August, 1981 LIDS-P-1138

JUMP BEHAVIOR OF CIRCUITS AND SYSTEMS

S.S. Sastry*
L.I.D.S.
M.I.T.
Cambridge, MA 02139

C. Desoer* and P. Varaiya
E.R.L.
Univ. of Calif.
Berkeley, CA 94720

INTRODUCTION

With particular reference to circuits we study the jump behavior, that is, the seemingly discontinuous change in state of systems driven by constrained (or implicitly defined) dynamics; i.e. $\dot{x} = f(x,y)$ $0 = g(x,y)$. To be specific, dynamics of a circuit are defined implicitly by specifying the velocities (time-derivatives) of capacitor voltages and inductor currents as well as the nonlinear resistive and Kirchoff constraints that the branch voltages and currents must satisfy. These constraints represent a constraint manifold over the base space of capacitor voltages and inductor currents. The process of integrating the circuit dynamics to obtain the transient response of the circuit consists of "lifting" the specified velocities to a vector field on the constraint manifold ("lifting" is the inverse operation of projecting). Lifting may not, however, be possible at points of singularity of the projection map, from the constraint manifold to the base space. We propose a way of resolving these singularities, consistent with the interpretation that the constraint manifold is a degeneration of very fast or singularly-perturbed dynamics. The physical meaning of this degeneration is the neglect of certain parasitic elements in the course of modelling. The detailed development is in [1].

1. CONSTRAINED DIFFERENTIAL EQUATIONS

Dynamics of circuits, power systems [2] and several other engineering systems are specified (implicitly) by constrained differential equations of the form

$$\Sigma \quad \begin{aligned} \dot{x} &= f(x,y) && (1.1)\\ 0 &= g(x,y) && (1.2) \end{aligned}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are smooth functions. Further, assume that 0 is a regular value of g. We try to interpret (1.1), (1.2) as describing implicity a dynamical system on the n-dimensional configuration manifold for Σ:

$$M = \{(x,y) : g(x,y) = 0\} \subset \mathbb{R}^{n+m}$$

*Research supported by DOE under grant ET-A01-2295T050 and by NSF under grant ENG-78-09032-A01.

The vector field X on M is specified by specifying its projection on the x-axis, namely,

$$\pi X(x,y) = f(x,y) \tag{1.3}$$

(here $\pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection map $(x,y) \to x$). At points at which $\pi TM(x,y)$, the projection of the tangent space to M at (x,y), is equal to $\mathbb{R}^n$ it is clear that f(x,y) uniquely specifies X. Difficulties arise when $\pi TM(x,y) \subsetneq \mathbb{R}^n$ and f(x,y) is transverse to $\pi TM(x,y)$. As specimens, two different kind of behavior are illustrated in Figure 1 at a point where M has a "fold".

(i) (Figure (1a)) f points out of the manifold M at (x_0,y_0) so that it would seem that the trajectory would jump off the manifold M, i.e., the y-coordinate changes discontinuously.

(ii) Figure (1b)) f points into the manifold M at (x_0,y_0) so that trajectories starting away from (x_0,y_0) do not tend towards (x_0,y_0).

These so called singular points of Σ are the points at which the implicit function theorem fails to hold in (1.2) in order to solve y as a function of x. At such points (x_0,y_0) it may not be possible to continuously extend an integral curve of Σ and it may be necessary to restart the integral curve of Σ at some (x_0,y_1) satisfying (1.2). We give a physically meaningful way of choosing this (x_0,y_1):

Empirical evidence leads us to postulate as in literature (a recent reference is [3]) that (1.2) is the degenerate limit as $\varepsilon \downarrow 0$ of

$$\varepsilon\dot{y} = g(x,y) \tag{1.4}$$

The system Σ is referred to as the degenerate system and the system (1.1), (1.4) for $\varepsilon > 0$ is referred to as the augmented system Σ_ε. For each $\varepsilon > 0$ the solution curves to Σ_ε are well defined.

The uniform limits of these solution curves as $\varepsilon \downarrow 0$ (provided they exist) are taken to be the solution concept for Σ. This is in keeping with the notion of consistent solutions in singular perturbation theory [4]. Thus, we have the following definition of jump behavior:

Definition 1. (Jump Behavior)

The solution of the system Σ described by (1), (2) is said to admit of jump from $(x_0,y_0) \in M$ to $(x_0,y_1) \in M$ if given $\delta > 0, \exists \varepsilon_0 > 0, t_0 > 0$ such that $\forall \varepsilon \in]0,\varepsilon_0]$

$$|x_\varepsilon - x_0| + |y_\varepsilon - y_0| < \delta$$

and for $t \in [\varepsilon t_0, \alpha]$

$$|x(t,\varepsilon) - \tilde{x}(t)| + |y(t,\varepsilon) - \tilde{y}(t)| < \delta$$

where $x(t,\varepsilon)$, $y(t,\varepsilon)$ is the trajectory of Σ_ε starting from $(x_\varepsilon, y_\varepsilon)$ at $t = 0$; $\tilde{x}(t)$, $\tilde{y}(t)$ is the trajectory of Σ starting from $(x_0,y_1) \in M$ at $t=0$ and defined on $[0,\alpha[$.

Remark: The intuitive content of our definition is that trajectories of the augmented system start close to one solution (x_0,y_0) of (1.2) and tend increasing rapidly towards trajectories starting for some other solution (x_0,y_1) of (1.2).

To get a feel for this definition rescale time in equation (1.1) and (1.4) and obtain with $\tau=t/\varepsilon$

$$\frac{dx}{d\tau} = \varepsilon f(x,y) \qquad (1.5)$$

$$\frac{dt}{d\tau} = g(x,y) \qquad (1.6)$$

so that in the limit that $\varepsilon \downarrow 0$ equations (1.5), (1.6) would only describe the dynamics of the frozen boundary layer system; $\mathcal{B}_{x_0}$

$$\frac{dy}{d\tau} = g(x_0,y) \quad : \mathcal{B}_{x_0} \qquad (1.7)$$

The assumptions required for the limits in Definition 1 above to exist are:

Assumption 1 (Complete Stability of $\mathcal{B}_{x_0}$)

For each $x_0 \in \pi M$, the system $\mathcal{B}_{x_0}$ is completely stable i.e. if $\xi(\tau,\tilde{y})$ is the trajectory of

$$\frac{dy}{d\tau} = g(x_0,y), \quad y(0) = \tilde{y}$$

then, $\lim_{\tau\to\infty} \xi(\tau,\tilde{y})$ exists and $\in\{\bar{y}: g(x_0,\bar{y}) = 0\}$.
Equivalently $\xi(\tau,\tilde{y})$ converges to an equilibrium point of $\mathcal{B}_{x_0}$ for each $\tilde{y}$. □

Assumption 2 (No Dynamic Bifurcation)

As (x_0,y_0) moves over M, the eigenloci of $D_2 g(x_0,y_0)$ cross the $j\omega$-axis only at the origin. □
The first observation that we make is that definition 1 allows for jump from non-singular points: First some notation: let y_0 by an equilibrium of the system $\mathcal{B}_{x_0}$ and let its attracting set or stable manifold be denoted

$$S_{y_0}^{x_0} = \{y: \lim_{\tau\to\infty} \xi(\tau, \tilde{y}) = y_0\} .$$

Theorem 1 (Jump Characterization from Non Singular Points).

Assume y_0 to be a hyperbolic equilibrium of $\mathcal{B}_{x_0}$ and let $\sigma(D_2 g(x_0,y_0)) \cap \mathbb{C}_+ \neq \emptyset$. Further let all sufficiently small neighbourhoods V of y_0 in $\{x_0\} \times \mathbb{R}^m$ be decomposed as

$$V = (V\cap S_{y_0}^{x_0}) \cup (V\cap S_{y_1}^{x_0}) \cup \dots (V \cap S_{y_p}^{x_0})$$

where $V\cap S_{y_i}^{x_0} \neq \emptyset$ for $i = 1,\dots,p$ and $S_{y_1}^{x_0}$ are the stable manifolds of the (hyperbolic) equilibria y_i of $\mathcal{B}_{x_0}$. Then Σ admits of jump from (x_0,y_0) to (x_0,y_1), (x_0,y_0) to $(x_0,y_2) \dots (x_0,y_0)$ to (x_0,y_p).

Comments: (i) The theorem is visualized in Fig. 2.
(ii) It is intuitive that a subset of M that does not admit of jumps is

$$M_a = \{(x,y) : g(x,y) = 0, \ \sigma(D_2 g(x,y)) \subset \mathbb{C}_-\} .$$

(iii) Of course, a similar theorem holds at singular points:

Theorem 2. (Jump Characterization from Singular Points

Let $\sigma(D_2 g(x_0,y_0)) \cap \{0\} \neq \emptyset$. Then Σ admits of jump from (x_0,y_0) if for all neighbourhoods V of (x_0,y_0) in $\{x_0\} \times \mathbb{R}^m$

$$V \subsetneq S_{y_0}^{x_0} .$$

Further, let all sufficiently small neighbourhoods V of (x_0,y_0) in $\{x_0\} \times \mathbb{R}^m$ be decomposed as

$$V = (V\cap S_{y_0}^{x_0}) \cup (V\cap S_{y_1}^{x_0}) \dots (V\cap S_{y_p}^{x_0})$$

where $V\cap S_{y_i}^{x_0} \neq \emptyset$ for $i = 1,\dots,p$ and the $S_{y_0}^{x_0}$ are stable manifolds of hyperbolic equilibria y_i of $\mathcal{B}_{x_0}$. Then Σ admits of jump from (x_0,y_0) to $(x_0,y_1),\dots,(x_0,y_p)$.

Comments: (i) The theorem is visualized in Fig. 3.
(ii) In general the hypothesis of the theorem (equation (1.7)) are verified by a study of the singularity using bifurcation theory. The detailed development is presented in [1]. Here, we show the pictures two of the singularities that occur if $D_2 g(x_0,y_0)$ has a single zero-eigenvalue.

Fold-Singularity

This is shown in Fig. 4. From the viewpoint of $\mathcal{B}_{x_0}$ two equilibria of $\mathcal{B}_{x_0}$ come together and annihilate each other. The flow in the vicinity of the fold boundary is as shown in Fig. 4.

Cusp-Singularity

This is shown in Fig. 5. From the viewpoint of $\mathcal{B}_{x_0}$ three equilibria of $\mathcal{B}_{x_0}$ fuse together and result in one equilibrium (conserving index). No jump is necessary at the cusp point and in the vicinity of the cusp point are two fold surfaces which have been studied above.

Other Singularities

A complete zoo of other singularities is possible, see for instance [5].

2. JUMP BEHAVIOR IN CIRCUITS AND PHYSICALLY MEASURABLE OPERATING POINTS

Consider the class of non-linear, time-invariant networks shown in Fig. 6:

(C)&(L). We assume the capacitors to be time-invariant charge controlled and inductors to be time-invariant flux controlled. Let $z \in \mathbb{R}^{n_c+n_\ell}$ represent charges on the capacitors ($z_1 \in \mathbb{R}^{n_c}$), fluxes in inductors ($z_2 \in \mathbb{R}^{n_\ell}$) and $x \in \mathbb{R}^{n_c+n_\ell}$ represent capacitor voltages ($x_1 \in \mathbb{R}^{n_c}$) and inductor currents ($x_2 \in \mathbb{R}^{n_\ell}$). Then, we assume

$$x = h(z) \tag{2.1}$$

with $h : \mathbb{R}^{n_c+n_\ell} \rightarrow \mathbb{R}^{n_c+n_\ell}$ a C^1 diffeomorphism.

(R) We assume that the linear time-invariant resistive n-port has a global hybrid representation i.e. if y is the hybrid vector of capacitor port currents (y_1) and inductor port voltages (y_2) with $x^T y$ representing power into the n-port then there exists a partition $A \cup B$ of $\{1,\ldots,n\}$ such that

$$\begin{aligned} x_A &= f_A(y_A, x_B) \\ y_B &= f_B(y_A, x_B) \end{aligned} \tag{2.2}$$

Using equations (2.1), (2.2) and Coulomb's, Faraday's law we have

$$\dot{x} = -Dh(h^{-1}(x)) \begin{bmatrix} y_A \\ f_B(y_A, x_B) \end{bmatrix} \tag{2.3}$$

$$0 = x_A - f_A(y_A, x_B) \tag{2.4}$$

Note that unless $A = \emptyset$ (when we have normal form equations) equations (2.3), (2.4) are a pair of constrained differential equations to which we apply the theory of the previous section. The physical significance of various assumptions and perturbations introduced in the foregoing will be presented at the meeting (see, also [1]). We only make a brief comment on the singular-perturbation assumption - (2.4) is the limit as $\varepsilon \downarrow 0$ of

$$\varepsilon \dot{y}_A = x_A - f_A(y_A, x_B)$$

This perturbation is shown dotted in Fig. 6, and is the multiport generalization of the following:

A current-controlled resistor is envisioned as the singularly perturbed limit as $\varepsilon \downarrow 0$ of the resistor in series with a small linear parasitic inductor because current is the controlling variable. The dual is true for a voltage-controlled resistor.

3. DETERMINISTIC AND NOISY CONSTRAINED DYNAMICAL SYSTEMS

In [6] we study noisy constrained systems of the form

$$\dot{x} = f(x,y) + \sqrt{\mu}\, \xi(t) \tag{3.1}$$

$$\varepsilon \dot{y} = g(x,y) + \sqrt{\lambda\varepsilon}\, \eta(t) \tag{3.2}$$

in the limit that $\varepsilon \downarrow 0$ (weakly convergent limits). Here $\xi(\cdot)$ and $\eta(\cdot)$ are independent vector valued white noise and λ, u scale their variance. It is remarkable that in the limit of noise variance tending to zero ($\lambda, \mu \downarrow 0$) the results for noisy constrained systems are quite different from those of the preceding section. Here we only illustrate the difference for the instance of a degenerate van der Pol oscillator.

Example 3.1 (Degenerate van der Pol oscillator).

$$\dot{x} = y$$

$$0 = -x - y^3 + y.$$

The phase portrait of the degenerate system including jumps from two fold singularities is shown in Fig. 7. Note the relaxation oscillation formed by including the two jumps.

Example 3.2 (Noisy degenerate van der Pol oscillator)

$$\dot{x} = y + \sqrt{\mu}\, \xi$$

$$\varepsilon \dot{y} = -x - y^3 + y + \sqrt{\lambda\varepsilon}\, \eta$$

for $\lambda, \mu > 0$ as $\varepsilon \downarrow 0$ the x-process converges (weakly on $C([0,T], \mathbb{R})$) to one satisfying

$$\dot{x} = \overline{y}^{\lambda}(x) + \sqrt{\mu}\ \ \xi$$

where $\overline{y}^{\lambda}$ is plotted for $\lambda_1, \lambda_2 > 0$ in Figure 8. In the further limit that $\varepsilon \downarrow 0$ followed by $\mu \downarrow 0$, x satisfies

$$\dot{x} = \begin{cases} \psi(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where $\psi(x)$ is shown heavy in Figure 8. Note the discontinuity of ψ at $x = 0$ and that <u>the relaxation oscillation is broken up by the presence of small noise</u>.

REFERENCES

1. Sastry, S.S., Desoer, C.A., and Varaiya, P., "Jump Behavior of Circuits and Systems", ERL Memo No., 80/44, Univ. of Cal. Berkeley, to appear <u>IEEE Trans</u>. CAS.

2. Sastry, S.S. and Varaiya, P., "Hierarchical Stability and Alert State Steering Control of Power Systems; <u>IEEE Trans. Circuits and Systems</u>, Vol. CAS-27 (1980), pp. 1102-1112.

3. Chua, L.O., Alexander, G.R., "The Effect of Parasitic Reactances on Non-Linear Networks", IEEE Trans. Circuits Theory, Vol.CT-18 (1971) pp. 520-532.

4. Fenichel, N., "Geometric Singular Perturbation. Theory for Ordinary Differential Equations", <u>Journal of Diff. Equations,</u> Vol. 31, (1979) pp. 53-98.

5. Thom, R., <u>Structural Stability and Marphogenesis</u>, Benjamin, 1975, Chap. 5, pp. 60-90.

6. Sastry, S. and Hijab, O., "Bifurcation in the Presence of Small Noise", LIDS-P-1089, M.I.T., to appear in <u>Systems and Control Letters.</u>

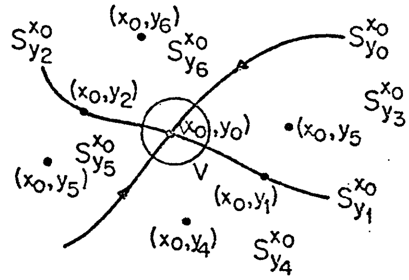

Figure 2. Jump from Non-Singular Points.

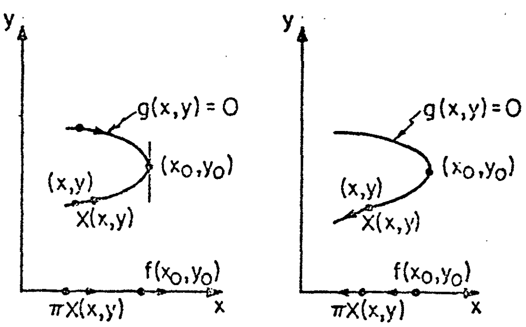

Figure 1. Illustrating the Nature of the Difficulty Obtaining $X(x,y)$ from $f(x,y)$.

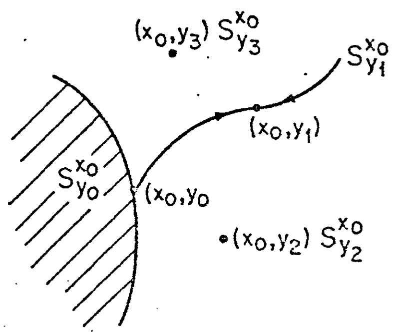

Figure 3. Jump from a (fold) Singularity

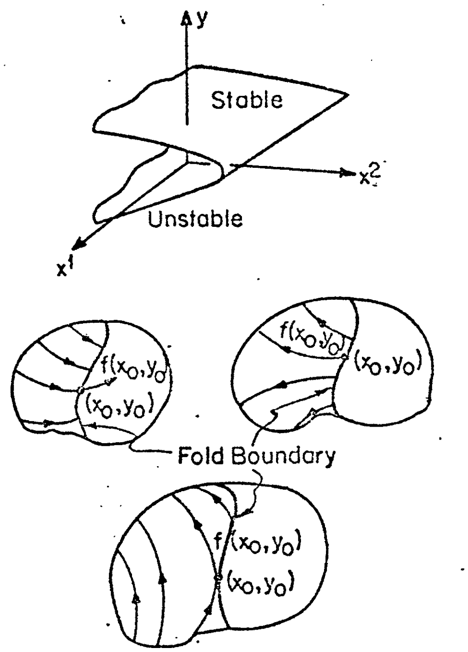

Figure 4. Fold Singularity and Flow near the Fold.

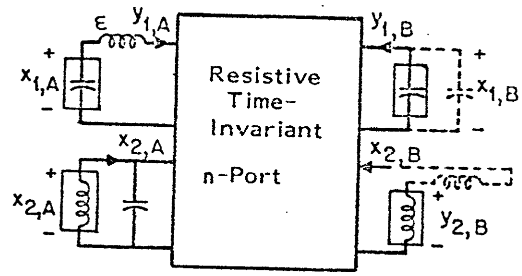

Figure 6. Nonlinear Circuit with Parasitics Introduced.

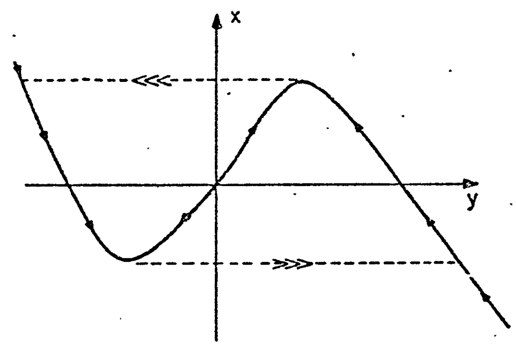

Figure 7. Degenerate Form of the Van Der Pol Oscillator Showing Jump Behavior.

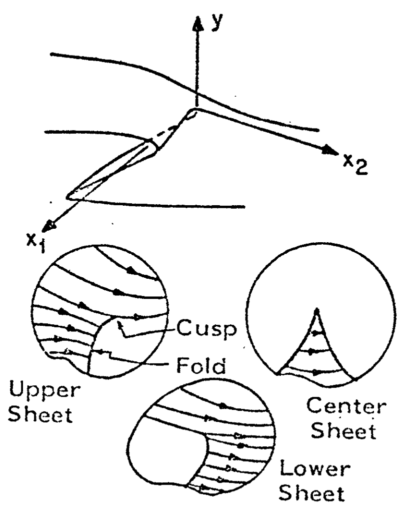

Figure 5. Cusp Singularity and Flow near the Cusp.

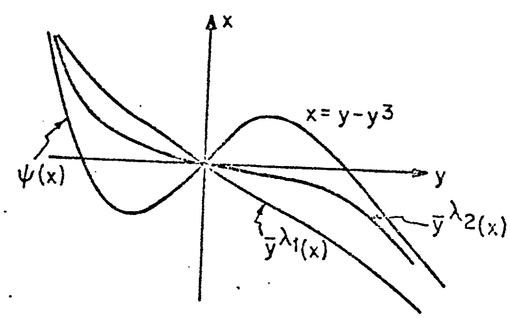

Figure 8. The Drift $\bar{y}^{\lambda}(x)$ for the Limit Diffusion of the Degenerate Van Der Pol Oscillator.