Solutions PS4 Macro III

1 Two-Sided Lack of Commitment: Stationary Allocations

We have

$$\begin{aligned} V^1 &= u\left(c^1\right) + \beta\left[pV^1 + (1-p)\,V^2\right] \\ V^2 &= u\left(c^2\right) + \beta\left[pV^2 + (1-p)\,V^1\right] \end{aligned}$$

clearly, $V^2\,(y,x) = V^1\,(x,y)$.

Add the two conditions to get

$$V^1 + V^2 = u\left(c^1\right) + u\left(c^2\right) + \beta\left[V^1 + V^2\right] \Leftrightarrow$$

$$V^1 + V^2 = \frac{1}{1-\beta}\left[u\left(c^1\right) + u\left(c^2\right)\right].$$

Plug this in the equation for V^1 to get

$$V_1 = u\left(c^1\right) + \beta p V_1 + \beta\frac{1-p}{1-\beta}\left[u\left(c^1\right) + u\left(c^2\right)\right],$$

and rearranging we get to the desired expression

$$\begin{aligned} V^1\left(c^1, c^2\right) &= \frac{1}{1-\beta}\left\{\omega u\left(c^1\right) + (1-\omega)\,u\left(c^2\right)\right\} \\ & \text{where } \omega = \frac{1-\beta p}{1+\beta-2p\beta} > \frac{1}{2} \end{aligned}$$

Notice that $\frac{\partial\omega}{\partial p} = \frac{-\beta(1+\beta-2p\beta)+2\beta(1-\beta p)}{(1+\beta-2p\beta)^2} = \frac{-\beta^2+\beta}{(1+\beta-2p\beta)^2} > 0$ and $\frac{\partial\omega}{\partial\beta} = \frac{-p(1+\beta-2p\beta)+(1-2p)(1-\beta p)}{(1+\beta-2p\beta)^2} = \frac{p-1}{(1+\beta-2p\beta)^2} < 0$. This is intuitive. For β close 1, we don't discount the future at all so V^1 is (almost) an average with weights .5 of both utility levels. When $\beta = 0$ we don't care about the future and thus $V^1 = u(c^1)$. the higher β, the more we care about the future and $u(c^2)$ plays a higher role relative to $u(c^1)$ in determining V^1.An the higher the most likely we stay on the same state and thus $u(c_1)$ has a higher weight.

(b) We call a stationary symmetric allocation **feasible** if it satisfies the resource and participation constraints:

$$c^1 + c^2 = e$$

$$\begin{aligned} V^1\left(c^1, c^2\right) &\geq V^1\left(y^1, y^2\right) \qquad (1) \\ V^2\left(c^1, c^2\right) &\geq V^2\left(y^1, y^2\right) \qquad (2) \end{aligned}$$

Notice that autarky is always feasible.

If (1) holds

$$\frac{1}{1-\beta}\left\{\omega u\left(c^1\right) + (1-\omega)\, u\left(c^2\right)\right\} \geq \frac{1}{1-\beta}\left\{\omega u\left(y^1\right) + (1-\omega)\, u\left(y^2\right)\right\}$$

$$\Leftrightarrow u\left(y^1\right) - u\left(c^1\right) \leq -\frac{1-\omega}{\omega} u\left(y^2\right) - u\left(c^2\right).$$

For (2) to hold we need (using the fact that $V^2\left(c^1, c^2\right) = V^1\left(c^2, c^1\right)$ and $V^2\left(y^1, y^2\right) = V^1\left(y^2, y^1\right)$)

$$\frac{1}{1-\beta}\left\{\omega u\left(c^2\right) + (1-\omega)\, u\left(c^1\right)\right\} \geq \frac{1}{1-\beta}\left\{\omega u\left(y^2\right) + (1-\omega)\, u\left(y^1\right)\right\}$$

$$\Leftrightarrow u\left(y^1\right) - u\left(c^1\right) \leq -\frac{\omega}{1-\omega} u\left(y^2\right) - u\left(c^2\right).$$

Notice that $\frac{\omega}{1-\omega} > \frac{1-\omega}{\omega}$ as $\omega > .5$, which implies that whenever (1) holds, (2) holds with strict inequality (this is true as long as $c^1 \leq y^1$, see discussion below. If $c^1 > y^1$, then the opposite argument holds).

(c) Full risk sharing $\Leftrightarrow c^1 = c^2 = \frac{e}{2}$ which is clearly feasible. From (b) we know that we only need to check (1). (1) is satisfied if

$$\left\{\omega u\left(\frac{e}{2}\right) + (1-\omega)\, u\left(\frac{e}{2}\right)\right\} \geq \frac{1}{1-\beta}\left\{\omega u\left(y^1\right) + (1-\omega)\, u\left(y^2\right)\right\}$$

$$\Leftrightarrow u\,(e/2) \geq \omega u\left(y^1\right) + (1-\omega)\, u\left(y^2\right) \qquad (3)$$

(d) A higher p and a lower β both increase ω. The higher is ω, the higher is the autarky value when you have the high pay-off, which makes harder for the full-risk sharing equilibrium to be feasible.

For the next part, we can write the condition as

$$\Leftrightarrow u\,(e/2) \geq \omega u\,(e/2 + b) + (1-\omega)\, u\,(e/2 - b)\,,$$

and take second order Taylor approximations around $e/2$ on the RHS to get

$$\begin{aligned} u\,(e/2) \quad \geq \quad & \omega u(e/2) + \omega u'(e/2) b + \frac{w}{2} b^2 u''(e/2) + \\ & (1-\omega)\, u\,(e/2) - (1-\omega) u'(e/2) b + \frac{1-w}{2} b^2 u''(e/2) + O(3) \end{aligned}$$

$$\Leftrightarrow u\left(e/2\right) \geq u(e/2) + (2\omega - 1)u'(e/2)b + \frac{1}{2}b^2 u''(e/2) \Leftrightarrow$$

$$(2\omega - 1)u'(e/2) \leq -\frac{1}{2}bu''(e/2),$$

and for b small enough (i.e. taking the limit when $b \to 0$)

$$(2\omega - 1)u'(e/2) \leq 0.$$

which is a contradiction. So for b small enough perfect risk sharing is not possible.

To proof the same thing for σ small enough, take the same expression

$$(2\omega - 1)u'(e/2) \leq -\frac{1}{2}bu''(e/2),$$

$\Leftrightarrow$

$$(2\omega - 1) \leq \frac{-\frac{1}{2}bu''(e/2)}{u'(e/2)}$$

and use the utility function to get

$$(2\omega - 1) \leq \frac{b\sigma}{e}$$

and taking the limit when $\sigma \to 0$

$$(2\omega - 1) \leq 0,$$

which again is a contradiction.
(e) We want to proof that the best symmetric allocation satisfies

$$c^1 + c^2 = e$$
$$\omega \left[u\left(c^1\right) - u\left(y^1\right)\right] + (1 - \omega)\left[u\left(c^2\right) - u\left(y^2\right)\right] = 0$$

and $y^2 \leq c^2 \leq c^1 \leq y^1$ (i.e. satisfies the above two equations and has less variability than autarky).

Assume $c^1 \leq y^1$. Notice that if full risk sharing is not binding then $IC1$ must be binding, i.e.$\omega \left[u\left(c^1\right) - u\left(y^1\right)\right] + (1 - \omega)\left[u\left(c^2\right) - u\left(y^2\right)\right] = 0$. Can we have $c^2 > c^1$? No,. if full risk sharing is not attainable, neither it is an allocation with $y^2 \leq c^1 \leq c^2 \leq y^1$. To see that notice that

$$\omega u\left(e/2 - c\right) + (1 - \omega)\, u\left(e/2 + c\right) < u\left(e/2\right) < \omega u\left(e/2 + b\right) + (1 - \omega)\, u\left(e/2 - b\right),$$

where the second inequality comes from full risk sharing not being attainable and the first from the fact that $\omega > .5$ and concavity of u.

Can we have $c^1 > y^1$? No because an allocation like this will give the same expected payoff than autarky with more volatility, and thus autarky is a better solution.
(f) Here we compute numerically the optimal allocation for the case where the utility function is of the CRRA form: $u\left(c\right) = c^{1-\sigma}/\left(1 - \sigma\right)$.

Use the following parameters[1] $\beta = .65$, $p = 0.75$, $y^1 = 0.641$ and $y^2 = 0.359$. Plot the optimal c^1 and c^2 as functions of σ for the range $\sigma \in [1, 5]$ (i.e. use a grid over σ with enough points between 1 and 5) [2].

See graphs.

2 Risk Free rate Puzzle

(a)See graphs attached.

There are two forces counteracting here. For high levels of γ, the elasticity of substitution is really low and we need really high levels of the risk free rate to convince the individual to save enough to achieve the required growth rate in consumption. on the other hand, high γ means high risk aversion and thus as $u''' > 0$, a stronger precautionary savings motive. This last effect makes individuals save more for any level of the interest rate, and therefore we require a lower one to achieve the desired growth rate in consumption. To see that clearly, see how when the precautionary savings motive is absent ($\text{Var}_c = 0$) the interest rate is increasing in γ. And for enough variance, the second effect dominates the first and we get a decreasing pattern.

(b) This graph reflects how the second effect dominates in this case for γ big enough. Notice that we need $\gamma > 30$ to be able to explain the value of the risk free rate.

(c) This again reflect the same facts. And it makes things worst as we are lowering the variance of consumption.

[1] These parameters imply a standard deviation for log-output of .29 and a first-order auto-correlation of .5, matching findings by Heaton and Lucas (1996) using the PSID.

[2] Hint: Make sure you first check for perfect risk sharing. If full risk sharing is available take that allocation. Otherwise compute the allocation that satisfies the requirements in part (e), which may imply autarky or some insurance (watch out: do not compute an allocation with more variability than autarky!).