# Solutions Problem Set 2

Macro III (14.453)

1. **Problem 1:** The general expression that can be derived for the CEQ-PIH case for the change in consumption is:

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[E_t y_{t+j} - E_{t-1}y_{t+j}\right]$$

so we only need to compute this expression for the different income processes. For parts a) b) and c) we can derive the general case for a given $\rho$ and then think about the different cases. So, assume $y_t = \bar{y} + \rho y_{t-1} + \varepsilon_t$, $0 \leq \rho \leq 1$.That means that (inverting the lag polynomial)

$$y_t = E_t y_t = \frac{1}{1-\rho}\bar{y} + \sum_{j=0}^{\infty}\rho^j \varepsilon_{t-j}$$

which implies that

$$E_{t-1}y_t = \frac{1}{1-\rho}\bar{y} + \sum_{j=1}^{\infty}\rho^j \varepsilon_{t-j}$$

and then

$$y_t - E_{t-1}y_t = \varepsilon_t.$$

You can easily see following the steps I just did that in general

$$E_t y_{t+j} - E_{t-1}y_{t+j} = \rho^j \varepsilon_t$$

and replacing this in the equation for the change in consumption

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{\rho^j}{(1+r)^j}\varepsilon_t = \frac{r}{1+r}\varepsilon_t\sum_{j=0}^{\infty}\frac{\rho^j}{(1+r)^j},$$

and given that $\frac{\rho}{1+r} < 1$ we have

$$\Delta c_t = \frac{r}{1+r}\varepsilon_t\frac{1}{1-\frac{\rho}{1+r}} = \frac{r}{1+r-\rho}\varepsilon_t.$$

So, assume $\rho = 0$ as in part a). In that case shocks are not persistent at all. If you have an increase in income today, that does not affect much your permanent income and thus you only want to consume a fraction $\frac{r}{1+r}$ of it (an annuity of it). If $\rho = 1$, that means that any shock in income translates directly to the same shock in permanent income, shocks in current income affect permanent income in the same value. In that case, as your income increases the same in all periods, you consume all the increase every period, $\Delta c_t = \frac{1}{1}\varepsilon_t = \varepsilon_t$. And for $0 < \rho < 1$ we have the intermediate case, $\frac{r}{1+r-\rho}$. We have persistence but not too strong (the more the closer to one) and thus consumption reaction will be half way between the two cases discussed before, $\frac{r}{1+r} < \frac{r}{1+r-\rho} < 1$. For the last case we have

$$y_t(1-L)(1-\rho L) = \overline{y} + \varepsilon_t \rightarrow$$

$$(1-L)y_t = \frac{1}{1-\rho}\overline{y} + \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j} \rightarrow$$

$$y_t = \frac{t}{1-\rho}\overline{y} + \sum_{j=0}^{\infty} \varepsilon_{t-j} + \sum_{j=1}^{\infty} \rho \varepsilon_{t-j} + \sum_{j=2}^{\infty} \rho^2 \varepsilon_{t-j} \ldots$$

which implies that

$$E_t y_{t+j} = y_{t+j} - \sum_{k=0}^{j-1} \varepsilon_{t+j-k} - \sum_{k=1}^{j-1} \rho \varepsilon_{t+j-k} - \ldots - \sum_{k=j}^{j-1} \rho^j \varepsilon_{t+j-k}$$

$$E_{t-1} y_{t+j} = y_{t+j} - \sum_{k=0}^{j} \varepsilon_{t+j-k} - \sum_{k=1}^{j} \rho \varepsilon_{t+j-k} - \ldots - \sum_{k=j}^{j} \rho^j \varepsilon_{t+j-k}$$

and thus

$$E_t y_{t+j} - E_{t-1} y_{t+j} = (1 + \rho + \ldots + \rho^j)\varepsilon_t = \frac{1-\rho^{j+1}}{1-\rho}\varepsilon_t$$

which gives us

$$\Delta c_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \frac{1-\rho^{j+1}}{1-\rho} \varepsilon_t =$$

$$\frac{r \varepsilon_t}{(1+r)(1-\rho)} \left[ \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} - \rho \sum_{j=0}^{\infty} \frac{\rho^j}{(1+r)^j} \right] =$$

$$\frac{1+r}{1+r-\rho}\varepsilon_t$$

And this is bigger than one. This is intuitive. Now a shock affects permanent income more than transitory and thus consumption should react more.The implications for the variance of consumption are straight forward. When income does not have much persistence ($\rho < 1$), then consumption reacts little to shocks and thus it varies less than income. When income has a lot of persistence ($\rho = 1$), the two variances are the same, consumption follows current income. In the last case, consumption has a higher variance than income. Here is when the excess smoothness puzzle appears. According to data, consumption varies less than income, while at the same time it can't be rejected that income behaves like in d). Thus the two facts are not consistent with the model. But can you really differentiate when income is a random walk or it follows a deterministic trend? The solution in the little game played in class.

2. **Problem 2:** the maximization problem is

$$\underset{c_t, A_{t+1}}{\mathsf{Max}}\, \mathsf{E}_0 \sum_{t=0}^{\infty} \beta^t \mathsf{u}\left(\mathsf{S}_t\right)$$
$$\text{s.t.}$$
$$\mathsf{S}_t = (1-\delta)\,\mathsf{S}_{t-1} + \mathsf{c}_t$$
$$\mathsf{A}_{t+1} = (1+\mathsf{r})\left(\mathsf{A}_t + \mathsf{y}_t - \mathsf{c}_t\right)$$

Note that from the accumulation equation we can write the stock of durables as

$$\mathsf{S}_t = \sum_{j=0}^{\infty} (1-\delta)^j \mathsf{c}_{t-j}$$

and replacing that on the utility function the final problem is

$$\underset{c_t, A_{t+1}}{\mathsf{Max}}\, \mathsf{E}_0 \sum_{t=0}^{\infty} \beta^t \mathsf{u}\left(\sum_{j=0}^{\infty} (1-\delta)^j \mathsf{c}_{t-j}\right)$$
$$\text{s.t.}$$
$$\mathsf{A}_{t+1} = (1+\mathsf{r})\left(\mathsf{A}_t + \mathsf{y}_t - \mathsf{c}_t\right) \;\; (\lambda_t \text{ is the associated multiplier for each restriction})$$

The $\mathsf{F.O.C.}$ with respect to $\mathsf{c}_t$ gives

$$\sum_{j=0}^{\infty} \beta^j \left(1-\delta\right)^j \mathsf{E}_t \mathsf{u}'\left(\mathsf{S}_{t+j}\right) = \lambda_t$$

while the condition for $\mathsf{A}_{t+1}$ is

$$\lambda_t = (1+\mathsf{r})\lambda_{t+1}.$$

and thus

$$\sum_{j=0}^{\infty} \beta^j \left(1-\delta\right)^j \mathsf{E}_t \mathsf{u}'\left(\mathsf{S}_{t+j}\right) = (1+\mathsf{r})\lambda_{t+1}$$

From the first equation we know (using the LIE)

$$\sum_{j=0}^{\infty} \beta^j (1-\delta)^j \, \mathsf{E}_t \mathsf{u}'(\mathsf{S}_{t+1+j}) = \lambda_{t+1}$$

and replacing this we get our result, equation (1) in the problem set. To get equation (2) in the problem set, there is not much to do, just following the steps explained is straightforward: the expression of (1) in $\mathsf{t}+1$ is ( once taken $\mathsf{E}_t$ and multiplying by $\beta(1-\delta)$)

$$\beta(1-\delta) \sum_{j=0}^{\infty} \beta^j (1-\delta)^j \, \mathsf{E}_t \mathsf{u}'(\mathsf{S}_{t+1+j}) =$$

$$\beta\beta(1-\delta)(1+\mathsf{r}) \sum_{j=0}^{\infty} \beta^j (1-\delta)^j \, \mathsf{E}_t \mathsf{u}'(\mathsf{S}_{t+2+j})$$

and substracting one from each other we get the result we were looking for,

$$\mathsf{u}'(\mathsf{S}_t) = \beta \mathsf{R} \mathsf{E}_t \mathsf{u}'(\mathsf{S}_{t+1})$$

The alternate route can be done in the following way: define

$$\tilde{\mathsf{A}}_t = \mathsf{A}_t + \mathsf{S}_{t-1}(1-\delta)$$

and substitute this in the budget constraint to get

$$\tilde{\mathsf{A}}_{t+1} - \mathsf{S}_t(1-\delta) = (1+\mathsf{r})\left(\tilde{\mathsf{A}}_t + \mathsf{y}_t - \mathsf{S}_{t-1}(1-\delta) - \mathsf{c}_t\right) \Rightarrow$$

$$\tilde{\mathsf{A}}_{t+1} = (1+\mathsf{r})\left[\tilde{\mathsf{A}}_t + \mathsf{y}_t - \underbrace{(\mathsf{S}_{t-1}(1-\delta) + \mathsf{c}_t)}_{S_t} + \frac{\mathsf{S}_t(1-\delta)}{1+\mathsf{r}}\right]$$

$$\tilde{\mathsf{A}}_{t+1} = (1+\mathsf{r})\left(\tilde{\mathsf{A}}_t + \mathsf{y}_t - \mathsf{S}_t\left[1 - \frac{(1-\delta)}{(1+\mathsf{r})}\right]\right).$$

If we rewrite the problem we have

$$\underset{S_t, A_{t+1}}{\mathsf{Max}} \, \mathsf{E}_0 \sum_{t=0}^{\infty} \beta^t \mathsf{u}(\mathsf{S}_t)$$

$$\text{s.t.}$$

$$\tilde{\mathsf{A}}_{t+1} = (1+\mathsf{r})\left(\tilde{\mathsf{A}}_t + \mathsf{y}_t - \mathsf{S}_t\left[1 - \frac{(1-\delta)}{(1+\mathsf{r})}\right]\right)$$

and the F.O.C. are

$$(\mathsf{S}_t): \beta^t \mathsf{u}'(\mathsf{S}_t) = \lambda_t (1+\mathsf{r})\left[1 - \frac{(1-\delta)}{(1+\mathsf{r})}\right]$$

$$(\mathsf{A}_{t+1}) \; \lambda_t = (1+\mathsf{r})\lambda_{t+1}.$$

and again writing the first condition in $t+1$ and substituting the $\lambda'$s in the second we get our result. Alternatively you could substitute the new budget constraint in the utility function, and the F.O.C. with respect to $A_{t+1}$ would give you the desired result.
With a quadratic utility function we know the marginal utility is linear, and so we get from the Euler condition

$$S_t = E_t S_{t+1}$$

or

$$S_{t+1} = S_t + \varepsilon_t, \; E_{t-1}\varepsilon_t = 0,$$

the stock of durables follows a Martingale distribution. Then from the accumulation equation,

$$\begin{aligned} c_t &= S_t - (1-\delta)S_{t-1} \\ c_{t-1} &= S_{t-1} - (1-\delta)S_{t-2} \end{aligned}$$

and substracting the second equation from the first

$$\Delta c_t = \Delta S_t - (1-\delta)\Delta S_{t-1}.$$

From the Martingale property we know

$$\Delta S_t = \varepsilon_t \text{ for all } t.$$

and thus we have our result,

$$\Delta c_t = \varepsilon_t - (1-\delta)\varepsilon_{t-1}.$$

And the intuition is clear, if you have a positive shock today and you get more durables, tomorrow you don't need to buy as much as today, as you still have the nondepreciated part of what you bought. In particular, when there is full depreciation, we are back to consumption being a random walk. Thus the presence of durables can make consumption to be dependent on past shocks

3. **Problem 3:** An equilibrium is a collection of factor prices such that firms maximize profits, and market of factors and goods clear at every period. Firms face the following problem,

$$\underset{K_t, L_t}{\text{Max}} \; K_t^{1/3} L_t^{2/3} - w_t L_t - (r_t + \delta) K_t$$

which gives us (remember $L = 1$)

$$\begin{aligned} \frac{K_t^{-2/3}}{3} &= r + \delta \Rightarrow K_t = \left(\frac{1}{3(\delta + r_t)}\right)^{\frac{3}{2}} \\ w_t &= K_t^{1/3}\frac{2}{3} = \frac{2}{3}\left(\frac{1}{3(\delta + r_t)}\right)^{\frac{1}{2}} \end{aligned}$$

Consumers maximize preferences. For the constrained agents the solution is clear, they consume their labor income every period,

$$c_t^1 = \frac{1}{2} w_t.$$

The unconstrained agents solve

$$\underset{\{c_t\}}{\mathrm{Max}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t.}$$

$$A_{t+1} = (1+r)\left(A_t + \frac{1}{2} w_t - c_t\right).$$

Solving as usual we get the standard Euler equation,

$$u'(c_t) = \beta(1+r)u'(c_{t+1}).$$

Markets have to clear which means $L_t = 1$ (we already used that before), and $A_{t+1} = (1+r)K_{t+1}$. In equilibrium we know that=

$$\begin{aligned} c_t^1 &= \frac{1}{2} w_t \\ w_t &= \frac{2}{3}\left(\frac{K_t}{L_t}\right)^{\frac{1}{3}} \frac{L_t}{L_t} = \frac{2}{3}\frac{Y_t}{L_t} = \frac{2}{3} Y_t \end{aligned}$$

and so

$$c_t^1 = \lambda w_t, \ \lambda = \frac{1}{3}.$$

Now let's solve for the steady state. From the Euler equation we know

$$u'(c_\infty) = \beta(1+r)u'(c_\infty) \Rightarrow \widehat{r}_\infty = \frac{\beta}{1-\beta}$$

and thus

$$\widehat{K}_\infty = \left(\frac{1}{3(\delta + \widehat{r}_\infty)}\right)^{\frac{3}{2}}$$
$$\widehat{w}_\infty = \frac{2}{3}\left(\frac{1}{3(\delta + r_\infty)}\right)^{\frac{1}{2}}$$
$$\widehat{Y}_\infty = \left(\widehat{K}_\infty\right)^{\frac{1}{3}}$$
$$\widehat{c}_\infty^1 = \lambda \widehat{w}_\infty$$
$$\widehat{c}_\infty^2 = \widehat{Y}_\infty - \delta \widehat{K}_\infty - \widehat{c}_\infty^1.$$

To see that the competitive equilibrium is not Pareto Optimal you just need to find the Euler equation for the second case (obtained as usual):

$$u'(c_t) = \beta(1 + r - \lambda(r+\delta))u'(c_{t+1}).$$

The two Euler equations are different and thus the equilibrium can't be optimum. Note that in steady state the P.O. solution has

$$\widetilde{\widetilde{\mathsf{r}}}_{\infty} = \frac{\beta}{(1-\beta)(1-\lambda)} + \frac{\lambda}{1-\lambda}\delta$$

which is lower than in the competitive equilibrium, which implies that agents are overaccumulating capital with respect to the optimum. They could have a lower capital stock and a higher consumption. This is surprising because it implies that capitalists would be happy if they were taxed and given this revenue. And constrained workers wouldn't want this kind of taxation on capital. The reason of this result is that the normal pecuniary externalities for an increase in the capital stock do not cancel out now. Assume a typical individual in a Ramsey economy, for a unit of additional capital his wage increases and the interest rate decreases. In equilibrium with a unit of labor supply the two effects cancel out. But now the individual has half the labor supply. So he will overacumulate capital. Mathematically, and additional unit of capital gives him (in capital labor ratio terms) if he has an income of $\mathsf{rk} + \mathsf{w} = \mathsf{f}'\mathsf{k} + \mathsf{f} - \mathsf{f}'\mathsf{k}$

$$\mathsf{f}'(\mathsf{k}) = \mathsf{r}$$

but if he has $\mathsf{rk} + \frac{1}{2}\mathsf{w}$ then he gets

$$\mathsf{f}''\mathsf{k} + \mathsf{f}' + \frac{1}{2}\mathsf{f}' - \frac{1}{2}\mathsf{f}''\mathsf{k} - \frac{1}{2}\mathsf{f}' = \mathsf{r} + \frac{1}{2}\mathsf{f}''\mathsf{k}$$

which is smaller for any value of $\mathsf{k}$ and so as he does no take it into account, he will tend to overacumulate with respect to the P.O.
If we compare this to an economy with no hand-to-moth consumers, the result is that we get the same capital stock in steady state as the Euler equation (and thus the steady state interest rate) is the same. The reason is that the hand-to-mouth consumers act as a lump-sum tax, it is the same as if we had a Ramsey economy and we took half of the labor income from the individuals. The marginal conditions for accumulation are not distorted in the steady state and thus they will accumulate the same and will consume less.