Solutions Problem Set 3

Macro III (14.453)

1. **Problem 1:** I will treat the interest rate as certain in all the results of the exercise. Notice that I can safely do that as there is no aggregate uncertainty.

   (a) This is pretty straightforward. We have

   $$c_t = \frac{r}{1+r}\left(A_t + y_t + \frac{1}{r}\overline{y}\right) - \pi$$
   $$A_{t+1} = (1+r)\left(A_t + y_t - c_t\right).$$

   Take the first equation and solve for A_t

   $$A_t = \frac{1+r}{r}[c_t + \pi] - y_t - \frac{1}{r}\overline{y}\ \forall t.$$

   Replace this on the second equation to get

   $$\frac{1+r}{r}[c_t + \pi] - y_t - \frac{1}{r}\overline{y} =$$
   $$(1+r)\left(\frac{1+r}{r}[c_{t-1} + \pi] - y_{t-1} - \frac{1}{r}\overline{y} + y_{t-1} - c_{t-1}\right),$$

   and rearranging terms

   $$\frac{1+r}{r}[c_t + \pi] - y_t - \frac{1}{r}\overline{y} = \frac{1+r}{r}c_{t-1} + \frac{(1+r)^2}{r}\pi - \frac{1+r}{r}\overline{y}$$
   $$\Leftrightarrow \frac{1+r}{r}\Delta c_t = (1+r)\pi + y_t - \overline{y}$$
   $$\Leftrightarrow \Delta c_t = r\pi + \frac{r(y_t - \overline{y})}{1+r} = r\pi + \frac{r}{1+r}\varepsilon_t$$

   (b) From the Euler equation we have

   $$\exp\{-c_t\} = \beta(1+r)E_t\left[\exp\{-c_{t+1}\}\right]$$
   $$E_t\left[\exp\{-\Delta c_{t+1}\}\right]\beta(1+r) = 1,$$

and using the result from before

$$\exp\{-r\pi\}E_t\left[\exp\{-\frac{r}{1+r}\varepsilon_{t+1}\}\right]\beta(1+r) = 1 \Rightarrow$$

$$\pi = \frac{1}{r}\left[\ln\{\beta(1+r)\} + \ln\{E_t\left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right]\}\right].$$

(c) If $\beta(1+r) = 1$,

$$\pi = \frac{1}{r}\left[\ln\{E_t\left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right]\}\right].$$

By Jensen's inequality, $E_t \exp > \exp E_t$. Thus

$$\pi = \frac{1}{r}\left[\ln\{E_t\left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right]\}\right] > \frac{1}{r}\ln\{\exp(-\frac{r}{1+r}E_t\varepsilon_{t+1})\} = 0.$$

Note that as the function exp is convex, a mean preserving spread of ε_t is going to increase $E_t\left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right]$,which implies that the bigger the uncertainty, the bigger is π.Intuitively that makes sense, the bigger the uncertainty the bigger is the precautionary motive (everything else equal) and thus the more you want to save.

(d) From the result in part a

$$\Delta c_t^i = r\pi + \frac{r}{1+r}\varepsilon_t^i,$$

and taking the average

$$\Delta C_t = \int_0^1 \Delta c_t^i dF_i = r\pi + \frac{r}{1+r}\int_0^1 \varepsilon_t^i dF_i,$$

and because there is no aggregate uncertainty $\int_0^1 \varepsilon_t^i dF_i = 0$. Thus

$$\Delta C_t = r\pi$$

Taking the equation for assets accumulation and replacing the consumption function we get

$$A_{t+1}^i = A_t^i + \varepsilon_t^i + (1+r)\pi,$$

and taking the average

$$A_{t+1} = \int_0^1 A_{t+1}^i dF_i = A_t + (1+r)\pi.$$

Both conditions together imply that in order to have finite and constant aggregate consumption and assets we need $\pi = 0$.The aggregate supply of capital (aggregate savings), is totally elastic at r^*.

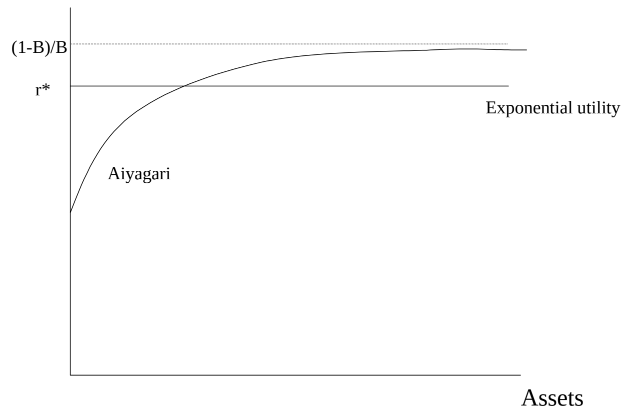

Figure 1:

To see that (see figure below), notice that from the accumulation equation, if $r > r^*$, the supply goes to infinity, while if $r < r^*$, then it goes to minus infinity. At $r = r^*$ the assets stay at the initial level always. The cross section distribution of consumption diverges over time because $\Delta c_t^i = \frac{r}{1+r}\varepsilon_t^i$. To see why is that, assume we start from a situation with complete equality, and ε can take two values, high and low, with probability .5 each. After one period, half the population had the good shock and half the bad one. After two periods we have 25% with two consecutive bad shocks and 25% with two consecutive good shocks. And if you keep going with this reasoning you easily see that the cross section diverges over time. A surprising result (at least for me when I solved the exercise) is the form of $A(r)$ if we compare it with the one in Aiyagari. Why does Aiyagari have continuity and we have discontinuity?. For $r < r^*$ is clear. Aiyagari has a borrowing constraint (either the natural or the exogenous one) that stops the agents from borrowing too much and thus assets for $r < r^*$will be finite. In our case this is not true, nothing stops the individual from borrowing as he can have negative consumption in the future. For $r > r^*$, from the proof in the class notes, we know that if absolute risk aversion goes to 0, assets are bounded above for a given $r < (1-\beta)/\beta$. The intuition for the result is that as assets become large, uncertainty from income is not important and the precautionary motive disappears. And this is what Ayagari has and this is the reason why assets are finite for $r^* < r < (1-\beta)/\beta$. But with CARA preferences by definition this is not true and consequently assets are not bounded above.

(e) We know that because of the precautionary savings the equilibrium interest rate is going to be lower than with complete markets. How much lower will tell us about how important is this precautionary motive. From the condition $\pi = 0$ we have

$$\ln\{\beta(1+r)\} + \ln\{E_t \left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right]\} = 0,$$

and notice that because of normality

$$E_t \left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right] = \exp\{.5 \left(\frac{r}{1+r}\right)^2 \sigma_\varepsilon^2.$$

The complete markets equilibrium interest rate is

$$r = \frac{1-\beta}{\beta} = 4,1666\%$$

and the incomplete markets one is given by

$$\ln\{\beta(1+r)\} + \ln\{E_t \left[\exp(-\frac{r}{1+r}\varepsilon_{t+1})\right]\} = 0$$

$$= \ln\{0.96(1+r)\} + 0.5 \left(\frac{r}{1+r}\right)^2 0.2^2 = 0$$

$$\ln(0.96) + \ln(1+r) + 0.5 \left(\frac{r}{1+r}\right)^2 0.2^2 \Rightarrow r^* = 4.1633\%.$$

Thus the precautionary motive is not very important. This might look surprising if compared to the results in Caballero(1991). In that article, the author computes aggregate savings in a OLG model with agents living for T periods. It is a partial equilibrium exercise where Caballero looks at the aggregate savings given that $\beta(1+r) = 1$.One of the results of the article is that savings will increase fast with T because of the precautionary savings motive. But the supply of capital (aggregate savings) becomes more and more elastic as T increases (as we saw before, it is perfectly elastic when T goes to infinity). Thus the general equilibrium effect won't be that big when T is big enough, the interest rate won't fall that much. So both results can be reconciled.

2. Problem 3:

   (a) The Pareto problem is

$$\max_{\{c_t^i(s^t)\}} \sum_{i\in I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t u^i \left(c_t^i \left(s^t\right)\right) \Pr\left(s^t\right)$$

subject to

$$Y(s_t) \equiv \sum_{i\in I} y^i(s_t) = \sum_{i\in I} c_t^i \left(s^t\right) \quad (\theta(s^t))$$

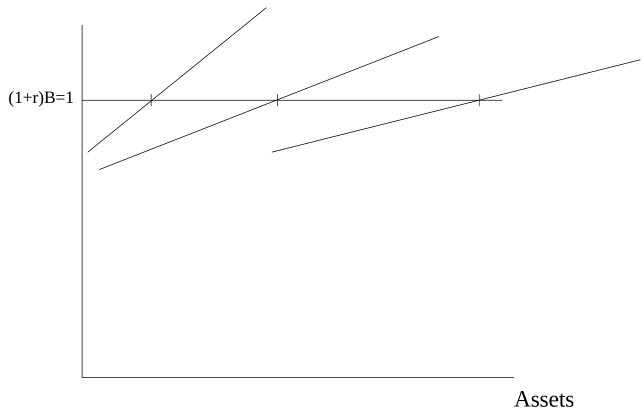

Figure 2:

for all t and $s^t \in S^t$. The F.O.C. are

$$c_t^i\left(s^t\right) : \lambda^i \beta^t u^{i\prime}\left(c_t^i\left(s^t\right)\right) \Pr\left(s^t\right) = \theta(s^t).$$

for all i t and $s^t \in S^t$.This implies

$$\lambda^i u^{i\prime}\left(c_t^i\left(s^t\right)\right) = \lambda^1 u^{1\prime}\left(c_t^1\left(s^t\right)\right)$$

for all i. So

$$c_t^i\left(s^t\right) = C_i(c_t^1\left(s^t\right), \frac{\lambda^1}{\lambda^i}).$$

Now replacing in the resources constraint,

$$Y\left(s_t\right) = \sum_{i \neq 1} C_i(c_t^1\left(s^t\right), \frac{\lambda^1}{\lambda^i}) + c_t^1\left(s^t\right),$$

which implies

$$\begin{aligned} c_t^1\left(s^t\right) &= G_1(Y(s_t), ..) = c_t^1\left(s_t\right) \\ c_t^i\left(s^t\right) &= G_i(Y(s_t), ..) = c_t^i\left(s_t\right). \end{aligned}$$

(b) The Pareto problem is

$$\max_{\{c_t^i(s^t), \{S_t(s^t)\}} \sum_{i \in I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u^i\left(c_t^i\left(s^t\right)\right) \Pr\left(s^t\right)$$

subject to

$$Y\left(s_t\right) - S_t(s^t) + (1 + r(s^t))S_{t-1}(s^{t-1}) = \sum_{i \in I} c_t^i\left(s^t\right) =$$

$$= C(s^t) \quad (\theta(s^t))$$

$$S_t(s^t) \geq 0$$

for all t and $s^t \in S^t$. The F.O.C. are

$$c_t^i\left(s^t\right) : \lambda^i \beta^t u^{i\,\prime}\left(c_t^i\left(s^t\right)\right) \Pr\left(s^t\right) = \theta(s^t).$$

for all i t and $s^t \in S^t$.This implies

$$\lambda^i u^{i\,\prime}\left(c_t^i\left(s^t\right)\right) = \lambda^1 u^{1\,\prime}\left(c_t^1\left(s^t\right)\right)$$

for all i. So

$$c_t^i\left(s^t\right) = H_i(c_t^1\left(s^t\right), \frac{\lambda^1}{\lambda^i}).$$

Now replacing in the constraint (for a given $C(s^t)$),

$$C\left(s^t\right) = \sum_{i\neq 1} H_i(c_t^1\left(s^t\right), \frac{\lambda^1}{\lambda^i}) + c_t^1\left(s^t\right),$$

which implies

$$\begin{aligned} c_t^1\left(s^t\right) &= G_1(C(s^t, ..) \\ c_t^i\left(s^t\right) &= G_i(C(s^t, ..) \end{aligned}$$

(c) From b) we know

$$\begin{aligned} \lambda^i u^{i\,\prime}\left(c_t^i\left(s^t\right)\right) &= \lambda^1 u^{1\,\prime}\left(c_t^1\left(s^t\right)\right) \Leftrightarrow \\ \lambda^i \exp\{-\gamma^i c^i\} &= \lambda^1 \exp\{-\gamma^1 c^1\} \Leftrightarrow \\ c^i &= \frac{\gamma^1}{\gamma^i} c^1 - \frac{1}{\gamma^i} B^{1,i},\ B^{1,i} = \log(\frac{\lambda^1}{\lambda^i}) \end{aligned}$$

and replacing that in the condition $\sum_{i\in I} c_t^i\,(s^t) = C(s^t)$ we get

$$C(s^t) = \gamma^1 \sum_{i\in I} \frac{1}{\gamma^i} c^1 - B^1,\ B^1 = \sum_{i\in I} \frac{1}{\gamma^i} B^{1,i},$$

which implies that

$$c^1 = \alpha^1 C(s^t) + \beta^1,\ \alpha^1 = \frac{\frac{1}{\gamma^1}}{\sum_{i\in I} \frac{1}{\gamma^i}},\ \beta^1 = \ \alpha^1 B^1.$$

Thus

$$\begin{aligned} c^i &= \frac{\gamma^1}{\gamma^i}(\alpha^1 C(s^t) + \beta^1) - \frac{1}{\gamma^i} B^{1,i} = \\ & \alpha^i C(s^t) + \alpha^i B^1 - \frac{1}{\gamma^i} B^{1,i}. \end{aligned}$$

Thus

$$c^i = \alpha^i C(s^t) + \beta^i,\ \alpha^1 = \frac{\frac{1}{\gamma^i}}{\sum_{i\in I} \frac{1}{\gamma^i}},\ \beta^i = \alpha^i B^1 - \frac{1}{\gamma^i} B^{1,i}.$$

Notice that
$$\sum_{i\in I}\alpha^i = \frac{\sum_{i\in I}\frac{1}{\gamma^i}}{\sum_{i\in I}\frac{1}{\gamma^i}} = 1,$$
and
$$\sum_{i\in I}\beta^i = B^1\sum_{i\in I}\alpha^i - \sum_{i\in I}\frac{1}{\gamma^i}B^{1,i} = B^1 - B^1 = 0.$$
Notice that individuals with high risk aversion have a lower α_1, the more risk averse they are the less fluctuations they face because the variable income is a smaller percentage of total income. What happens with transfers as a function of the degree of risk aversion? Among those with a positive transfer, those with high risk aversion get a smaller one. The same happens with negative transfers. And individual with a higher Pareto weight will have the same α^i than those with the same risk aversion, but he will get a higher transfer.

(d) From the FOC
$$\lambda^i(c^i)^{-\sigma} = \lambda^1(c^1)^{-\sigma} \Leftrightarrow c_i = \left(\frac{\lambda^1}{\lambda^i}\right)^{-\frac{1}{\sigma}} c_1$$
and replacing this in $\sum_{i\in I} c_t^i(s^t) = C(s^t)$ we get
$$c^1 = \frac{\left(\frac{1}{\lambda^1}\right)^{-\frac{1}{\sigma}}}{\sum\left(\frac{1}{\lambda^i}\right)^{-\frac{1}{\sigma}}}C(s^t),$$
and generally
$$c^i = \alpha^i C(s^t),\ \alpha^i = \frac{\left(\frac{1}{\lambda^i}\right)^{-\frac{1}{\sigma}}}{\sum\left(\frac{1}{\lambda^i}\right)^{-\frac{1}{\sigma}}}.$$
and
$$\sum\alpha^i = \frac{\sum\left(\frac{1}{\lambda^i}\right)^{-\frac{1}{\sigma}}}{\sum\left(\frac{1}{\lambda^i}\right)^{-\frac{1}{\sigma}}} = 1.$$
And the bigger the Pareto weight, the bigger the piece the individual gets from the total consumption.

(e) Notice that the expression in part c) is one of the specifications Townsend was using for testing risk sharing. In particular he regressed individual consumption on average consumption and individual income, and according to the model, the coefficient on individual income should be 0 if there is perfect risk sharing. And if we look at the expression in part d), takimg logs we have
$$\log c_t^i = const + \log C_t.$$

So according to this preferences, if we regress the log of individual consumption on a constant term, the log of average consumption and log of individual income, the coefficient on aggregate consumption should be one and that on income should be 0 if there is perfect risk sharing. Thus the exercise gives the justification for the specifications tested by Townsend.

3. See graphs attached.

   (a) Two main things can be stressed here. First of all, the policy function for consumption (assets) is (are) lower (higher) for any value of cash in hand when the individual is more risk averse. This just reflects that the precautionary savings motive is stronger the more risk averse is the individual. Secondly, the upper bound on assets is higher when the individual is more risk averse. Remembering the discussion in the first exercise, for the more risk averse individual the coefficient of relative risk aversion takes longer to go to 0 and thus the precautionary motive matters for longer.

   (b) Consumption is smoother than income, and is smoother the more risk averse is the individual, again reflecting a bigger precautionary savings motive. For the same reason asset holdings are bigger for the more risk averse individual. Notice that the agent is almost never constrained (never for sigma=3). This is reflecting the fact that the agent faces an implicit borrowing constraint. The fact that consumption can never be 0 prevents him for saving too little (in the case of sigma=3, the implicit borrowing constraint is so high that the agent is never constrained). This is important because it tell us that we should not try to evaluate the presence of borrowing constraints by looking at how many times an agent is effectively constrained, because the agent is precisely trying not to be.

   (c) The argument is similar to the one before. If there is the possibility of income being 0, the agent will always keep some positive savings to avoid 0 consumption and thus he will never be constrained.