Problem Set #3
Macroeconomic Theory III

1 Precautionary Savings in General Equilibrium

Let utility be given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $u(c) = -\exp\{-c\}$. Assume the standard intertemporal budget constraint

$$A_{t+1} = (1+r)(A_t + y_t - c_t).$$

(note: we do not necessarily impose $\beta(1+r) = 1$ here). Assume that y_t is *i.i.d.* across time and agents. Let $y_t = \bar{y} + \varepsilon_t$ where ε_t is i.i.d. and $E_{t-1}\varepsilon_t = 0$. We do not impose a borrowing constraint on this problem, A_t can take any value, although a no-Ponzi condition should be thought as being implicitly imposed for the problem to be well defined (you will not have to think about this no-Ponzi condition explicitly for solving the problem though).

(a) Show that the consumption function,

$$c_t = \frac{r}{1+r}\left[A_t + y_t + \frac{1}{r}\bar{y}\right] - \pi$$

for some π implies,

$$\Delta c_t = \frac{r}{1+r}\left[y_t - \bar{y}\right] + r\pi$$

(b) Use the Euler equation and your results in (a) to show that the consumption function in (a) is optimal for some π (hint: use the Euler equation to guess and verify the optimality of the above consumption function) which depends on r and the distribution of ε.

(c) Show that $\pi > 0$ if $\beta(1+r) = 1$. Compare this to the CEQ-PIH case. How does π depend on the uncertainty in y_t?

(d) Assume there is a constant measure 1 of individuals in the population. Argue that for aggregate consumption and assets to be constant and also finite in the long run we require that $\pi = 0$. What does this imply about average long-run capital holdings as a function of $r : A(r)$? What is happening to the cross-section of consumption? Does this distribution converge?

(e) Use your results in d to compute the equilibrium interest rate r^e for $\beta = .96$, $\bar{y} = 1$ and with ε distributed normal with mean zero and standard deviation equal to 0.2 (this distributional assumption allows you to find an explicit expression for $E\exp(-\varepsilon)$). Compare this to the interest rate that prevails without uncertainty.

2 Income Fluctuation Problem – Numerical Computation

This exercise asks you to compute numerically an income fluctuations problem. The problem is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to:

$$x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

where x_t represents "cash in hand" and y_t labor income. The income process is assumed to be i.i.d. with only two possible realizations $y_l = 1.5w$ and $y_h = .5w$ with 1/2 probability each. Set $\beta = .95$, $r = 2\%$ and $w = 1$.

Perform all calculations below for $\sigma = 1/2$ and $\sigma = 3$. Since you have been given the basic code do not hand in the Matlab code. Instead, stress the intuition for the results you obtain.

We solve this problem by iterating on the Bellman equation

$$v(x) = \max_{0 \le c \le x} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta E v\left((1+r)(x - c_t) + y_t\right) \right\}$$

starting from a decent guess (like the one obtained from consuming all future current income and the interest on current cash in hand – so that x_t remains constant).

(a) Solve the optimal consumption problem obtaining the consumption function $c(x)$. Plot the function for consumption, asset holdings and cash-in-hand for tomorrow (for both realizations of tomorrow's income shock).

(b) Use this policy function together with random shocks to simulate the evolution of cash-in-hand, income, consumption and assets for 200 periods. Plot the simulated series for consumption, income and assets. Is consumption smoother than income? How high are asset holdings? For what fraction of periods is the agent liquidity constrained (i.e. $x_t - c_t = 0$)? How do your results depend on σ? Discuss your results.

(c) (Carroll, 1997) Modify the income process to have the following characteristic: there is a small probability $p = .005$ of income being zero. If this event does not occur income is drawn from the same distribution as before.

Argue that, with the preferences above, the borrowing constraint will never bind: we always have $a_t = x_t - c_t > 0$. If we allow for some borrowing, so that we replace the constraint $a_t \geq 0$ with $a_t \equiv x_t - c_t \geq -b$ for some positive $b > 0$, argue that this condition will never bind and that in fact $a_t > 0$.

3 Perfect Risk Sharing

Consider a finite group I of individuals. Income for each individual is determined each period as a function of the current state of nature[1] $s_t \in S$ (where S is a finite set): $y_t^i(s_t)$. Denote aggregate income by $Y_t(s_t) \equiv \sum_{i \in I} y_t^i(s_t)$. Let utility for individual i be given by $E \sum_{t=0}^{\infty} \beta^t u^i(c_t)$ which under our assumption on uncertainty is,

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c_t\left(s^t\right)\right) \Pr\left(s^t\right).$$

Parts (a) and (b) are just a refresher from the lectures.

(a) Assume that there is no aggregate savings technology, that the state of nature is observable and that there are no commitment problems.

[1]Note that s_t summarizes the entire distribution of current income and possibly contains additional information, e.g. forcasts of future income.

Write out the Pareto problem for given Pareto weights $\left\{\lambda^i\right\}_{i\in I}$. Show that at the optimum consumption for individual i, $c_t^i\left(s^t\right)$, can be written as depending only on aggregate income in that period – once we control for $Y_t\left(s^t\right)$ consumption does not depend additionally on s^t.

(b) We now generalize the previous result. Assume there is a "storage technology": if in period $t-1$ an amount $S_t\left(s^{t-1}\right)\geq 0$ was put aside for storage, then in period t an amount $\left(1+r_t\left(s_t\right)\right)S_t\left(s^{t-1}\right)$ is available (for consumption or storage) in addition to any current income $Y_t\left(s^t\right)$. Show that a similar result as in (a) holds but that now we must condition on total consumption $C\left(s^t\right)\equiv\sum_{i\in I}c_t^i\left(s^t\right)$. (note that we impose the non-negativity constraint on storage, thus our result in a can be thought as a special case where $r_t\equiv 0$ so that at the optimum $S_t=0$ and thus $C_t=Y_t$).

(c) Let the utility function be of the CARA form

$$u^i\left(c\right)=\frac{-1}{\gamma^i}\exp\left\{-\gamma^i c\right\}$$

show that consumption takes the form: $c_t^i=a^iC_t+b^i$ where a^i and b^i are constants and $\sum a^i=1$ and $\sum b^i=0$. How does the distribution of γ^i affect a^i and b^i? How does the distribution of Pareto weights λ^i affect a^i and b^i?

(d) Let the utility function be of the CRRA form

$$u^i\left(c\right)=\frac{c^{1-\sigma}}{1-\sigma}$$

where the risk aversion σ is assumed to be the same for all individuals. Show that consumption takes the form $c_t^i=\alpha^iC_t$ with the constants α^i satisfying $\sum_{i\in I}\alpha^i=1$. How do the constants α^i depend on the Pareto weights λ^i?