Solutions Problem Set 1

Macro III (14.453)

First of all a short note on the preferences we have.

$$v_t = \left[(1-\beta)c_t^\rho + \beta(E_t v_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}$$

Note that in that case we no longer restrict (as in the usual framework) the coefficient of risk aversion to be inversely related to the elasticity of intertemporal substitution. Now (1-ρ)$^{-1}$ is the elasticity of intertemporal substitution while α captures risk aversion. This will be important when discussing the results of the problem set.

1. In this question you are asked to proof that if $\{v_t\}$ is implied by $\{c_t\}$, then $\{\lambda v_t\}$ is implied by $\{\lambda c_t\}$. To do that we only need to verify our guess. Assume this is true for $t+j$, $j > 0$. Substitute in our lifetime utility and check that it also holds for time t.

$$\widetilde{v}_t = \left[(1-\beta)\lambda^\rho c_t^\rho + \beta(E_t \lambda^\alpha v_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \underset{\lambda \text{ is constant}}{=}$$

$$\left[(1-\beta)\lambda^\rho c_t^\rho + \lambda^\rho \beta(E_t \lambda^\alpha v_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} = \lambda\left[(1-\beta)c_t^\rho + \beta(E_t \lambda^\alpha v_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} =$$

$$= \lambda v_t \qquad Q.E.D.$$

To proof the next part, do the same thing,

$$\widetilde{v}_t = \left[(1-\beta)\psi^\rho + \beta(E_t \psi^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \underset{\psi \text{ is constant}}{=} [(1-\beta)\psi^\rho + \psi^\rho \beta]^{\frac{1}{\rho}}$$

$$= \psi[(1-\beta)+\beta]^{\frac{1}{\rho}} = \psi \qquad Q.E.D.$$

2. Now we want to build a measure of the cost of fluctuations as the one described in the lecture. η will be our measure and tells us how much do we

have to increase the consumption of the agent every period to compensate him for having variability, both because he does not like uncertainty and he does not like to substitute over time. When you have a deterministic and constant path of consumption $\{\overline{c}\}$, the ex ante lifetime utility is given by $u(\overline{c}) = (E_t\overline{c}^{\alpha})^{\frac{1}{\alpha}} = \overline{c}$ given the definition of $u()$ and the second result derived in 1.The ex ante lifetime utility of $(1+\eta)c$, given our first result, is given by $[E_t\{(1+\eta)v_t(c_t)\}^{\alpha}]^{\frac{1}{\alpha}} = (1+\eta)[E_t\{v_t(c_t)\}^{\alpha}]^{\frac{1}{\alpha}} = (1+\eta)u_0$. For the individual to be indifferent it has to be the case that

$$\begin{aligned} u(\overline{c}) &= u((1+\eta)c_t) \Leftrightarrow \\ \overline{c} &= (1+\eta)u_0 \Leftrightarrow \\ (1+\eta) &= \frac{\overline{c}}{u_0} \qquad Q.E.D. \end{aligned}$$

3. To solve this part, write our ex post lifetime utility for all periods with $\alpha = \rho$,

$$\begin{aligned} v_0 &= [(1-\beta)c_0^{\rho} + \beta(E_0 v_1^{\rho})]^{\frac{1}{\rho}} \\ v_1 &= [(1-\beta)c_1^{\rho} + \beta(E_1 v_2^{\rho})]^{\frac{1}{\rho}} \end{aligned}$$

...

Substitute v_1 in v_0 to get

$$\begin{aligned} v_0 &= \left[(1-\beta)c_0^{\rho} + \beta\{E_0\left[(1-\beta)c_1^{\rho} + \beta(E_1 v_2^{\rho})\right]^{\frac{1}{\rho}}\}^{\rho}\right]^{\frac{1}{\rho}} \\ &= \left[(1-\beta)c_0^{\rho} + \beta(1-\beta)E_0 c_1^{\rho} + \beta^2 E_0 v_2^{\rho}\right]^{\frac{1}{\rho}}, \end{aligned}$$

where I have used the law of iterated expectations to replace E_1 with E_0.If we keep replacing for $v_2, v_3, ...$we get to the general expression for a given time t

$$v_0 = \left[(1-\beta)\{c_0^{\rho} + \beta E_0 c_1^{\rho} + \beta^2 E_0 v_2^{\rho} + ... + \frac{\beta^t}{1-\beta}E_0 v_t^{\rho}\right]^{\frac{1}{\rho}},$$

and taking the limit when $t \to \infty$

$$v_0 = \left[(1-\beta)\left(c_0^{\rho} + E_0\sum_{t=1}^{\infty}\beta^t c_t^{\rho}\right)\right]^{\frac{1}{\rho}}.$$

given that $\beta < 1$ and that v_t is finite, which is a monotonic transformation of

$$v_0 = \left(\frac{c_0^\rho}{\rho} + E_0 \sum_{t=1}^{\infty} \beta^t \frac{c_t^\rho}{\rho} \right) ...Q.E.D.$$

The intuition for this is clear. When we assume $\alpha = \rho$, we no longer have a distinction between the coefficients that capture risk aversion and intertemporal substitution and thus, we are back to the standard expected utility framework with CRRA preferences.

4. Now our consumption path is deterministic and intuitively the coefficient that captures risk aversion should not play any role, lifetime utility should not depend on risk aversion if there isn't any. That is the reason why, even without assuming $\alpha = \rho$ we get the same expression than before. Algebraically, note that the α cancel in our expression as we know that v is deterministic. Thus we have

$$v_t = \left[(1-\beta) c_t^\rho + \beta v_{t+1}^\rho \right]^{\frac{1}{\rho}},$$

which is the same expression we had in 3.(without the expectation term), so doing the same steps we get to the same result,

$$v_0 = \left(\frac{c_0^\rho}{\rho} + \sum_{t=1}^{\infty} \beta^t \frac{c_t^\rho}{\rho} \right).$$

Now we have to compute the lifetime utility for $c_{lh} = \{c_l, c_h, c_l, c_h, ...\}$ and $c_{hl} = \{c_h, c_l, c_h, c_l, c_h, ...\}$. I'll solve the first case, the second can be found following the same steps. From our expression we have

$$\begin{aligned} v_0(c_{lh}) = \left[(1-\beta) \left(c_l^\rho + \beta c_h^\rho + \beta^2 c_l^\rho + ... \right) \right]^{\frac{1}{\rho}} = \\ \left[(1-\beta) \left(c_l^\rho + \beta c_h^\rho \right) (1 + \beta^2 + \beta^4 + ...) \right]^{\frac{1}{\rho}} = \\ \left[\frac{(1-\beta)}{1-\beta^2} (c_l^\rho + \beta c_h^\rho) \right]^{\frac{1}{\rho}} = v_{0lh} \end{aligned}$$

and symmetrically,

$$v_{0hl} = \left[\frac{(1-\beta)}{1-\beta^2} (c_h^\rho + \beta c_l^\rho) \right]^{\frac{1}{\rho}}.$$

As it was noticed before, the solution does not depend on α because the consumption path is deterministic.

But as we still have different consumption levels every period, it depends on the elasticity of substitution.

When we take the limit as $\beta \to 1$, the intuition is clear. The individual does not discount the future any more, he likes it as much as the present. So for him it does not make any difference in which order he gets the consumption levels and thus both paths of consumption give the same level of utility. Note that as we still have variability in consumption over time, the utility levels will depend on the elasticity of intertemporal substitution. Algebraically, applying L'Hopital Rule, $\lim_{\beta\to 1}\frac{1-\beta}{1-\beta^2} = \lim_{\beta\to 1}\frac{-1}{-2\beta} = \frac{1}{2}$ and then

$$v_0 = \lim_{\beta \to 1} v_{0hl} = \lim_{\beta \to 1} v_{0lh} = \left[\frac{1}{2}(c_h^\rho + c_l^\rho)\right]^{\frac{1}{\rho}} ...Q.E.D.$$

Using our equation for the welfare loss for $\overline{c} = \frac{1}{2}(c_h + c_l)$ and $u_0 = v_0$ obtained in 2 we get

$$(1+\eta) = \frac{\frac{1}{2}(c_h + c_l)}{\left[\frac{1}{2}(c_h^\rho + c_l^\rho)\right]^{\frac{1}{\rho}}}$$

Again the answer does not depend on α. Notice that if the individual has ∞ elasticity of intertemporal substitution ($\rho \to 1$) then there is no loss at all. This is reasonable since it does not matter to him that the consumption varies over time. And the less willing he is to substitute over time, the bigger the loss.

If instead of a deterministic path we have uncertainty, the parameter α will play a role. This is the case when we assume that consumption path can take any of the two ex ante with probability $\frac{1}{2}$.

$$u_0 = [E_0\{v_0\}^\alpha]^{\frac{1}{\alpha}} =$$

$$\left[\frac{1}{2}\left(\frac{(1-\beta)}{1-\beta^2}(c_h^\rho + \beta c_l^\rho)\right)^{\frac{\alpha}{\rho}} + \frac{1}{2}\left(\frac{(1-\beta)}{1-\beta^2}(c_l^\rho + \beta c_h^\rho)\right)^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}}.$$

When $\beta \to 1$ is totally indifferent between the two paths. So u_0 won't depend on α as he is not facing any uncertainty. Algebraically

$$\lim_{\beta \to 1} u_0 = \left[\frac{1}{2}\left(\frac{1}{2}(c_h^\rho + \beta c_l^\rho)\right)^{\frac{\alpha}{\rho}} + \frac{1}{2}\left(\frac{1}{2}(c_l^\rho + \beta c_h^\rho)\right)^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}} =$$

$$\left[\frac{1}{2}(c_h^\rho + c_l^\rho)\right]^{\frac{1}{\rho}}.$$

5. Now we have the opposite case. Once uncertainty is resolved in time 0, we have a deterministic and constant path of consumption. Intuitively, ex ante lifetime utility should not depend on the elasticity of intertemporal substitution but should depend on risk aversion as we have ex ante uncertainty. And it shouldn't depend on the discount factor as present is the same than future. Algebraically, we now that v_{0h} and v_{0l} are, using the result derived in the first part of the exercise,

$$v_{0h} = c_h \text{ and } v_{0l} = c_l.$$

This implies

$$u_0 = [E_0\{v_0\}^\alpha]^{\frac{1}{\alpha}} = \left[\frac{1}{2}(c_h^\alpha + c_l^\alpha)\right]^{\frac{1}{\alpha}}$$

6. Now we have both uncertainty and variability over time and thus, lifetime utility should depend on both parameters, α and β. Write the expression for u_0,

$$\begin{aligned} u_0 &= [E_0\{v_0\}^{\alpha}]^{\frac{1}{\alpha}} = \left[\frac{1}{2}v_0(c_l)^{\alpha} + \frac{1}{2}v_0(c_h)^{\alpha}\right]^{\frac{1}{\alpha}} \\ &= \left[\frac{1}{2}\{(1-\beta)c_l^{\rho} + \beta(E_0 v_1^{\alpha})^{\frac{\rho}{\alpha}}\}^{\frac{\alpha}{\rho}} + \frac{1}{2}\{(1-\beta)c_h^{\rho} + \beta(E_0 v_1^{\alpha})^{\frac{\rho}{\alpha}}\}^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}}, \end{aligned}$$

where I used the expression for v_0 given initially. Given that consumption is *i.i.d.* we have that

$$(E_0 v_1^{\alpha})^{\frac{1}{\alpha}} = (E_1 v_1^{\alpha})^{\frac{1}{\alpha}} = u_1$$

Because of stationary of the process and infinite horizon, the problem is the same in 0 and 1, i.e. $u_0 = u_1$.

Thus we can write

$$u_0 = \left[\frac{1}{2}\{(1-\beta)c_l^{\rho} + \beta u_0^{\rho}\}^{\frac{\alpha}{\rho}} + \frac{1}{2}\{(1-\beta)c_h^{\rho} + \beta u_0^{\rho}\}^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}} \ ...Q.E.D.$$

And as argued before, it depends on both parameters because the individual faces both uncertainty and fluctuations in consumption.

7. Solving with excel or mathlab, we obtain the following results for η:

$\alpha \backslash \rho$	1	.5	-1
1	0.0000%	0.0096%	0.0384%
.5	0.0004	0.0100	0.0388%
-1	0.0016	0.0112%	0.0400%

When the agent is risk neutral ($\alpha = 1$) and has a elasticity of intertemporal substitution of infinite ($\rho = 1$), there is no loss. for a given ρ, the more risk avert (smaller α), the bigger the loss. And for a given α, the less willing to substitute, the bigger the loss. Notice that the results vary more with ρ, that is, the consumer is affected more by variability on the path of consumption than for uncertainty when the tolerance to any of those things is small.