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COLLISIONAL EFFECTS ON TRAPPED PARTICLE MODES IN TANDEM MIRRORS

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ABSTRACT

The effects of collisions on trapped particle modes in tandem mirrors are analyzed. Two regimes are considered, a low collisionality regime, $\omega \sim \omega_*^i > \nu_e$ and a high collisionality regime, $\nu_i < \omega < \nu_e$. The magnetic geometry of the equilibrium is left arbitrary and a pitch angle scattering operator is used to model the effects of collisions. For $\omega > \nu_e$ electron collisions are found to destabilize an otherwise stable negative energy wave. Because of a boundary layer phenomenon the growth rate scales as $(\nu_e|\omega_*|)^{1/2} (B_{min}/B_{max})^{1/2} L_a/(L_a + L_c)$ where $B_{min}(B_{max})$ are the minimum (maximum) values of the magnetic field and $L_a(L_c)$ is the length of the anchor (central cell) region. For $\nu_i < \omega < \nu_e$ two modes are obtained: (a) a flute mode whose stability is determined by the flux tube integral of the beta weighted curvature drive and (b) a dissipative trapped ion mode driven unstable by the difference in collisionality between electrons and ions. The flute mode persists as $\omega \sim \nu_i < \nu_e$ while the dissipative trapped ion mode is damped by increasing ion collisionality.

I. INTRODUCTION

The present designs for tandem mirrors contain regions of unfavorable curvature linked to stable minimum B regions. At sufficiently low beta the field-line bending energy prevents the localization of an MHD mode to a bad curvature region and forces the eigenfunction to be flutelike through the machine. MHD stability is then determined by the average curvature drive which is designed to be favorable. Using a collisionless, high mode number theory it was shown, however, that such configurations were unstable to electrostatic modes which localize in regions of bad curvature and fall to near zero in regions of good curvature.[1] This localization is effected without the energy cost of creating perturbed magnetic field. The growth rate of such instabilities becomes comparable to the MHD growth rate as the number of particles linking the regions of good and bad curvature becomes small. In this paper we consider the effects of collisions on such modes.

We first consider a situation in which the collision frequency is small compared to the mode frequency. This is of interest for a case in which the trapped particle mode has been stabilized by the charge separation effects due to the spatial separation of electron and ion bounce points. This spatial separation of bounce points is incorporated in the current MFTF-B design. Such a stabilization mechanism creates a negative energy wave which can be destabilized by the dissipative effects of electron collisions. In Section III we calculate this destabilization using a boundary layer analysis.

In Section IV we consider the limit in which the collision frequency of electrons is much greater than the mode frequency. In this regime, which is of relevance to present experiments, there are two modes: an interchange mode whose stability depends on the beta weighted curvature drive and the dissipative trapped ion mode. This mode has been studied theoretically in tokamaks[2-10] and experimentally in the Columbia Linear Machine.[11] In the Columbia experiment the mode was found to saturate at levels of $\delta n/n<25\%$.

We begin in Section II with a discussion of the bounce averaged collisional drift kinetic equation and its boundary conditions in the context of a model equilibrium. We finish the paper with a summary and discussion of the results in Section V.

II. EQUILIBRIUM AND PERTURBED EQUATIONS

We consider a tandem mirror equilibrium consisting of cells linked by passing particles. Within each cell the magnetic field is assumed to vary with a scale length L_j where j labels the cell (central cell, plug, anchor, etc.). The cells are separated by field maxima whose scale length ΔL_B is assumed to be small compared to the cell scale length L_j. We assume the potential to be a constant except at the end of the machine where sharp positive and negative electrostatic maxima confine particles. The equilibrium distribution functions are taken to be equal temperature and density Maxwellians for both species.

In this model equilibrium configuration energy scattering is less important than pitch angle scattering and is therefore neglected. In particular, a pitch angle scattering event can convert a trapped particle into a passing particle and thus modify the response of the distribution function to the perturbing potential.

The perturbed distribution function $\tilde{f}$, is given by [4]

$$\tilde{f} = f \exp(iS(\alpha,\beta) - i\omega t)$$

where

$$f = q\phi \frac{\partial F_o}{\partial \varepsilon} + J_o h . \tag{1}$$

In Eq. (1) h is the non-adiabatic portion of the perturbed distribution function and is the solution to the drift kinetic equation

$$(\omega - \omega_d + iv_{||}\underset{\sim}{b} \cdot \nabla')h = - \omega \left[\frac{\partial F_o}{\partial \varepsilon} - \frac{\underset{\sim}{b} \times \nabla S \cdot \nabla' F_o}{m\Omega}\right] J_o q\phi + iC(h) \quad (2)$$

where C(h) is pitch angle scattering operator,(10)

$$iC(h) \equiv +i\nu \frac{(1 - \lambda B)^{1/2}}{B} \frac{\partial}{\partial\lambda}\left[\lambda(1 - \lambda B)^{1/2} \frac{\partial h}{\partial\lambda}\right].$$

We list below the definitions of the terms which appear in Eq. (1) - (2):

$$\underset{\sim}{b} \equiv \frac{\underset{\sim}{B}}{|\underset{\sim}{B}|} = \frac{\nabla\alpha \times \nabla\beta}{|\underset{\sim}{B}|}$$

$$\omega_d \equiv \frac{\nabla S \cdot \underset{\sim}{b} \times (mv_{||}^2 \underset{\sim}{b} \cdot \nabla\underset{\sim}{b} + \mu\nabla B + q\nabla\Phi_o)}{m\Omega}$$

$$\Omega = qB/mc$$

$$v_{||} = (2/m)^{1/2} (\varepsilon - \mu B - q\Phi_o)^{1/2}$$

$$\varepsilon = mv^2/2 + q\Phi_o$$

$$\mu = \frac{mv_\perp^2}{2B} \quad (3)$$

$$\lambda \equiv \frac{\mu}{\varepsilon}$$

$$F_o = F_o(\varepsilon, \alpha, \beta)$$

$$J_0 \equiv J_0\left(\frac{v_\perp |\nabla S|}{\Omega}\right)$$

$$\nu_e = 2\left(\frac{T}{\varepsilon}\right)^{3/2} \frac{\omega_{pe}^{2} e^2 m_e^{1/2}}{(2T)^{3/2}} \ \ln \Lambda \left\{ 1 + H\left[\left(\frac{\varepsilon}{T}\right)^{1/2}\right]\right\}$$

$$\nu_i = 2\left(\frac{T}{\varepsilon}\right)^{3/2} \frac{\omega_{pi}^2 e^2 m_i^{1/2}}{(2T)^{3/2}} \ln\Lambda \ H\left[\left(\frac{\varepsilon}{T}\right)^{1/2}\right]$$

$$H(z) = \frac{1}{\sqrt{\pi}}\frac{1}{z}\, e^{-z^2} + \left(1 - \frac{1}{2z^2}\right)\frac{2}{\sqrt{\pi}}\int_0^z dt\ e^{-t^2} \quad .$$

In Eq. (2) the prime on the spatial gradient signifies that ε and μ are to be held fixed in the differentiation. All perturbed quantities, $\tilde{\xi}$, are assumed to vary like $\xi = \xi(\varepsilon,\mu,\alpha,\beta,\ell)$ exp $(iS - i\omega t)$ where $S = \overset{o}{m}\beta + \overline{S}(\alpha)$ is a constant along a field line and $|\nabla S|\, \xi >> |\nabla\xi|$. This reflects a perturbation with short perpendicular wavelengths compared to equilibrium scale lengths while allowing arbitrary parallel wavelengths. The wave frequency, ω, is assumed to be less than the gyro frequency. The equilibrium distribution function $F_0(\varepsilon,\alpha,\beta)$ is independent of ℓ the distance along a field line. For simplicity we have restricted ourselves to a purely electrostatic perturbation and have ignored the compressional magnetic perturbations.

Because the equilibrium potential Φ_o is a constant axially it plays no significant role in Eq. (2). We therefore eliminate it by introducing the Doppler shifted frequency $\omega' \equiv \omega - \omega_E$ where $\omega_E = \overset{o}{m} c\, (\partial\Phi_o/\partial\alpha)$. This corresponds to a transformation to a frame moving at the local $E \times B$ velocity in which the local electric field vanishes. For notational simplicity we suppress the prime on ω in the analysis that follows.

We now consider cases where the transit time of particles through the anchor region is short compared to a wave period or an effective collision time. Expanding h in powers of ω/ω_b the lowest order equation is

$$iv_{||}\underset{\sim}{b} \cdot \nabla h_o = 0, \tag{4}$$

that is, h_o is a constant along a field line: $h_o = h_o(\lambda;\varepsilon,\alpha,\beta)$. The next order equation averaged over a particle bounce motion yields the constraint equation which determines h_o

$$(\omega - \bar{\omega}_d)h_o = -(\omega - \omega_*)J_o q\bar{\phi} \frac{\partial F_o}{\partial \varepsilon} + i \frac{\nu(\varepsilon)}{\hat{\tau}(\lambda)} \frac{\partial}{\partial \lambda}\left[D(\lambda) \frac{\partial h_o}{\partial \lambda}\right] \tag{5}$$

where

$$\hat{\tau}(\lambda) \equiv \int \frac{d\ell}{(1 - \lambda B)^{1/2}}$$

$$D(\lambda) \equiv \int \frac{d\ell}{B} \lambda(1 - \lambda B)^{1/2} \tag{6}$$

$$\bar{f}(\lambda) \equiv \frac{1}{\hat{\tau}} \int \frac{d\ell\, f(\ell)}{(1 - \lambda B)^{1/2}}$$

and

$$\omega_* \frac{\partial F_o}{\partial \varepsilon} = \frac{\underset{\sim}{b} \times \nabla S \cdot \nabla' F_o}{m\Omega} - m^o c \left(\frac{\partial \Phi_o}{\partial \alpha}\right) \frac{\partial F_o}{\partial \varepsilon} .$$

In the field line integral, the limits of integration are the bounce points, ℓ_b, where $B(\ell_b) = 1/\lambda$.

Equation (5) applies to three classes of particles:

(1) passing particles for whom $0<\lambda<1/B_{max}$

(2) particles trapped in the central cell for whom

$(1/B_{max}) < \lambda < (1/B^{cc}_{min})$

(3) particles trapped in the anchor for whom

$(1/B_{max}) < \lambda < (1/B^{a}_{min})$

where B_{max} is the maximum field point and B^{j}_{min} is the field minimum in region j. We distinguish the distribution function $h_o(\lambda;\varepsilon,\alpha,\beta)$, normalized time $\hat{\tau}(\lambda)$, and diffusion coefficient $D(\lambda)$ for each class by the subscripts p, t(cc) and t(a) respectively.

The perturbed potential ϕ is determined self consistently through the quasi-neutrality condition,

$$0 = \sum_{i,e} q \left[\int d^3v \; q\phi \frac{\partial F_o}{\partial \varepsilon} + \int d^3v \; J_o h \right] . \tag{7}$$

Writing the velocity integral in terms of λ and ε

$$\int d^3v = \frac{4\pi B}{m^2} \left(\frac{m}{2}\right)^{1/2} \int_0^{\infty} d\varepsilon \varepsilon^{1/2} \int_0^{1/B} d\lambda \; \frac{1}{(1 - \lambda B)^{1/2}}$$

we see that in regions where the magnetic field varies slowly the eigenfunction ϕ will also vary slowly and that ϕ will change where particles bounce.

We now consider the boundary conditions on h_o. Since h_o is independent of gyrophase $(\partial h_o/\partial \theta) = 0$ at $\theta = 0$ where $\cos \theta = v_{||}/|\underset{\sim}{v}|$. The angle θ is the polar angle in the $\left(v_{||}, \underset{\sim}{v}_{\perp}\right)$ velocity coordinate system. In terms of the (ϵ,λ) coordinate system the boundary condition at $\theta = 0$ implies that

$$\frac{\partial h_o}{\partial \lambda} \xrightarrow[\lambda \to 0]{} 0 . \tag{8}$$

Because of the high bounce frequency assumption, h_o is a constant along a field line and thus is equal for positive and negative going particles. Thus h_o is symmetric in the $(v_{||}, \underset{\sim}{v}_{\perp})$ velocity coordinate system about the plane $v_{||} = 0$ which corresponds to $\theta = \pi/2$. This implies that $(\partial h_o/\partial\theta) = 0$ at $\theta = \pi/2$ which in the (ϵ,λ) coordinate system becomes

$$(1 - \lambda B)^{1/2} \frac{\partial h_o}{\partial \lambda} \xrightarrow[\lambda \to \frac{1}{B}]{} 0. \tag{9}$$

In particular for deeply trapped particles

$$D(\lambda) \frac{\partial h_o}{\partial \lambda} \xrightarrow[\lambda \to \frac{1}{B^j_{min}}]{} 0. \tag{10}$$

The boundary condition on h_o at the boundaries between passing and trapped particles is determined by the condition that the sum of the fluxes into the boundary vanish. This is complicated by the existence of two boundary layers at $\lambda = 1/B_{max}$ the separatrix between passing and trapped particles. We will argue that the contributions of these boundary layers to the flux condition is small and can be ignored.

The inner boundary layer is due to the logarithmic divergence in the bounce period of particles which stagnate at the magnetic field maxima. We denote the width of this layer by $\delta\lambda_{log}$. It can be shown that

$$\delta\lambda_{log} B_{min} \sim \exp(-L/\Delta L_B)$$

where ΔL_B is the magnetic field scale length near the maximum and L is the length of the center cell or anchor.

The second boundary layer is due to the collisional pitch-angle spreading of distribution function perturbations during a particle transit time. Over the bulk of the distribution function these transit time effects contribute an order $(\nu/\hat{\omega}_b)$ modification to the infinite bounce frequency distribution function, where $\hat{\omega}_b \equiv (T/m)^{1/2}/L$. We will neglect this contribution. At the boundary between classes of particles, however, the infinite bounce frequency distribution function has an unphysical discontinuous derivative. Within a narrow layer about the separatrix these discontinuities are resolved by the finite collisional spreading during a particle transit time. We denote the width of this layer $\delta\lambda_t$ and estimate its width by comparing the parallel streaming term to the collision operator,

$$\delta\lambda_t \sim \frac{1}{\bar{B}} \left[\frac{\nu}{\hat{\omega}_b} \frac{\bar{B}}{B_{max}} \left(1 - \frac{\bar{B}}{B_{max}} \right) \right]^{1/2} \qquad (11)$$

where $\bar{B}$ is a typical field strength within the cell. For a square well $\bar{B} = B_{min}$. We assume that the equilibrium parameters are such that the logarithmic layer lies inside the transit time layer. However we assume that the transit time layer, $\delta\lambda_t$, is itself small compared to either the width of the passing particle region of pitch-angle space or to the width over which collisions modify the infinite bounce frequency distribution function during a wave period. We can estimate this last width by comparing the wave frequency, ω, to the collision operator. In the high collision frequency limit $\nu \gtrsim \omega$, the entire distribution function is affected. In the low collisionality regime we again have a boundary layer phenomenon about the separatrix between passing and trapped particles. Comparing the wave frequency to the collision operator gives the width of the collisional boundary layer

$$\Delta\lambda_{coll}\, \overline{B} \sim \left[\left(\frac{\nu}{\omega}\right) \left(1 - \frac{\overline{B}}{B_{max}}\right) \frac{\overline{B}}{B_{max}}\right]^{1/2} . \qquad (12)$$

Comparing the expressions for $\Delta\lambda_{coll}$ and $\delta\lambda_t$ we see that the transit time layer is contained within the collisional layer

$$\frac{\Delta\lambda_{coll}}{\delta\lambda_t} \sim \left(\frac{\hat{\omega}_b}{\omega}\right)^{1/2} >> 1.$$

We now consider the flux condition and show that the effects of the transit time layer can be neglected.

In order to calculate the flux condition we begin with the local statement of particle conservation by the collision operator

$$\int d^3v\; C(h) = 0 \qquad (13)$$

where both C and h have not yet been bounce averaged. Integrating over a flux tube gives

$$0 = \int_0^\infty d\varepsilon\, \varepsilon^{1/2}\, \frac{4\pi\nu}{m} \left(\frac{m}{2}\right)^{1/2} \left\{ \int_0^{\overline{\lambda}_*} d\lambda\, \frac{\partial}{\partial\lambda} \left[D_p \frac{\partial h}{\partial\lambda}\right]\right.$$

$$+ \sum_j \int_{\lambda_*^+}^{\lambda_{max}^j} d\lambda \; \frac{\partial}{\partial\lambda}\left[D_{t(j)} \; \frac{\partial h}{\partial\lambda}\;\right] \Bigg\} + I_1 \tag{14}$$

where

$$I_1 \equiv \int \frac{d\ell}{B} \int_S d^3v \; C(h) \equiv \frac{4\pi\nu}{m^2} \left(\frac{m}{2}\right)^{1/2} \int_0^\infty d\varepsilon\varepsilon^{1/2} \; \hat{I},$$

$$\lambda_*^{\pm} = \frac{1}{B_{max}} \pm \delta\lambda_t$$

$$\lambda_{max}^j = \frac{1}{B_{min}^j}$$

and S is the transit time boundary layer region. In the first two integrals, the infinite bounce frequency equation is valid so $h = h_0$ a constant along a field line. Using the boundary conditions on h_0 at $\lambda = 0$ and $\lambda = 1/B_{min}$, Eq.(8) and Eq.(10), gives

$$0 = D_p \frac{\partial h_p}{\partial\lambda}\bigg|_{\lambda = \lambda_*^-} - \sum_j D_{t(j)} \frac{\partial h_{t(j)}}{\partial\lambda}\bigg|_{\lambda = \lambda_*^+} + \hat{I}_1 \tag{15}$$

where h_p and $h_{t(j)}$ refer to the passing and trapped portions of $h_0(\lambda)$, and

$$\hat{I}_1 \approx \sum_{\mathrm{sgn}\,(v_{||})} \int_{\lambda_*^-}^{\lambda_*^+} \frac{d\lambda}{2} \int \frac{d\ell}{B} \frac{\partial}{\partial\lambda} \left[\lambda (1 - \lambda B)^{1/2} \frac{\partial h(\lambda, \ell)}{\partial \lambda} \right]. \quad (16)$$

Physically the quantities $D(\partial h/\partial\lambda)$ are the collisional fluxes of particles into the transit time boundary layer centered on the separatrix between passing and trapped regions of velocity space, while the quantity $\hat{I}$ represents the rate of change in the number of particles within the transit time boundary layer integrated over a flux tube. We now argue that the latter contribution is small and that therefore the fluxes into the transit time boundary sum to zero to lowest order.

In the low collisionality regime the fluxes into the boundary, $D(\partial h/\partial\lambda)$, are of order $D^\circ\, h/\Delta\lambda_{coll}$ where the diffusion coefficient D° is $D^\circ = L(\bar{B}\, B_{max})^{-1} \times (1 - \bar{B}/B_{max})^{1/2}$ and $\Delta\lambda_{coll}$ is given in Eq. (12). The integral $\hat{I}_1$ can be estimated by using the kinetic equation and can be shown to be smaller than the surface flux by $(\omega/\hat{\omega}_b)^{1/2}$. In the high collisionality regime the surface fluxes are of order $D^\circ h_o \bar{B}$ and $\hat{I}_1$ is smaller than the surface flux by a factor of

$$(\omega/\hat{\omega}_b)^{1/2} (\omega/\nu)^{1/2} (B_{max}/\bar{B})^{1/2} (1 - \bar{B}/B_{max})^{-1/2}$$

which we will assume to be small. Thus in both cases we drop the factor $\hat{I}_1$ giving as the flux condition

$$D_p \frac{\partial h_p}{\partial \lambda} \bigg|_{\lambda = \lambda_*^-} = \sum_j D_{t(j)} \frac{\partial h_{t(j)}}{\partial \lambda} \bigg|_{\lambda = \lambda_*^+} . \quad (17)$$

The final boundary condition we require is the continuity of h_o at the separatrix $\lambda = B_{max}^{-1}$. The distribution function h varies by an amount

$$\delta h \sim \delta\lambda_t \frac{\partial h_o}{\partial \lambda}\Bigg|_{\lambda = \lambda_*^{\pm}}$$

within the transit boundary layer. Since $|\partial h_o/\partial\lambda| < |h_o/\Delta\lambda_{coll}|$ this implies that $|\delta h/h_o| < \delta\lambda_t/\Delta\lambda_{coll} \sim (\omega/\hat{\omega}_b)^{1/2} \ll 1$. Thus we require that

$$h_p = h_{t(j)}\Bigg|_{\lambda \to B_{max}^{-1}} \quad . \tag{18}$$

The bounce averaged drift kinetic equation (Eq.(5)) together with the boundary conditions (Eq.(8), Eq.(10), Eq.(17) and Eq. (18)) and the quasi-neutrality condition Eq. (7) completes the formal specification of the problem. We now examine the solutions in two regimes.

III. Low Collisionality Limit

We consider first the situation in which the electron collision frequency is less than the mode frequency. In this situation electron collisions are unable to relax the bulk of the perturbed electron distribution function in pitch angle within a wave period. To lowest order the non-adiabatic perturbed electron distribution function is

$$h_o = -\frac{(\omega - \omega_*^e)}{(\omega - \bar{\omega}_d)}\, q_e\bar{\phi}\, \frac{\partial F_o}{\partial \varepsilon} \tag{19}$$

This function varies rapidly however near the separatrix between passing and trapped particles over a width comparable to $\delta\lambda_{log}$, the logarithmic stagnation boundary layer. The effect of collisions is to smooth out this rapid variation in h_o over a collisional boundary layer. We can estimate the width of this layer by assuming that h_o changes by unity over an interval $\Delta\lambda_{coll}$ and requiring the collision operator to be comparable to the wave frequency over this layer. Evaluating $\hat{\tau}(\lambda)$ and $D(\lambda)$ at a point within the collisional boundary layer gives for the collisional width

$$\Delta\lambda_{coll} \sim \frac{1}{\bar{B}} \left[\frac{\nu}{|\omega|} \left(1 - \frac{\bar{B}}{B_{max}} \right) \frac{\bar{B}}{B_{max}} \right]^{1/2}$$

where $\bar{B}$ is a typical magnetic field strength within the cell. We further assume that the boundary layer width is narrower than the passing particle width in pitch-angle space, that is, $\Delta\lambda_{coll} < B^{-1}_{max}$.

In order to calculate the contribution of collisions to the trapped particle mode growth rate, we construct a quadratic form by multiplying the quasi-neutrality relation by ϕ^* and integrating along a flux tube.

$$0 = \sum_{i,e} \left[q^2 \int \frac{d\ell}{B} \int d^3v \; |\phi|^2 \frac{\partial F_o}{\partial \varepsilon} + q \int \frac{d\ell}{B} \int d^3v \phi^* h J_o \right] \tag{20}$$

In the second integral we note that both ϕ^* and h are bounded everywhere. Thus if we exclude the region of phase space which includes the transit time strip we are in error by terms of order $\delta\lambda_t \bar{B}$. In this treatment we will neglect such terms. Using the bounce averaged equation for h_o, Eq. (5), which is valid outside the transit time strip we write

$$h_o = - \frac{(\omega - \omega_*)}{(\omega - \bar{\omega}_d)} q\bar{\phi} \frac{\partial F_o}{\partial \varepsilon} J_o + \frac{i\nu}{\omega} \frac{1}{\hat{\tau}} \frac{\partial}{\partial \lambda} \left(D \frac{\partial h_o}{\partial \lambda} \right) \tag{21}$$

where we have assumed $\omega_d \ll \omega$ and neglected the drift frequency in the collision term. In what follows we only deal with h_o, the non-adiabatic perturbed distribution function to lowest order in ω/ω_b. For notational simplicity, we will suppress the zero subscript. For electrons $J_o = 1$, while for ions $J_o \approx 1 - (v_\perp^2 k_\perp^2)/(4\Omega^2)$ and $\nu_i \approx 0$. Inserting this in the quadratic form we obtain

$$0 = I_1 + iI_2 \tag{22}$$

where

$$I_1 \equiv \sum_{i,e} q^2 \frac{4\pi}{m^2}\left(\frac{m}{2}\right)^{1/2} \int_0^\infty d\varepsilon\varepsilon^{1/2} \int_* d\lambda\, \hat{\tau} \left(-\frac{\partial F_o}{\partial \varepsilon}\right)\times$$

$$\left\{\left[\left(\overline{|\phi|^2} - |\bar{\phi}|^2\right) + \frac{k_\perp^{\,2} v_\perp^{\,2}}{2\Omega_i^2} |\bar{\phi}|^2\right]\omega^2 + \omega(\omega_* - \bar{\omega}_d)\times\right.$$

$$\left.\left(1 - \frac{k_\perp^{\,2} v_\perp^{\,2}}{2\Omega_i^2}\right) |\bar{\phi}|^2 + \omega_*\bar{\omega}_d\, |\bar{\phi}|^2\right\} \tag{23}$$

and

$$I_2 \equiv e\omega \frac{4\pi}{m^2}\left(\frac{m}{2}\right)^{1/2} \int_0^\infty d\varepsilon\, \varepsilon^{1/2}\, \nu(\varepsilon) \int_* d\lambda\, \bar{\phi}^* \frac{\partial}{\partial\lambda} D \frac{\partial h}{\partial\lambda} \ . \tag{24}$$

In the integration over pitch angle the transit-time strip is excluded

$$\int_* d\lambda = \int_0^{\lambda_*^-} + \sum_j \int_{\lambda_*^+}^{\lambda_{max}^j} d\lambda \tag{25}$$

where

$$\lambda_*^- \equiv \frac{1}{B_{max}} - \delta\lambda_t$$

$$\lambda_*^+ \equiv \frac{1}{B_{max}} + \delta\lambda_t$$

$$\lambda_{max}^i \equiv \frac{1}{B_{min}^i} \ . \tag{26}$$

We analyze this quadratic form using a perturbative approach. We write the exact solution to the collisional problem as

$$\phi = \phi_o + \phi_1$$

$$\omega = \omega_0 + \omega_1 \tag{27}$$

where ϕ_0 and ω_0 are the exact solution to the collisionless problem and ϕ_1, ω_1 are the collisional modifications which we assume are small. We write the quadratic form (Eq. 22) - (24))in a condensed notation as

$$0 = \langle\phi^*(A\omega^2 + B\omega + C)\phi\rangle + i\omega e\langle\phi^*\bar{C}(h)\rangle \tag{28}$$

where $A\omega^2 + B\omega + C$ is the collisionless integral operator and the angular brackets denote a flux tube integration. Substituting for ϕ and ω, (Eq. (27)), gives to lowest order

$$0 = \langle\phi_0^*(A\omega_1^2 + \omega_1(2A\omega_0 + B))\phi_0\rangle + i\omega_0 e\langle\phi_0^*\bar{C}(h)\rangle$$

$$+ \langle\phi_1^*(A\omega_0^2 + B\omega_0 + C)\phi_1\rangle$$

$$+ \omega_1 \left[\langle \phi_1^*(2A\omega_0 + B)\phi_0\rangle + \langle \phi_0^* (2A\omega_0 + B) \phi_1 \rangle\right]. \tag{29}$$

We examine two cases. If $\langle\phi_0^*(2A\omega_0 + B)\phi_0\rangle > \langle\phi_0^*A\phi_0\rangle\omega_1$ then to lowest order

$$\omega_1 = \frac{- i\omega_0 e\langle\phi_0^*\bar{C}(h)\rangle}{\langle\phi_0^*(2A\omega_0 + B)\phi_0\rangle} . \tag{30}$$

As the collisionless mode nears marginal stability the denominator of this expression vanishes and ω_1/ω_0 appears to grow without bound. In this case we return to Eq. (29) assuming that $|\omega_1/\omega_0| > |\phi_1/\phi_0|$ and obtain as an estimate of the growth rate near marginal stability,

$$\omega_1^2 = - \frac{i\omega_0 e\langle\phi_0^*\bar{C}(h)\rangle}{\langle\phi_0^*A\phi_0\rangle} . \tag{31}$$

In both of these cases we must evaluate the integral

$$I_2 \equiv e\omega_0\langle\phi_0^*\bar{C}(h)\rangle \tag{32}$$

where h is calculated using Eq. (21) with $\phi = \phi_o$ and $\omega = \omega_o$. For notational simplicity, however, we suppress the subscripts on ϕ_o and ω_o in the discussion that follows. In addition, we are concerned with the physical situation in which the collisionless trapped particle mode is stable and therefore take ω_o to be real.

We evaluate I_2 by performing a partial integration in λ. Writing, for example, only the λ integral over the untrapped region of pitch-angle space we obtain

$$I_3 = \int_0^{\lambda_*^-} d\lambda \quad \bar{\phi}^* \frac{\partial}{\partial\lambda} D \frac{\partial h}{\partial\lambda}$$

$$= \bar{\phi}^* D \frac{\partial h}{\partial\lambda}\bigg|_{\lambda = \lambda_*^-} - \int_0^{\lambda_*^- - \Delta\lambda_{coll}} d\lambda \quad \frac{\partial\bar{\phi}^*}{\partial\lambda} D \frac{\partial h}{\partial\lambda}$$

$$- \int_{\lambda_*^- - \Delta\lambda_{coll}}^{\lambda_*^-} d\lambda \quad \frac{\partial\bar{\phi}^*}{\partial\lambda} D \frac{\partial h}{\partial\lambda} . \tag{33}$$

In the first integral, which extends over the bulk pitch angle distribution, we may substitute for h the collisionless value

$$h = -\left(1 - \frac{\omega_*^e}{\omega}\right) q_e \bar{\phi} \quad \frac{\partial F_o}{\partial\varepsilon} \tag{34}$$

where we have neglected the particle drift frequency $\bar{\omega}_d$. In the second integral which extends over the collisional boundary layer, h departs from the collisionless value, but $\bar{\phi}(\lambda)$ is approximately constant and the integral is small compared to the surface term. Explicitly

$$\left\| \int_{\bar{\lambda}_* - \Delta\lambda_{coll}}^{\bar{\lambda}_*} d\lambda \quad \frac{\partial\bar{\phi}^*}{\partial\lambda} \; D \frac{\partial h}{\partial\lambda} \right\| < \left\| \int_{\bar{\lambda}_* - \Delta\lambda_{coll}}^{\bar{\lambda}_*} d\lambda \quad \frac{\partial\bar{\phi}^*}{\partial\lambda} \right\| \left\| D \frac{\partial h}{\partial\lambda} \right\|_{max}$$

$$= \left\| \bar{\phi}^* \Big|_{\lambda = \bar{\lambda}_* - \Delta\lambda_{coll}}^{\lambda = \bar{\lambda}_*} \right\| \left\| D \frac{\partial h}{\partial\lambda} \right\|_{max}$$

$$\lesssim \left(\frac{\Delta L_{coll}}{L}\right) \left\| \bar{\phi}^* \right\|_{\lambda_*} \left\| D \frac{\partial h}{\partial\lambda} \right\|_{max} \tag{35}$$

where $\Delta L_{coll} \equiv |Z_c - Z_m|$, $B(Z_m) = B_{max}$, $B^{-1}(Z_c) = B^{-1}_{max} + \Delta\lambda_{coll}$ and L is the cell length. As we show later the maximum value of $|D(\partial h/\partial\lambda)|$ occurs at $\lambda = \bar{\lambda}_*$. Thus the second integral is small by at least $(\Delta L_{coll}/L)$ compared to the surface term and can be neglected. A similar manipulation can be performed for the integral over the trapped region of velocity space. We therefore can write

$$I_2 = e\omega \frac{4\pi}{m^2} \left(\frac{m}{2}\right)^{1/2} \int_0^\infty d\varepsilon \; \varepsilon^{1/2}\nu(\varepsilon) \times$$

$$\left\{ \int_0^{\bar{\lambda}_* - \Delta\lambda_{coll}} d\lambda \quad \left|\frac{\partial\bar{\phi}_p}{\partial\lambda}\right|^2 \; D_p \; \left(1 - \frac{\omega_*^e}{\omega}\right) q_e \; \frac{\partial F_o}{\partial\varepsilon} \right.$$

$$+ \sum_j \int_{\lambda_*^+ + \Delta\lambda_{coll}}^{\lambda^j_{max}} d\lambda \quad \left|\frac{\partial\bar{\phi}_{t(j)}}{\partial\lambda}\right|^2 \; D_{t(j)} \left(1 - \frac{\omega_*^e}{\omega}\right) q_e \; \frac{\partial F_o}{\partial\varepsilon}$$

$$+ \bar{\phi}^*_p D_p \frac{\partial h_p}{\partial \lambda}\Bigg|_{\lambda = \lambda_*^-} - \sum_j \bar{\phi}^*_{t(j)} D_{t(j)} \frac{\partial h_{t(j)}}{\partial \lambda}\Bigg|_{\lambda = \lambda_*^+} \Bigg\} \qquad (36)$$

In order to evaluate the last two surface terms we need to explicitly solve for h in the collisional boundary layer.

In analyzing the boundary layer we recall that we are concerned only with the region in which the bounce averaged equation, Eq. (5), is valid and are outside the layer in which $\hat{\tau}$ diverges logarithmically. Thus although h varies by unity $\hat{\tau}$, $D(\lambda)$ and $\bar{\phi}$ are all approximately constants within the boundary layer which we denote as $\hat{\tau}^\circ$, D° and $\bar{\phi}^\circ$ respectively. We write the bounce averaged equation in the boundary layer as

$$\omega h^{in} = -(\omega - \omega_*^e)\, q_e\, \bar{\phi}^\circ \frac{\partial F_\circ}{\partial \varepsilon} + i\nu \frac{D^\circ}{\hat{\tau}^\circ} \frac{\partial^2}{\partial \lambda^2} h^{in} \qquad (37)$$

We note that each of the three classes of electrons has a boundary layer at $\lambda = B^{-1}_{max}$ and that therefore Eq. (37) represents three boundary layer problems for h^{in}, the inner solution for each class. In each boundary layer we require that h^{in} asymptotically approach the collisionless solution valid away from the boundary $\lambda = B^{-1}_{max}$. We connect the three solutions, h^{in}_p, $h^{in}_{t(c)}$ and $h^{in}_{t(a)}$, by requiring that h^{in} have the same value at $\lambda = B^{-1}_{max}$ for the three classes and that the flux condition, Eq. (17) be satisfied. Defining the quantity $\hat{\phi}^\circ$ for notational convenience

$$\hat{\phi}^\circ \equiv -\left(1 - \frac{\omega_*^e}{\omega}\right) q_e\, \bar{\phi}^\circ \frac{\partial F_\circ}{\partial \varepsilon} \qquad (38)$$

we obtain

$$h^{in}_p = \hat{\phi}^\circ_p + (h^\circ - \hat{\phi}^\circ_p)\exp\left[-\sigma\left(\frac{|\omega|\hat{\tau}^\circ_p}{\nu D^\circ_p}\right)^{1/2}\left(\lambda_* - \lambda\right)\right] \qquad (39)$$

for $\lambda < 1/B_{max}$ and

$$h^{in}_{t(j)} = \hat{\phi}^\circ_{t(j)} + (h^\circ - \hat{\phi}^\circ_{t(j)})\exp\left[-\sigma\left(\frac{|\omega|\hat{\tau}^\circ_{t(j)}}{\nu D^\circ_{t(j)}}\right)^{1/2}\left(\lambda - \lambda_*\right)\right] \qquad (40)$$

for $\lambda > 1/B_{max}$ where $\sigma \equiv (1 - i\,\mathrm{sgn}(\omega))/\sqrt{2}$ and $\lambda_* = \bar{B}^{-1}_{max}$.

We determine the value of h°, the value of h at $\lambda = B^{-1}_{max}$, by imposing flux conservation, Eq. (17). Ignoring terms of order $\delta\lambda_t/\Delta\lambda_{coll} \sim (\omega/\hat{\omega}_b)^{1/2}$ we obtain

$$D^\circ_p\,(h^\circ - \hat{\phi}^\circ_p)\left(\frac{\hat{\tau}^\circ_p}{D^\circ_p}\right)^{1/2} = -\sum_j D^\circ_{t(j)}\left(h^\circ - \hat{\phi}^\circ_{t(j)}\right)\left(\frac{\hat{\tau}^\circ_{t(j)}}{D^\circ_{t(j)}}\right)^{1/2} \quad (41)$$

and thus

$$h^\circ = \frac{\hat{\phi}^\circ_p\,(\hat{\tau}^\circ_p D^\circ_p)^{1/2} + \sum_j \hat{\phi}^\circ_{t(j)}\,(\hat{\tau}^\circ_{t(j)}\,D^\circ_{t(j)})^{1/2}}{(\hat{\tau}^\circ_p D^\circ_p)^{1/2} + \sum_j (\hat{\tau}^\circ_{t(j)} D^\circ_{t(j)})^{1/2}} \quad (42)$$

Returning to the integral I_2, we can now evaluate the surface terms,

$$S \equiv \bar{\phi}^*_p D_p \left.\frac{\partial h_p}{\partial\lambda}\right|_{\lambda = \lambda^-_*} - \sum_j \bar{\phi}^*_{t(j)} D_{t(j)} \left.\frac{\partial h_{t(j)}}{\partial\lambda}\right|_{\lambda = \lambda^+_*}$$

$$= (\bar{\phi}^\circ_p)^*\, D^\circ_p\, \sigma(h^\circ - \hat{\phi}^\circ_p)\left(\frac{|\omega|}{\nu}\frac{\hat{\tau}^\circ_p}{D^\circ_p}\right)^{1/2}$$

$$+ \sum_j (\bar{\phi}^\circ_{t(j)})^*\, D^\circ_{t(j)}\, \sigma(h^\circ - \hat{\phi}^\circ_{t(j)})\left(\frac{|\omega|}{\nu}\frac{\hat{\tau}^\circ_{t(j)}}{D^\circ_{t(j)}}\right)^{1/2} + O(\delta\lambda_t B_{max}) . \quad (43)$$

Substituting the expression for h°, Eq. (42), in the expression for S, Eq. (43), and substituting the result in Eq. (36) gives the following expression for I_2

$$I_2 = e^2\omega^2 \frac{4\pi}{m^2}\left(\frac{m}{2}\right)^{1/2} \int_0^\infty d\varepsilon\,\varepsilon^{1/2}\left(1 - \frac{\omega^e_*}{\omega}\right)\left(-\frac{\partial F_0}{\partial\varepsilon}\right) \times$$

$$\left\{ \frac{\nu}{\omega}\int d\lambda \left|\frac{\partial\bar{\phi}}{\partial\lambda}\right|^2 D\right.$$

$$+ \left|\frac{\nu}{\omega}\right|^{1/2} \frac{\mathrm{sgn}(\omega) - i}{\sqrt{2}} \times$$

$$\left\{\left[\left|\bar{\phi}_p^{\circ}\right|^2 (\hat{\tau}_p^{\nu} D_p^{\circ})^{1/2} + \sum_j \left|\bar{\phi}_{t(j)}^{\circ}\right|^2 (\hat{\tau}_{t(j)}^{\circ} D_{t(j)}^{\circ})^{1/2}\right]\right.$$

$$\left. - \frac{\left|\phi_p^{\circ} (\hat{\tau}_p^{\circ} D_p^{\circ})^{1/2} + \sum_j \bar{\phi}_{t(j)}^{\circ} (\hat{\tau}_{t(j)}^{\circ} D_{t(j)}^{\circ})^{1/2}\right|^2}{(\hat{\tau}_p^{\circ} D_p^{\circ})^{1/2} + \sum_j (\hat{\tau}_{t(j)}^{\circ} D_{t(j)}^{\circ})^{1/2}}\right\}\Bigg\} \qquad (44)$$

where the integral over λ extends only over the region outside of the collisonal boundary layer. The first term in I_2 represents the effects of collisions on the bulk perturbed distribution function while the second is due to the boundary layer. Note that both terms in I_2 vanish if ϕ is flute-like through the machine. We recall from Eq. (30) that the collisional growth rate is given by

$$\gamma = - \frac{\mathrm{Re}\{I_2\}}{\frac{\partial I_1}{\partial \omega}} \qquad (45)$$

In order to estimate typical growth rates, we consider a simplified model square well equilibrium with passing electron and ion bounce points at $z = L_{be}$ and $z = L_{bi}$ respectively. The anchor region where the curvature is favorable begins at $z = L_c$ and extends to L_{bi}. We make the further assumption that the eigenfunction drops to near zero in the anchor region and that the anchor and center cell magnetic field strengths are equal. Because the magnetic field strength is flat within each region the effect of collisions on the bulk vanishes and we obtain the following estimates:

$$I_1 \simeq e^2 \frac{L_c}{B_o} \left\{\omega^2 \left[n_{pass} \left(\frac{L_{bi} - L_c}{L_{bi}}\right) + \frac{L_{be} - L_c}{L_{be}} + k_{\perp}^2 \rho_i^2 n_o\right]\right.$$

$$\left. - \omega \hat{\omega}_*^i \left[n_{pass} \frac{L_c (L_{bi} - L_{be})}{L_{bi} L_{be}} + k_{\perp}^2 \rho_i^2 (1 + \eta_i) n_o\right] + \gamma_{MHD}^2 k_{\perp}^2 \rho_i^2 n_o\right\}$$

$$I_2 \simeq e^2 \frac{L_c}{B_o} \omega^2 \left\{\left|\frac{\nu_e}{\omega}\right|^{1/2} \frac{(\mathrm{sgn}(\omega) - i)}{\sqrt{2}}\right.$$

$$\times\left(\frac{B_o}{B_{max}}\right)^{1/2} \frac{(L_{be} - L_c)}{L_{be}}$$

$$\times \frac{4\pi}{m^2}\left(\frac{m}{2}\right)^{1/2} \int_0^\infty d\varepsilon\varepsilon^{1/2} F_o(1 - \frac{\omega_*^e}{\omega}) \left(\frac{T}{\varepsilon}\right)^{3/4} \left[1 + H\left[\left(\frac{\varepsilon}{T}\right)^{1/2}\right]\right]^{1/2} \quad (46)$$

where

$$\gamma_{MHD}^2 = 2\frac{(\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \nabla' P)(\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \nabla \underset{\sim}{b})}{n_o m_i k_\perp^2}$$

$$\left|\frac{\bar{\nu}_e}{\omega}\right|^{1/2} = \left|\frac{2\omega_{pe^2} e^2 m_e^{1/2}}{|\omega|(2T)^{3/2}} \ln\Lambda\right|^{1/2}$$

$$\hat{\omega}_*^i \equiv \frac{T\,\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \nabla' n_o}{n_o m_i \Omega_i} = -\hat{\omega}_*^e$$

$$\eta_e \equiv \frac{d(\ell nT)}{d(\ell nn_o)} .$$

We wish to consider a situation which is stable to trapped particle modes and thus take $L_{bi} > L_{be}$. Setting I_1 to zero yields two real roots with $\omega_o/\hat{\omega}_*^i > 0$. The real part of I_2 carries the same sign as $\hat{\omega}_*^i$ and thus the root with smaller magnitude is destabilized by collisions. Writing $I_1 = A\omega^2 + B\omega + C$, the value of $(\partial I_1/\partial\omega)$ for this root is

$$\left.\frac{\partial I_1}{\partial\omega}\right|_{\omega=\omega_o} = -\,\mathrm{sgn}(\hat{\omega}_*^i)\frac{(B^2 - 4AC)^{1/2}}{2A} \quad (47)$$

Thus the growth rate depends on how much the underlying curvature driven trapped particle has been stabilized by charge separation. For the case that $\omega_o \approx (\partial I_1/\partial\omega) \approx \hat{\omega}_*^i$ we estimate the growth rate as

$$\gamma \simeq |\hat{\omega}_*^i| \left|\frac{\nu_e}{\hat{\omega}_*^i}\right|^{1/2} \left(\frac{B_o}{B_{max}}\right)^{1/2} \frac{(L_{be} - L_c)}{L_{be}} . \qquad (48)$$

When the collisionless mode is marginally stable we obtain from Eq. (31) the following collisional growth rate

$$\gamma \simeq |\omega_o| \left[\frac{\left|\frac{\bar{\nu}_e}{\omega_o}\right|^{1/2} \left(\frac{B_o}{B_{max}}\right)^{1/2} \frac{L_{be} - L_c}{L_{be}} \left(1 - \frac{\omega_*^e}{\omega_o}\right)}{\frac{n_p}{n_o}\left(\frac{L_{bi} - L_e}{L_{bi}} + \frac{L_{be} - L_c}{L_{be}}\right) + k_\perp^2 \rho_i^2} \right]^{1/2} . \qquad (49)$$

For $L_{bi} - L_{be} \ll L_{bi}, L_{be}$ and $k_\perp^2 \rho_i^2 < n_p/n_o \, (L_{bi} - L_c)/L_{bi}$,

this expression reduces to

$$\gamma \simeq |\omega_o| \left|\frac{\bar{\nu}_e}{\omega_o} \frac{B_{max}}{B_o}\right|^{1/4} \left(1 - \frac{\omega_*^e}{\omega_o}\right)^{1/2}$$

where $|\omega_o| \sim |\hat{\omega}_*^i| \, (L_{bi} - L_{be})/4(L_{be} - L_e)$ and we have assumed $\eta_e = 0$. Thus when the collisionless mode is well stabilized the collisional growth rate due to the dissipative effect of electron collisions is small. For plasma parameters for which the collisionless mode is marginally stable, however, the collisional growth rate can be a substantial fraction of the mode real frequency which in turn is approximately $\hat{\omega}_*^i$.

IV. High Collisionality Limit

We turn now to the limit in which $\omega_b \gg \nu_e \gg \omega \gg \nu_i$. For simplicity we again consider an equilibrium with a constant electrostatic potential except for large positive and negative confining peaks at the end of the tandem. In this limit the difference between electron and ion bounce points does not fundamentally alter the physics and so we assume that electrons and ions turn at the same point.

The electron non-adiabatic distribution function is given in the high bounce frequency limit by the solution to the bounced averaged collisional drift kinetic equation, Eq. (5) and Eq. (6),

$$(\omega - \overline{\omega_d^e})h = -(\omega - \omega_*^e)q\overline{\phi}\,\frac{\partial F_o}{\partial \varepsilon} + i\overline{C}(h). \qquad (50)$$

we will analyze this equation using a perturbative approach exploiting the two small parameters $\delta_1 \sim \omega/\nu_e$ and $\delta_2 \sim \omega_d^e/\omega$. To lowest order in both parameters the collision operator dominates yielding the following equation for the zero order non-adiabatic distribution function $h_o^{(e)}$.

$$i\overline{C}(h_o^{(e)}) = 0 \qquad (51)$$

The solution to Eq. (51) is that h_o is proportional to the the Maxwellian equilibrium distribution function

$$h_o^{(e)} = h_{o,o}^{(e)}\, F_o \qquad (52)$$

and is thus independent of pitch angle.

This implies specifically that the non-adiabatic perturbed distribution function, $h_{o,o}^{(e)}F_o$, for electrons trapped in each region is equal to the non-adiabatic perturbed distribution function for passing particles, and thus that $h_{o,o}^{(e)}$ is equal to the same constant for all classes of particles.

We now turn to the ion equations. In this case there are again two small parameters $\delta_3 \sim k_\perp^2\rho_i^2 \sim \omega_d^i/\omega \sim \delta_2$ and $\delta_4 \sim (\nu_i/\omega)$. By analogy to the results of the preceeding sections for the low collisionality limit of electrons we expect the ions to exhibit a boundary layer behaviour leading to a contribution to the growth rate of order $(\nu_i/\omega)^{1/2}$. We write the perturbed potential as

$$\phi = \phi_o + \phi_1 \tag{53}$$

where ϕ_o is the potential to zero order in all the small parameters and ϕ_1 is the modification induced by the various small effects. Away from the collisional boundary layer we can write the ion perturbed response to lowest order as

$$\tilde{f}_o^{(i)} = -\frac{e\phi_o}{T} F_o + \left(1 - \frac{\omega_*^i}{\omega}\right) \frac{e\bar{\phi}_o}{T} F_o . \tag{54}$$

and in the boundary layer as

$$\hat{f}^{(i)} = -\frac{e\phi_o}{T} F_o + h_{in}^{(i)} . \tag{55}$$

Thus the quasi-neutrality condition to lowest order is

$$0 = -\frac{2e\phi_o}{T} n_o + \int d^3v \left(1 - \frac{\omega_*^i}{\omega}\right) \frac{e\bar{\phi}_o}{T} F_o + h_{o,o}^{(e)} n_o . \tag{56}$$

We have added and subtracted the collisionless non-adiabatic response in the integral over the collisional boundary layer and treat the difference, $\delta f_{BL}^{(i)}$ in the next order equation, where

$$\delta f_{BL}^{(i)} \equiv \int_{BL} d^3v \left[h_{in}^{(i)} - \left(1 - \frac{\omega_*^i}{\omega}\right) \frac{e\bar{\phi}_o}{T} F_o \right].$$

The subscript BL indicates that the velocity integral extends over the collisional boundary layer.

Eq. (56) is an inhomogeneous integral equation for ϕ_o. We distinguish two cases depending on whether ω is an eigenvalue of the homogeneous equation. In case (a) we assume that ω is not an eigenvalue of the homogeneous equation; then the solution to the inhomogeneous equation, Eq. (56), is that ϕ_o is a constant, $\phi_o = \phi_o^o$ and

$$h_{o,o}^{(e)} = -\frac{2e\phi_o^o}{T} + \left(1 - \frac{\hat{\omega}_*^i}{\omega}\right)\frac{e\phi_o^o}{T} \tag{57}$$

where

$$\hat{\omega}_*^i = \int d^3v \frac{F_o}{n_o} \omega_*^i = \frac{T\, \underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \nabla n_o}{n_o m_i \Omega_i} \quad . \tag{58}$$

The eigenfrequency ω is undetermined at this order. Thus case (a) yields a flute mode.

In case (b) ω is an eigenvalue of the homogeneous equation which corresponds to Eq. (56),

$$0 = -\frac{2e\phi_h^o}{T} n_o + \int d^3v \left(1 - \frac{\omega_*^i}{\omega_o}\right)\frac{e\bar{\phi}_h^o}{T} F_o. \tag{59}$$

For notational clarity we denote the eigenfunction of the homogeneous equation as ϕ_h^o with corresponding eigenvalue ω_o. The inhomogeneous equation, Eq. (56), is an inhomogeneous Fredholm integral equation of the second kind. In general such an equation has no well behaved solutions if ω is an eigenvalue of the corresponding homogeneous equation except in the case that ϕ_h^o is orthogonal to the inhomogeneous term. Before considering this orthogonality constraint we first show that ω_o is real. This can be shown by multiplying the homogeneous equation by $(\phi_h^o)^*$ and integrating along a flux tube.

Solving for ω_o gives

$$\omega_o = -\frac{\int \frac{d\ell}{B} \int d^3v \; \omega_*^i \left|\bar{\phi}_h^o\right|^2 \frac{eF_o}{T}}{\int \frac{d\ell}{B} \int d^3v \left(2\overline{|\phi_h^o|^2}\right) - \left|\bar{\phi}_h^o\right|^2 \frac{eF_o}{T}} . \tag{60}$$

This shows that ω_o is real and decreases as the number of nodes in ϕ_h increases. Thus $(-\omega_o/\hat{\omega}_*^i)$ is bounded from above. Since ω_o is real we may choose ϕ_h^o to be real as well. We also note that the phase velocity of this mode is in the direction of the electron diamagnetic drift. The phase velocity of an unstable collisionless trapped particle mode with equal electron and ion bounce points is in the direction of the ion diamagnetic drift. This suggests that the non-flute like mode we are considering does not go over into the fast growing collisionless trapped particle mode, but rather into a collisionally driven trapped particle mode.

In order to derive the constraint on ϕ_h^o, we multiply the inhomogeneous equation, Eq. (56), by ϕ_h^o, and the homogeneous equation, Eq. (59), by ϕ_o. Integrating each along a flux tube and subtracting gives the condition

$$\int \frac{d\ell}{B}\, \phi_h^o = 0. \tag{61}$$

Eq. (61) is a necessary condition for the existence of solutions to the inhomogeneous equation if ω is an eigenvalue of the homogeneous equation. We now show that in fact the homogeneous solution ϕ_h^o does satisfy this constraint.

We first integrate the homogeneous equation Eq. (59) along a flux tube

$$0 = -2 \int \frac{d\ell}{B} \int d^3v \overline{\phi}_h^o F_o + \int \frac{d\ell}{B} \int d^3v \left(1 - \frac{\omega_*^i}{\omega_o}\right) \overline{\phi_h^o}\, F_o . \tag{62}$$

Since $\overline{\phi}_h^o$ depends only on pitch angle we can perform the energy integrals in Eq. (60) and Eq. (62). Substituting for $(1 - \hat{\omega}_*^i/\omega_o)$ from the quadratic form, Eq. (60), in the flux tube average of the homogeneous equation gives,

$$0 = \frac{\int \frac{d\ell}{B} \int d^3v \overline{\phi}_h^o F_o \times \int \frac{d\ell}{B} \int d^3v \left[\overline{(\phi_h^o)^2} - (\overline{\phi}_h^o)^2\right] F_o}{\int \frac{d\ell}{B} \int d^3v (\overline{\phi}_h^o)^2\, F_o} . \tag{63}$$

This can only be satisfied if

$$\int \frac{d\ell}{B} \int d^3v \overline{\phi}_h^o F_o = \int \frac{d\ell}{B} \int d^3v \phi_h^o F_o = n_o \int \frac{d\ell}{B}\, \phi_h^o = 0, \tag{64}$$

or if ϕ_h^o is a constant. Thus if any non-constant solutions to the homogeneous equation exist they satisfy the constraint that their flux tube integral vanishes and in such a case solutions to the inhomogeneous equation exist even if ω is an eigenvalue of the homogeneous equation. By inspection we see that we can write the general solution to the inhomogeneous equation in case (b) as

$$\phi_o = \phi_o^o + \phi_h^o \tag{65}$$

where ϕ_o^o is related to $h_{o,o}^{(e)}$ by Eq. (57). As noted earlier, ω_o is real, therefore to calculate a growth rate we must go to higher order. As we shall see below case (b) leads to the dissipative trapped ion mode which has been studied theoretically in the context of the Tokamak geometry[(2-10)] and experimentally in the Columbia Linear Machine [(11)].

Case (a)

We return now to calculate the eigenfrequency for the flute mode of case (a). We consider electrons first and write,

$$\phi = \phi_o^o + \phi_1$$

$$h^{(e)} = -2 \frac{e\phi_o^o}{T} + \left(1 - \frac{\omega_*^i}{\omega}\right) \frac{e\phi_o^o}{T} F_o + h_1^{(e)}$$

$$= - \left(1 - \frac{\omega_*^e}{\omega}\right) \frac{e\phi_o^o}{T} F_o + h_1^{(e)} \tag{66}$$

where use has been made that $\omega_*^i = -\omega_*^e$. Substituting into Eq. (50) gives

$$h_1^{(e)} = - \frac{\bar{\omega}_d^e}{\omega} \left(1 - \frac{\omega_*^e}{\omega}\right) \frac{e\phi_o^o}{T} F_o - \left(1 - \frac{\omega_*^e}{\omega}\right) \frac{e\bar{\phi}_1}{T} F_o$$

$$+ \frac{\bar{\omega}_d^e}{\omega} h_1^{(e)} + i \frac{\bar{C}}{\omega} (h^{(e)}) \tag{67}$$

The term $(\bar{\omega}_d^e/\omega)h_1^{(e)}$ is an order δ_2 correction to $h_1^{(e)}$ and can be neglected self-consistently in this order. Integrating over velocity and along a flux tube gives the integral of the first order correction to the electron non-adiabatic perturbed density

$$\int\frac{d\ell}{B}\int d^3v h_1^{(e)} = -\frac{e\phi_o^o}{T}\int\frac{d\ell}{B}\int d^3v\,\frac{\bar{\omega}_d^e}{\omega}\left(1-\frac{\omega_*^e}{\omega}\right)F_o$$

$$-\int\frac{d\ell}{B}\int d^3v\left(1-\frac{\omega_*^e}{\omega}\right)\frac{e\phi_1}{T}F_o. \qquad (68)$$

We now consider the first order ion response. We note that since the mode is flute-like the lowest order non-adiabatic perturbed ion distribution function is independent of pitch angle. Thus the ion collision operator operating on $h_o^{(i)}$ vanishes. The effect of ion collisions on this mode will thus be of higher order, specifically $(\nu_i/\omega)^{1/2}(\omega_d/\omega)$, and does not affect the lowest order eigenfrequency. We write the ion response as

$$h^{(i)} = \left(1-\frac{\omega_*^i}{\omega}\right)\frac{e\phi_o^o}{T}F_o + h_1^{(i)} \qquad (69)$$

where

$$h_1^{(i)} = \left(1-\frac{\omega_*^i}{\omega}\right)\left[\frac{\bar{\omega}_d^i}{\omega}\frac{e\phi_o^o}{T}+\frac{e\phi_1}{T}+(\bar{J}_o-1)\frac{e\phi_o^o}{T}\right]F_o. \qquad (70)$$

Thus the quasi-neutrality condition correct to first order in the various small parameters is

$$0 = -2\frac{e\phi_1}{T}n_o + \int d^3v\left(1-\frac{\omega_*^i}{\omega}\right)\frac{e\bar{\phi}_1}{T}F_o$$

$$+ \int d^3v \left(1 - \frac{\omega_*^i}{\omega}\right)\left(\frac{\bar{\omega}_d^i}{\omega} - \frac{\overline{k_\perp^2 v_\perp^2}}{2\Omega^2}\right)\frac{e\phi_o^o}{T} F_o$$

$$- \int d^3v h_1^{(e)}. \tag{71}$$

Integrating over a flux tube and using the expression for the flux tube integral of the non-adiabatic electron response, Eq. (68), gives a quadratic in ω,

$$(\omega^2 - \hat{\omega}_*^i(1 + \eta_i)\omega + \gamma_{MHD}^2)\frac{e\phi_o^o}{T} = 0 \tag{72}$$

where

$$\gamma_{MHD}^2 = \frac{\int \frac{d\ell}{B} \int d^3v(\omega_*^i \bar{\omega}_d^i + \bar{\omega}_*^e \bar{\omega}_d^e)\frac{F_o}{Tn_o}}{\int \frac{d\ell}{B} \frac{k_\perp^2}{m_i \Omega_i^2}}$$

$$= \int \frac{d\ell}{B} \frac{2}{m_i \Omega_i^2 n_o} \frac{(\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \nabla P)\,[\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})]}{\int \frac{d\ell}{B} \frac{k_\perp^2}{\Omega_i^2}}. \tag{73}$$

The drive term is the usual beta weighted line averaged curvature and by assumption the machine has been designed to make this negative in order to achieve MHD stability. Thus we conclude that in the high collisionality limit one mode of the system is a stable flute mode. Collisions have served to couple the response of the central cell and anchor and thereby prevented the localized perturbations characteristic of the trapped particle mode.

We note that this mode remains unchanged as we increase the ion collisionality since the lowest order solution for the ion non-adiabatic perturbed distribution function is independent of pitch angle. The remaining modes of the system which are non-flute-like are not driven by local bad curvature but by the difference between the electron and ion collision frequencies. We turn to these now.

Case (b)

We begin our analysis of case (b) by writing,

$$\phi = \phi_o^o + \phi_h^o + \phi_1 \tag{74}$$

$$\omega = \omega_o + \omega_1$$

$$h^{(e)} = -\left(1 - \frac{\omega_*^e}{\omega_o}\right) \frac{e\phi_o^o}{T} F_o + h_1^{(e)} \tag{75}$$

where ϕ_h^o and ω_o are the eigenfunction and eigenvalue of the homogeneous integral equation,

$$0 = \frac{-2e\phi_h^o}{T} n_o + \int d^3v \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e\bar{\phi}_h^o}{T} F_o . \tag{76}$$

Substituting Eq. (74) and Eq. (75) in the electron equation, Eq. (50), gives

$$(\omega_o + \omega_1 - \bar{\omega}_d^e)h_1^{(e)} = -\bar{\omega}_d^e \left(1 - \frac{\omega_*^e}{\omega_o}\right)\frac{e\phi_o^o}{T} F_o$$

$$- (\omega_o - \omega_*^e) \frac{e}{T} (\bar{\phi}_h^o + \bar{\phi}_1)F_o - \omega_1 \frac{e}{T}(\bar{\phi}_h^o + \bar{\phi}_1)F_o - \frac{\omega_1\omega_*^e}{\omega_o} \frac{e\phi_o^o}{T} F_o$$

$$+ i\nu_e C(h_1^{(e)}) \tag{77}$$

Recalling that $(\nu_e/\omega) >> 1$ the lowest order equation assuming

$$|h_1| < |(1 - \omega_*^e/\omega_o)(e\phi_o^o/T)F_o|$$

is

$$(\omega_o - \omega_*^e)\frac{e\bar{\phi}_h^o}{T} F_o = i\nu_e \bar{C}(h_1^{(e)}) \, . \tag{78}$$

We note that if we integrate this equation over velocity space and along a flux tube, both sides vanish since the flux tube integral of ϕ_h^o vanishes. Thus if we integrate the exact electron equation over velocity and along a flux tube we annihilate the lowest order piece leaving a constraint on the integral of $h_1^{(e)}$. Dropping terms which are second order small we obtain

$$\int \frac{d\ell}{B} \int d^3v h_1^{(e)} = - \int \frac{d\ell}{B} \int d^3v F_o \left\{ \left[\frac{\bar{\omega}_d^e}{\omega_o} \left(1 - \frac{\omega_*^e}{\omega_o}\right) + \frac{\omega_1 \omega_*^e}{\omega_o^2} \right] \frac{e\phi_o^o}{T} \right.$$

$$\left. - \left(1 - \frac{\omega_*^e}{\omega_o}\right) \frac{e\bar{\phi}_1}{T} \right\} \, . \tag{79}$$

Turning to the ion equations we first consider the region away from the boundary. Writing

$$h^{(i)} = \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} (\phi_o^o + \bar{\phi}_h^o) F_o + h_1^{(i)} \tag{80}$$

and assuming $|h_1^{(i)}| << |(1 - \omega_*^i/\omega_o)e(\phi_o^o + \bar{\phi}_h^o)(F_o/T)|$, we obtain

$$h_1^{(i)} = \frac{\overline{\omega}_d^i}{\omega_o}\left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T}(\phi_o^o + \overline{\phi}_h^o)F_o + \frac{\omega_1\omega_*^i}{\omega_o^2}\frac{e}{T}(\phi_o^o + \overline{\phi}_h^o)F_o$$

$$+ \left(1 - \frac{\omega_*^i}{\omega_o}\right)\frac{e\overline{\phi}_1}{T}F_o$$

$$- \left(1 - \frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\left(\phi_o^o \overline{\frac{k_\perp^2 v_\perp^2}{2\Omega_i^2}} + \overline{\phi_h^o \frac{k_\perp^2 v_\perp^2}{2\Omega_i^2}}\right)F_o$$

$$+ \frac{i\nu_i}{\omega_o}\overline{C}(h^{(i)}) \ . \qquad (81)$$

In the boundary layer we write

$$h^{(i)} = \left(1 - \frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\phi_o^o F_o + h_{in}^{(i)} \qquad (82)$$

where $h_{in}^{(i)}$ satisfies the equation

$$\omega_o h_{in}^{(i)} = (\omega_o - \omega_*^i)\frac{e\overline{\phi}_h^o}{T}F_o + i\nu_i\overline{C}(h_{in}^{(i)}). \qquad (83)$$

We neglect all higher order terms in obtaining this equation since we only need $h_{in}^{(i)}$ to zero order in (ω_d/ω) and $(k_\perp\rho_i)^2$. We note that integrating this equation over velocity and along a flux tube yields the constraint on $h_{in}^{(i)}$,

$$\int\frac{d\ell}{B}\int d^3v h_{in}^{(i)} = 0. \qquad (84)$$

Gathering these results, Eq. (75) - Eq. (82), we write the condition of quasi-neutrality to first order as

$$0 = -\frac{2e\phi_1}{T} n_o + \int d^3vF_o \left[\left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e\overline{\phi}_1}{T} \right.$$

$$+ \frac{\overline{\omega}_d^i}{\omega_o} \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} (\phi_o^o + \overline{\phi}_h^o)$$

$$- \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} \left(\phi_o^o \overline{\frac{k_\perp^2 v_\perp^2}{2\Omega_i^2}} + \phi_h^o \overline{\frac{k_\perp^2 v_\perp^2}{2\Omega_i^2}} \right)$$

$$\left. + \frac{\omega_1 \omega_*^i}{\omega_o^2} \frac{e}{T} (\phi_o^o + \overline{\phi}_h^o) \right] + \frac{i\nu_i}{\omega_o} \int_{\text{bulk}} d^3v\overline{C}(h^{(i)})$$

$$+ \int_{\substack{\text{boundary} \\ \text{layer}}} d^3v \left[h_{in}^{(i)} - \left(1 - \frac{\omega_*^i}{\omega_o}\right) \right] \frac{e}{T} \overline{\phi}_h^o F_o$$

$$- \int d^3v h_1^{(e)} \quad . \qquad (85)$$

We first obtain an expression for ϕ_o^o in terms of ϕ_h^o the solution to the zero order homogeneous integral equation. We integrate the quasi-neutrality condition over a flux tube and use the electron constraint derived earlier Eq. (79) to yield

$$0 = \frac{e\phi_o^o}{T} \int \frac{d\ell}{B} \left[\frac{k_\perp^2 T}{m\Omega_i^2} \left(1 - \frac{\hat{\omega}_*^i}{\omega_o} (1 + \eta_i) \right) \right.$$

$$\left. + \frac{2T}{m_i^2 \Omega_i^2 n_o} \frac{\left(\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \nabla P \right) \left[\underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) \right]}{\omega_o^2} \right]$$

$$+ \int \frac{d\ell}{B} \int d^3v \left[\left(\frac{k_\perp^2 v_\perp^2}{2\Omega_i^2} - \frac{\bar{\omega}_d^i}{\omega_o} \right) \left(1 - \frac{\omega_*^i}{\omega_o} \right) \phi_h^o \right] F_o \quad . \qquad (86)$$

In obtaining this we exploit the vanishing of the flux tube integral of ϕ_h^o and the fact that $\omega_*^i = -\omega_*^e$.

Our final task is to derive an expression for ω_1. This we accomplish by multiplying the quasi-neutrality condition by ϕ_h^o and integrating over a flux tube. We eliminate the terms in ϕ_1 with the expression obtained by multiplying Eq. (76), the homogeneous integral equation for ϕ_h^o by ϕ_1 and integrating over a flux tube.

This yields the following expression

$$0 = \int \frac{d\ell}{B} \int d^3 vF_o \left(\frac{\omega_1 \omega_*^i}{\omega_o^2} \right) \frac{e}{T} (\bar{\phi}_h^o)^2$$

$$+\int\frac{d\ell}{B}\int d^3v\, F_o\left(1-\frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\left[\frac{\overline{\omega}_d}{\omega_o}\left(\overline{\phi}_h^o\right)^2-\left(\phi_o^o\,\overline{\frac{k_\perp^2v_\perp^2}{2\Omega_i^2}}+\overline{\phi_h^o\frac{k_\perp^2v_\perp^2}{2\Omega_i^2}}\right)\phi_h^o\right]$$

$$+\frac{i\nu_i}{\omega_o}\int\frac{d\ell}{B}\int_{\text{bulk}}d^3v\,\phi_h^o\overline{C}(h^{(i)})$$

$$+\int\frac{d\ell}{B}\int_{\substack{\text{boundary}\\ \text{layer}}}d^3v\,\phi_h^o\left[h_{in}^{(i)}-\left(1-\frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\overline{\phi}_h^o\right]$$

$$-\int\frac{d\ell}{B}\int d^3v\,\phi_h^o h_1^{(e)}. \tag{87}$$

Using the flux tube integral of the boundary layer ion equation we can combine the second and third terms which are due to ion collisions to give the following expression for ω_1

$$\frac{\omega_1}{\omega_o}\int\frac{d\ell}{B}\int d^3v\,\frac{\omega_*^i}{\omega_o}\frac{e}{T}(\overline{\phi}_h^o)^2F_o = -\int\frac{d\ell}{B}\int d^3v\, F_o\left(1-\frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\left[\frac{\overline{\omega}_d}{\omega_o}(\overline{\phi}_h^o)^2\right.$$

$$\left.-\left(\phi_o^o\,\overline{\frac{k_\perp^2v_\perp^2}{2\Omega_i^2}}+\overline{\phi_h^o\,\frac{k_\perp^2v_\perp^2}{2\Omega_i^2}}\right)\phi_h^o\right]$$

$$-\frac{i\nu_i}{\omega_o}\int\frac{d\ell}{B}\int d^3v\,\phi_h^o\overline{C}(h^{(i)})+\int\frac{d\ell}{B}\int d^3v\,\phi_h^o h_1^{(e)} \tag{88}$$

where the ion velocity integral is over the entire velocity space except for the transit-time layer. Note that the particle curvature drift and the ion finite Larmor radius terms only contribute to modify the real part of the frequency. The ion and electron collisional terms, however, will contribute to mode damping and growth respectively for a sufficiently flat temperature profile.

We proceed now to evaluate the ion collisional term as in the previous section dealing with the low collisionality limit. Using a pitch angle scattering operator, performing a partial integration and neglecting the integral over the boundary layer in comparison to the surface term we obtain the result

$$\frac{i\nu_i}{\omega_o}\int\frac{d\ell}{B}\int d^3v\,\overline{\phi_h^o C}(h^{(i)}) =$$

$$-\ i\,\frac{4\pi}{m^2}\left(\frac{m}{2}\right)^{3/2}\int_0^\infty d\varepsilon\,\varepsilon^{1/2}\,\frac{\nu_i(\varepsilon)}{\omega_o}\ \times$$

$$\left\{\int_0^{\lambda_*^- - \Delta\lambda_{coll}} d\lambda\left(1-\frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\,D_p\left(\frac{\partial\overline{\phi}^o_{h,p}}{\partial\lambda}\right)^2 F_o\right.$$

$$+\ \sum_j\int_{\lambda_*^+ + \Delta\lambda_{coll}}^{\lambda^j_{max}} d\lambda\left(1-\frac{\omega_*^i}{\omega_o}\right)\frac{e}{T}\,D_{t(j)}\left(\frac{\partial\overline{\phi}^o_{h,t}}{\partial\lambda}\right)^2 F_o$$

$$\left.-\left[\overline{\phi}^o_{h,p}\,D_p\,\frac{\partial h^{(i)}_{in,p}}{\partial\lambda}\Bigg|_{\lambda=\lambda_*^-} - \sum_j\overline{\phi}^o_{h,t}D_{t(j)}\,\frac{\partial h^{(i)}_{in,t}}{\partial\lambda}\Bigg|_{\lambda=\lambda_*^+}\right]\right\}. \qquad (89)$$

Returning to the boundary layer equation we note that the values of $\bar{\phi}_h^{\,o}$, D and $\hat{\tau}$ are constant to order $(\Delta L_B/L)$ within the boundary layer. Defining $\bar{\phi}_h^{o,o}$, D^o and $\hat{\tau}^o$ as those constants the solution to the boundary layer equation is

$$h_{in,p}^{(i)} = \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} \bar{\phi}_{h,p}^{o,o} F_o + \left[h^o - \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} \bar{\phi}_{h,p}^{o,o} F_o\right] \times$$

$$\exp\left[-\sigma \left(\frac{|\omega_o| \hat{\tau}_p^o}{\nu_i D_p^o}\right)^{1/2} (\lambda_*^- - \lambda)\right] \qquad (90)$$

for $0 < \lambda < \lambda_*^-$

$$h_{in,t(j)}^{(i)} = \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} \bar{\phi}_{h,t(j)}^{o,o} F_o +$$

$$\left[h^o - \left(1 - \frac{\omega_*^i}{\omega_o}\right) \frac{e}{T} e\bar{\phi}_{h,t(j)}^{o,o} F_o\right] \exp\left[-\sigma \left(\frac{|\omega_o| \hat{\tau}_{t(j)}^o}{\nu_i D_{t(j)}^o}\right)^{1/2} (\lambda - \lambda_*^+)\right] (91)$$

for $\lambda_*^+ < \lambda < \lambda_{max}^j$

where $\sigma \equiv (1 - i\ \mathrm{sgn}(\omega))/\sqrt{2}$ and h^o is the value of $h^{(i)}$ at the boundary between trapped and passing particles. We determine the value of h^o by imposing particle flux conservation. To leading order in $(\omega/\omega_b)^{1/2}$ Eq. (17) yields

$$\left[h^{o} - \left(1 - \frac{\omega_{*}^{i}}{\omega_{o}}\right)\frac{e}{T}\,\bar{\phi}_{h,p}^{o,o}\,F_{o}\right]\,(\hat{\tau}_{p}^{o}D_{p}^{o})^{1/2} =$$

$$- \sum_{j}\left[h^{o} - \left(1 - \frac{\omega_{*}^{i}}{\omega_{o}}\right)\frac{e}{T}\,\bar{\phi}_{h,t(j)}^{o,o}\,F_{o}\right]\,(\hat{\tau}_{t(j)}^{o}D_{t(j)}^{o})^{1/2} \quad . \qquad (92)$$

Solving for h^{o}, substituting in the boundary layer solutions $h_{in}^{(i)}$ and using these to evaluate the surface terms in the ion collisional contribution to the first order quadratic form gives

$$-\frac{i\nu_{i}}{\omega_{o}}\int\frac{d\ell}{B}\int d^{3}v\,\phi_{h}^{o}\bar{C}(h^{(i)}) =$$

$$+\frac{i4\pi}{m^{2}}\left(\frac{m}{2}\right)^{1/2}\int_{o}^{\infty}d\varepsilon\;\varepsilon^{1/2}\;\frac{\nu_{i}(\varepsilon)}{\omega_{o}}\left(1 - \frac{\omega_{*}^{i}}{\omega_{o}}\right)\frac{e}{T}\,F_{o}\times$$

$$\left[\int_{0}^{\lambda_{*}^{-} - \Delta\lambda_{coll}} d\lambda\; D_{p}\left(\frac{\partial\bar{\phi}_{h,p}^{o}}{\partial\lambda}\right)^{2} + \sum_{j}\int_{\lambda_{*}^{+} + \Delta\lambda_{coll}}^{\lambda_{max}^{j}} D_{t(j)}\left(\frac{\partial\bar{\phi}_{h,t(j)}^{o}}{\partial\lambda}\right)^{2}\right]$$

$$+\frac{i4\pi}{m^{2}}\left(\frac{m}{2}\right)^{1/2}\int_{o}^{\infty}d\varepsilon\;\varepsilon^{1/2}\left(\frac{\nu_{i}(\varepsilon)}{|\omega_{o}|}\right)^{1/2}\frac{(\mathrm{sgn}(\omega_{o}) - i)}{\sqrt{2}}\left(1 - \frac{\omega_{*}^{i}}{\omega_{o}}\right)\frac{e}{T}\,F_{o}\times$$

$$\left\{(\bar{\phi}_{h,p}^{o,o})^{2}(\hat{\tau}_{p}^{o}D_{p}^{o})^{1/2} + \sum_{j}(\bar{\phi}_{h,t(j)}^{o,o})^{2}(\hat{\tau}^{o}_{t(j)}D_{t(j)}^{o})^{1/2}\right.$$

$$- \frac{\left[\bar{\phi}^{o,o}_{h,p} (\hat{\tau}^o_p D^o_p)^{1/2} + \sum_j \bar{\phi}^{o,o}_{h,t(j)} (\hat{\tau}^o_{t(j)} D^o_{t(j)})^{1/2}\right]^2}{(\hat{\tau}^o_p D^o_p)^{1/2} + \sum_j (\hat{\tau}^o_{t(j)} D^o_{t(j)})^{1/2}} \quad . \tag{93}$$

The first term proportional to (ν_i/ω_o) is due to the effects of collisions on the bulk of the pitch angle distribution, while the last term proportional to $(\nu_i/\omega)^{1/2}$ is due to the boundary layer. As we shall see the ion collisional terms are stabilizing for a sufficiently flat temperature gradient. We turn now to the electron destabilizing term.

We will analyze the electron term which is proportional to $\int d\ell \;/\; B \int d^3v \phi^o_h h^{(e)}_1$ by rewriting it in terms of $|\partial h^{(e)}_1/\partial\lambda|^2$ and then integrating the electron equation explicitly to obtain the pitch angle derivative of the electron perturbed distribution function. We begin with the first order electron equation,

$$(\omega_o - \omega^e_*) \frac{e\bar{\phi}^o_h}{T} F_o = i\bar{C}(h^{(e)}_1) = \frac{i\nu_e}{\tau} \frac{\partial}{\partial\lambda} D \frac{\partial h^{(e)}_1}{\partial\lambda} \tag{94}$$

We first note that since $\overline{\phi^o_h}$ is purely real, $h^{(e)}_1$ must be purely imaginary. Multiplying the electron equation by $h^{(e)}_1/(\omega_o - \omega^e_*)$, integrating over velocity space and a flux tube, performing a partial integration in λ and using the particle flux condition Eq. (17) and continuity of h, gives

$$\int \frac{d\ell}{B} \int d^3v \frac{e\phi^o_h}{T} h^{(e)}_1 = + i \int \frac{d\ell}{B} \int d^3v \frac{\nu_e}{\tau} F_o \frac{D}{(\omega_o - \omega^e_*)} \left| \frac{\partial h^{(e)}_1}{\partial\lambda} \right|^2 . \tag{95}$$

Returning to the equation for $h_1^{(e)}$, Eq. (94), we write out the bounce average of ϕ_h^o explicitly

$$(\omega_o - \omega_*^e)\int \frac{d\ell}{(1 - \lambda B)^{1/2}} \frac{e\phi_h^o}{T} F_o = i\nu_e \frac{\partial}{\partial\lambda} D \frac{\partial h_1^{(e)}}{\partial\lambda}. \tag{96}$$

Integrating both sides gives

$$-2 \ (\omega_o - \omega_*^e)\int \frac{d\ell}{B} (1 - \lambda B)^{1/2} \frac{e\phi_h^o}{T} F_o = i\nu_e D \frac{\partial h_1^{(e)}}{\partial\lambda} + C. \tag{97}$$

We determine the value of the integration constant C by applying the boundary condition that $(\partial h_1^{(e)}/\partial\lambda)$ be finite at $\lambda = 0$ and $\lambda = (1/B_{min})$. Noting that $D(\lambda = 1/B_{min}) = D(\lambda = 0) = 0$ and that

$$\int \frac{d\ell}{B} (1 - \lambda B)^{1/2} \frac{e\phi_h^o}{T} \xrightarrow[\lambda \to \frac{1}{B_{min}}]{} 0 \tag{98}$$

and

$$\int \frac{d\ell}{B} (1 - \lambda B)^{1/2} \frac{e\phi_h^o}{T} \xrightarrow[\lambda \to 0]{} \int \frac{d\ell}{B} \frac{e\phi_h^o}{T} = 0 \tag{99}$$

we conclude that $C = 0$ for both trapped and passing species. Thus combining Eq. (88), Eq. (95), and Eq. (97) we obtain the following expression for the electron contribution to the growth rate

$$\int\frac{d\ell}{B}\int d^3v\phi_h^o h_1^{(e)} = i\int\frac{d\ell}{B}\int d^3vF_o\,\frac{4(\omega_o-\omega_*^e)}{\hat{\tau}\nu_e D(\lambda)}\,\frac{e}{T}\left[\int\frac{d\ell}{B}\,\phi_h^o\,(1-\lambda B)^{1/2}\right]^2 . \quad (100)$$

We are now in a position to write an expression for ω_1 in terms of the solution of the zero order homogeneous integral equation, Eq. (59). Returning to the quadratic form Eq. (88) we substitute for the electron and ion collisional contributions from Eq. (100) and Eq. (93) and use Eq. (60) to write ω_o in terms of ϕ_h^o. We can write the resulting expression as

$$\omega_1 = \delta\omega_r + i\gamma_{ion} + i\gamma_{electron} \quad (101)$$

where

$$\delta\omega_r = \mathrm{sgn}(\omega_o)\gamma_{ion}^{boundary}$$

$$+\frac{\int\frac{d\ell}{B}\int d^3vF_o\;(\omega_o-\omega_*^i)\left[\frac{\bar{\omega}_d}{\omega_o}\;(\overline{\phi_h^o})^2-\left(\phi_o^o\,\overline{\frac{k_\perp^2v_\perp^2}{2\Omega_i^2}}+\phi_h^o\,\overline{\frac{k_\perp^2v_\perp^2}{2\Omega_i^2}}\right)\phi_h^o\right]}{\int\frac{d\ell}{B}\int d^3v\left[2\overline{(\phi_h^o)^2}-(\overline{\phi_h^o})^2\right]F_o}$$

$$\gamma_{ion} = \gamma_{ion}^{boundary} + \gamma_{ion}^{bulk}$$

$$\gamma_{ion}^{bulk} = -\frac{4\pi}{m^2}\left(\frac{m}{2}\right)^{1/2}\int_0^\infty d\epsilon\,\epsilon^{1/2}\nu_i(\epsilon)\left(1-\frac{\omega_*^i}{\omega_o}\right)F_o$$

$$\times\left[\int_0^{\lambda_*^- - \Delta\lambda_{coll}} d\lambda\; D_p\left(\frac{\partial\overline{\phi}^o_{h,p}}{\partial\lambda}\right)^2 + \sum_j\int_{\lambda_*^+ + \Delta\lambda_{coll}}^{\lambda_{max}^j} d\lambda\; D_{t(j)}\left(\frac{\partial\overline{\phi}^o_{h,t(j)}}{\partial\lambda}\right)^2\right]$$

$$\times \left\{ \int \frac{d\ell}{B} \int d^3v \left[\overline{2(\phi_h^o)^2} - (\overline{\phi_h^o})^2 \right] F_o \right\}^{-1}$$

$$\gamma_{ion}^{boundary} = -.3(\overline{\nu}_i |\omega_o|)^{1/2} \left(1 - \frac{\hat{\omega}_*^i}{\omega_o} (1 - .57\ \eta_i) \right) n_o \times$$

$$\left\{ (\overline{\phi}_{h,p}^{o,o})^2 (\hat{\tau}_p^o D_p^o)^{1/2} + \sum_j (\overline{\phi}_{h,t(j)}^{o,o})^2 (\hat{\tau}_{t(j)}^o D_{t(j)}^o)^{1/2} \right.$$

$$\left. - \frac{\left[\overline{\phi}_{h,p}^{o,o} (\hat{\tau}_p^o D_p^o)^{1/2} + \sum_j \overline{\phi}_{h,t(j)}^{o,o} (\hat{\tau}_{t(j)}^o D_{t(j)}^o)^{1/2} \right]^2}{(\tau_p^o D_p^o)^{1/2} + \sum_j (\hat{\tau}_{t(j)}^o D_{t(j)}^o)^{1/2}} \right\}$$

$$\times \left\{ \int \frac{d\ell}{B} \int d^3v [\overline{2(\phi_h^o)^2} - (\overline{\phi_h^o})^2]\ F_o \right\}^{-1} \tag{102}$$

$$\gamma_{electron} = -2.52 \left(\frac{\omega_o^2}{\overline{\nu}_e} \right) \left[1 - \frac{\hat{\omega}_*^e}{\omega_o} (1 + 1.4\eta_e) \right] n_o \times$$

$$\sum_{p,t(j)} \int \frac{d\lambda}{\lambda} \frac{\left[\int \frac{d\ell}{B} \phi_h^o (1 - \lambda B)^{1/2} \right]^2}{\int \frac{d\ell}{B} (1 - \lambda B)^{1/2}} \times \left\{ \int \frac{d\ell}{B} \int d^3v\ F_o \left[\overline{2(\phi_h^o)^2} - (\overline{\phi_h^o})^2 \right] \right\}^{-1}$$

The energy integrals over the collision frequencies have been expressed for the ion boundary layer damping and electron growth rates using the numerical values of Rosenbluth, Ross and Kostamorov[10]. The quantities $\overline{\nu}_i$ and $\overline{\nu}_e$ in Eq. (102) are

$$\bar{\nu}_e = \frac{1}{\sqrt{2}} \frac{4\pi e^4 n_o}{m_e^{1/2} T^{3/2}} \ln\Lambda$$

$$\approx 7.7 \times 10^{-6} \frac{n_o \ln\Lambda}{T^{3/2}}$$

$$\bar{\nu}_i = \frac{1}{\sqrt{2}} \frac{4\pi e^4 n_o}{m_i^{1/2} T_i^{3/2}} \ln\Lambda \approx 1.8 \times 10^{-7} \frac{n_o \ln\Lambda}{T^{3/2}} \qquad (103)$$

where in the numerical expression n_o is in cm^{-3} and T is to be expressed in eV.

We note that Eq. (101) and Eq. (102) are not variational for ω_1 in terms of ϕ_h^o; rather ϕ_h^o and ω_o are determined as the solution to the homogeneous integral equation Eq. (76) and then used to evaluate Eq. (101) and Eq. (102). We can, however, construct a maximizing variational quadratic form for ω_o, Eq. (60). Thus our procedure to evaluate ω_1 approximately is as follows: choose an appropriate set of trial functions for ϕ_h^o, vary them until the quadratic form for ω_o is maximized, then use the maximizing trial function and the corresponding value of ω_o to evaluate ω_1 using Eq. (101) and Eq. (102).

In order to derive a very rough estimate of the growth rates involved we consider a square-well model where the magnetic field is a constant, B_o, except for abrupt maxima where the field rises to B_{max}. We denote the value of ϕ_h^o in each region as ϕ_j where j labels the region. Subtracting the quasi-neutrality relations Eq. (76) for each region gives

$$\omega_o = \frac{\hat{\omega}_*^e}{2\frac{n_o}{n_T} - 1} \qquad (104)$$

where n_T is the trapped density. The constraint that $\int d\ell\ (\phi_h^o/B) = 0$ implies that $\overline{\phi}_h^o = 0$ for passing particles. Using these results in Eq. (102) gives the following expression for the imaginary part of ω_1,

$$\mathrm{Im}\ (\omega_1) = \gamma_{ion}^{bulk} + \gamma_{ion}^{boundary} + \gamma_{electron}$$

$$\gamma_{ion}^{bulk} = 0$$

$$\gamma_{ion}^{boundary} = -.3(\overline{\nu}_i|\hat{\omega}_*^e|)^{1/2}\left[1 + \left(2\frac{n_o}{n_T} - 1\right)\ (1 - .57\eta)\right] \times$$

$$\frac{n_o}{n_T}\left(\frac{B_o}{B_{max}}\right)^{1/2}\frac{1}{\left(2\frac{n_o}{n_T} - 1\right)^{3/2}}$$

$$\gamma_{electron} = 5\,\frac{(\hat{\omega}_*^e)^2}{\overline{\nu}_e}\ \frac{\left[2\frac{n_p}{n_o} + \left(2\frac{n_o}{n_T} - 1\right)1.4\eta\right]}{\left(2\frac{n_o}{n_T} - 1\right)^3}\ \frac{n_o}{n_T}$$

$$\times\left\{\ln\left[\frac{1 + \left(1 - \frac{B_o}{B_{max}}\right)^{1/2}}{\left(\frac{B_o}{B_{max}}\right)^{1/2}}\right] - \left(1 - \frac{B_o}{B_{max}}\right)^{1/2}\right\} \qquad (105)$$

where

$$\frac{n_T}{n_o} = \left(1 - \frac{B_o}{B_{max}}\right)^{1/2}$$

$$n_p = n_o - n_T$$

$$\eta = \frac{d\ln T}{d\ln n} \quad . \tag{106}$$

We note that as η is increased the ions become less stabilizing while the electrons become more stabilizing. For $\eta = [2(n_o/n_T)/(2n_o/n_T - 1)]/.57$ the ions are no longer stabilizing. Assuming that η is less than this and that the magnetic geometry is kept fixed then $\gamma_{electron}/\gamma_{ion}^{boundary} \propto (T^{3.75}/n_o^{1.5})$. More explicitly, for plasma parameters such that

$$|\omega_o| \sim |\hat{\omega}_*^e| > \nu_i , \tag{107}$$

linear theory predicts stability if

$$\gamma_{ion}^{boundary} + \gamma_{electron} < 0. \tag{108}$$

For the square well model this implies

$$\overline{\nu_i} > a^2 \frac{|\hat{\omega}_*^e|^3}{\overline{\nu_e}^2} \tag{109}$$

$$a = \frac{5\left[\frac{2n_p}{n_o} + \left(2\,\frac{n_o}{n_T} - 1\right)1.4\eta\right] \times \left\{ \ln\left[\frac{1 + \left(1 - \frac{B_o}{B_{max}}\right)^{1/2}}{\left(\frac{B_o}{B_{max}}\right)^{1/2}}\right] - \left(1 \quad \frac{B_o}{B_{max}}\right)^{1/2}\right\}}{.3\left[1 + \left(2\frac{n_o}{n_T} - 1\right) \times \left(1 - .57\eta\right)\right]\left(\frac{B_o}{B_{max}}\right)^{1/2}\left(2\,\frac{n_o}{n_T} - 1\right)^{3/2}} \qquad .(110)$$

This mode had been studied experimentally in the Columbia Linear Machine with good agreement with theory. The real frequency increases with n_T/n_o while the saturated mode amplitude decreases with increasing density [(11)].

V. SUMMARY

We have presented the collisional effects on trapped particle modes in the high and low collisionality limits using a perturbative approach. The magnetic equilibrium geometry is taken to be arbitrary although the equilibrium electrostatic potential is taken to be a constant axially.

We can summarize the physical effects of collisions as follows. In the drift kinetic equation a perturbing potential acts as a perturbing local spatial source of particles. This number perturbation is then carried along unperturbed orbits. In a collisionless trapped particle mode distribution function perturbations in the center cell are communicated to the anchor only through the streaming of passing particles. With the addition of pitch-angle collisions the local perturbing source is carried along unperturbed orbits and diffused in pitch angle. The collisional diffusion creates an additional mechanism by which distribution function perturbations can flow from the center cell into the anchor. In the low collisionality case the dissipative nature of this relaxation destabilizes an otherwise stable negative energy wave.

If the collision frequency is sufficiently large the perturbed distribution function is forced to be nearly isotropic in pitch angle. This leads to two possible modes. The first is a flute-like mode whose stability is determined by the flux tube integral of the beta weighted curvature. Because of this mode's flute-like nature the lowest order response of both ions and electrons is a constant times the equilibrium Maxwellian distribution. Physically the ions and electrons $E \times B$ drift together and no net charge perturbation results. The eigenfrequency is determined by taking into account the Doppler shifts due to the curvature drifts of electrons and ions. Because of the constraint that collisions do not change the net number of particles in a flux tube, collisional effects do not enter into determining the mode stability. Thus this mode persists even for ion collision frequencies large compared to the mode frequency.

The second mode in this regime, the dissipative trapped ion mode is driven by the difference in electron and ion collision frequencies. Thus a density increase which raises both the ion and electron collision frequencies without affecting the lowest order mode eigenfrequency, leads to a damping of the mode. In this case the explicit form of the collision operator is important in determining the growth rate. For a calculation of this growth rate using a collisional operator which includes energy drag the reader is referred to Ref. (12).

Because of the perturbative nature of the analysis followed here, the behavior of a system with $\omega_* \sim \nu_e$ cannot be determined. Using a two region, magnetic square well model an expression suitable for arbitrary collisionality and amenable to numerical analysis has been developed elsewhere. The reader is referred to Ref. (12) for details.

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