''?
- fortunately, if $u''' > 0$ then $v''' > 0$ [Sibley (1975)]
- example: $u(c) = -\frac{1}{\gamma}\exp(-\gamma c)$

guess and verify that $v(x) = -A\frac{1}{\gamma}\exp\left(-\gamma\frac{r}{1+r}x\right)$

- consumption function

$$c(x) = \frac{r}{1+r}\left[x + \frac{1}{r}y^*\right]$$

where,

$$y^* \equiv \frac{1+r}{-\gamma r}\log E\exp\left(\frac{-\gamma r}{1+r}y\right)$$

a kind of certainty equivalent

- aggregation example from problem set (after seeing Aiyagari's model)