1 Two-Sided Lack of Commitment: Stationary Allocations

Motivation: In class we showed that for the symmetric case with 2 agents and 2 shocks the optimal allocation eventually reached a "memory-less" (history independent) allocation, i.e. consumption for each agent at time t depends only on the current state, s_t.

Although such an extreme history independence result is special to the 2x2 case, it does illustrate a more robust property of limited commitment models: the cross-sectional distribution of consumption does not diverge – inequality is bounded. This is a crucial difference with the asymmetric-information models.

Based on our result, for this 2x2 case, that in the long run the allocation is stationary, we now seek to characterize optimal stationary symmetric distributions. The idea is to learn how the possibilities for risk sharing depend on the parameters of our model.

(a) Given $\left(c^1, c^2\right)$ let $V^1\left(c^1, c^2\right)$ and $V^2\left(c^1, c^2\right)$ be the unique solutions to:

$$
\begin{aligned}
V^1 &= u\left(c^1\right) + \beta\left[pV^1 + (1-p)V^2\right] \\
V^2 &= u\left(c^2\right) + \beta\left[pV^2 + (1-p)V^1\right]
\end{aligned}
$$

clearly, $V^2(y,x) = V^1(x,y)$.

Show that,

$$
\begin{aligned}
V^1\left(c^1, c^2\right) &= \frac{1}{1-\beta}\left\{\omega u\left(c^1\right) + (1-\omega)u\left(c^2\right)\right\} \\
&\text{where } \omega = \frac{1-\beta p}{1+\beta-2p\beta} > \frac{1}{2}
\end{aligned}
$$

show that ω is **decreasing in β and increasing in p**.

(b) We call a stationary symmetric allocation **feasible** if it satisfies the resource and participation constraints:

$$c^1 + c^2 = e$$

$$
\begin{aligned}
V^1\left(c^1, c^2\right) &\geq V^1\left(y^1, y^2\right) \quad (1) \\
V^2\left(c^1, c^2\right) &\geq V^2\left(y^1, y^2\right) \quad (2)
\end{aligned}
$$

Notice that autarky is always feasible.

Show that in any symmetric allocation (2) never binds. That is, show that whenever (1) holds then (2) automatically holds with strict inequality.

(c) Show that full risk sharing is attainable if and only if:

$$u(e/2) \geq \omega u\left(y^1\right) + (1-\omega)u\left(y^2\right) \quad (3)$$

(d) Here we use the comparative static results for ω found in part (a) and the result in part (c) to examine the parameters that affect the feasibility of risk sharing.

How do β and p affect the likelihood of full-risk sharing being feasible?

Show that for small enough spread between y^1 and y^2 (holding e constant) full risk sharing is not possible. Let utility take the form $u(c) = c^{1-\sigma}/(1-\sigma)$ show that if σ is sufficiently close to 0 full risk is not feasible.

(e) If full risk sharing is not attainable we are interested in the best allocation that is feasible.

Using your results from part (b) show that if (3) is not satisfied the best symmetric allocation satisfies

$$c^1 + c^2 = e$$
$$\omega \left[u\left(c^1\right) - u\left(y^1\right) \right] + (1-\omega) \left[u\left(c^2\right) - u\left(y^2\right) \right] = 0$$

and $y^2 \leq c^2 \leq c^1 \leq y^1$ (i.e. satisfies the above two equations and has less variability than autarky).

(f) Here we compute numerically the optimal allocation for the case where the utility function is of the CRRA form: $u(c) = c^{1-\sigma}/(1-\sigma)$.

Use the following parameters[1] $\beta = .65$, $p = 0.75$, $y^1 = 0.641$ and $y^2 = 0.359$. Plot the optimal c^1 and c^2 as functions of σ for the range $\sigma \in [1, 5]$ (i.e. use a grid over σ with enough points between 1 and 5) [2].

2 Risk Free rate Puzzle

Motivation: The equity premium puzzle is the fact that, with CRRA ($u'(c) = c^{-\gamma}$), we require a high value of γ for the difference between the risk-free rate and stock's Euler equations to hold.

The risk-free rate puzzle is the fact that it not easy to match the risk-free rate's Euler equation – given the low real risk-free rate (about 1%) observed in the data[3]. It takes either a huge γ (as with the equity premium puzzle) or a small γ (inconsistent with resolving the equity premium puzzle) to match the risk-free rate puzzle.

(a) First we will set parameters to get a feel for the two forces at work. Set $\beta = .98$ and $\bar{c}_\Delta = 1.01$. Using the Euler equation for a risk-free asset compute r^f for $\sigma_c \in [0, .1]$ (equivalently $\sigma_c^2 \in [0, .01]$) and $\gamma \in [0, 5]$ using a fine enough grid on both intervals and computing r^f for all combinations from both grids. If possible, plot the resulting 3-D figure (see figure 10.1 on page 257 in Ljungqvist and Sargent's book; if possible set the perspective of your figure so that it look like their figure).

[1] These parameters imply a standard deviation for log-output of .29 and a first-order auto-correlation of .5, matching findings by Heaton and Lucas (1996) using the PSID.

[2] Hint: Make sure you first check for perfect risk sharing. If full risk sharing is available take that allocation. Otherwise compute the allocation that satisfies the requirements in part (e), which may imply autarky or some insurance (watch out: do not compute an allocation with more variability than autarky!).

[3] Hall, Flavin and others tested precisely the risk free rate's Euler equation. However, they typically did not use real interest rate return data, they did not test the Euler equation "in the levels": they allowed for an arbitrary constant while testing for the martingale implications of the theory. Examining this constant is what the risk-free rate puzzle is all about.

Show with your calculations that for low enough values of σ_c^2, r^f is increasing in γ (you can create a 2D plot as a function of γ for $\sigma_c = 0$) Why?

Show with your calculations that for high enough values of σ_c^2, r^f is decreasing in γ (again, plotting a 2D figure of r^f as a function of γ for some $\sigma_c = .1$) Why?

What are the two effects here? (hint: in your explanation you should use the fact that with CRRA marginal utility is a convex function of consumption)

(b) Now we set parameters to match the variability of aggregate consumption. Set $\beta = .99$ and $\bar{c}_\Delta = 1.018$. Note that, setting a lower β would only make matching $r^f = 1\%$ harder (some people have advocated allowing $\beta > 1$ to resolve the risk-free rate puzzle – we will not pursue this here).

Using $\sigma_c = 3.5637\%$ plot r^f for $\gamma \in [0, 40]$. This value for σ_c is the sample standard deviation of consumption of the growth rate of non-durables and services from the Mehra-Prescott data, which covers 1889-1978 ($\hat{\sigma}_c$ is higher for pre-war data than for post-war data. Measurement differences or real facts? There is a large debate over this).

(c) Plot the value of r^f with the post-war $\sigma_c = 1.24\%$ (this number is taken from Aiyagari (1993)) for $\gamma \in [0, 225]$.