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Matt Landreman and Peter J. Catto

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Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge MA 02139 USA

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Effects of the radial electric field in a quasisymmetric stellarator

Matt Landreman, Peter J Catto, MIT PSFC

Massachusetts Institute of Technology, Plasma Science and Fusion Center, 167 Albany St., Cambridge, MA 02139, USA
E-mail: landrema@mit.edu

Abstract

Recent calculations have shown that a radial electric field can significantly alter the ion flow, neoclassical ion heat flux, bootstrap current, and residual zonal flow in a tokamak, even when the $\mathbf{E}\times\mathbf{B}$ drift is much smaller than the ion thermal speed. Here we show the novel analytical methods used in these calculations can be adapted to a quasisymmetric stellarator. The methods are based on using the conserved helical momentum ψ_* instead of the poloidal or toroidal flux as a coordinate in the kinetic equation. The calculations also employ a model collision operator which keeps only the velocity-space derivatives normal to the trapped-passing boundary, even as this boundary is shifted and deformed by the $\mathbf{E}\times\mathbf{B}$ drift. We prove the isomorphism between quasisymmetric stellarators and tokamaks extends to the finite-$\mathbf{E}\times\mathbf{B}$ generalizations of both neoclassical theory and residual zonal flow. The electric field in the HSX stellarator may be sufficient for these finite-$\mathbf{E}\times\mathbf{B}$ effects to be significant.

1. Introduction

One important concept in modern stellarator design is quasisymmetry [1-5]. A magnetic field is defined to be quasisymmetric when the magnitude $B = |\mathbf{B}|$ varies on a flux surface only through a fixed linear combination of the Boozer angles [3]. Remarkably, magnetic fields can be found [6,7] which have this symmetry property even though the fields are not axisymmetric in conventional cylindrical coordinates. Quasisymmetry can also be defined in a coordinate-independent manner [8]. In a quasisymmetric field, Noether's theorem implies the existence of a conserved quantity. Therefore the particle orbits become integrable, so there are no direct orbit losses, and radial transport is reduced. Another noteworthy consequence of quasisymmetry is the "isomorphism" [1,2] between a quasisymmetric plasma and an axisymmetric one, whereby conventional neoclassical transport formulae for the quasisymmetric case can be obtained from the corresponding formulae for axisymmetry by making certain substitutions. These isomorphism rules will be reviewed in detail in section 3. However, not all plasma quantities are related by the isomorphism. For example, it is pointed out in [9] that classical transport fluxes do not obey the substitution rules, since classical transport arises from gyromotion rather than drift motion.

In this paper we show the isomorphism does extend to neoclassical transport and to the residual zonal flow even when these quantities are modified by a radial electric field E_ψ. These "finite-E_ψ" modifications were calculated recently for tokamaks by Kagan *et al* in [10-16]. Here, the radial electric field is defined by $E_\psi = -\left(\partial\Phi / \partial\psi_{\mathrm{p}}\right)\left|\nabla\psi_{\mathrm{p}}\right|$ where Φ is the electrostatic potential and $2\pi\psi_{\mathrm{p}}$ is the poloidal flux. The finite-E_ψ calculations in a tokamak are based on several novel analytical techniques. First, new orderings are used for gradient scale-lengths and for E_ψ based on the small parameter B_{p} / B, where B_{p} is the poloidal field. Secondly, the canonical angular momentum Ψ_* is used as the "radial" coordinate in the kinetic equation, and the transit average is redefined to be an integral along a constant-Ψ_* path instead of along a flux surface. Finally, a new model collision operator is used, one in which velocity-space derivatives are taken normal to the trapped-passing boundary, accounting for the modification of this boundary by E_ψ. In this paper we will show how to generalize all of these elements – the orderings, B_{p} / B, Ψ_*, and the model collision operator – to a quasisymmetric stellarator.

In a tokamak, the finite-E_ψ effects are important primarily in a high confinement mode (H-mode) pedestal. If an H-mode transition is ever observed on a future large quasisymmetric stellarator, as might have occurred in NCSX [17], the finite-E_ψ effects derived herein will be applicable to the pedestal of that device. However, the finite-E_ψ effects may be observable in a stellarator without H-mode, perhaps even in the only presently operating quasisymmetric device, HSX [7]. Finite-E_ψ effects will be more important in a quasisymmetric stellarator than in the equivalent tokamak due to the larger effective ripple in the former. A field generated by real magnets will inevitably deviate somewhat from perfect quasisymmetry, and even small deviations can cause a large enough nonambipolar particle flux [8,18] to generate a large E_ψ. Furthermore, HSX uses electron cyclotron heating, which leaves the ions cold, while we will show the electric field scale at which finite-E_ψ effects set in is proportional to the ion thermal speed. We will show quantitatively that E_ψ in HSX may indeed be large enough for the effects evaluated herein to appear.

The next section gives further background on the finite-E_ψ effects in tokamaks and on quasisymmetry. Detailed quantitative analysis begins in section 3 with a review of the isomorphism between neoclassical transport in a quasisymmetric plasma and an axisymmetric one. In section 4 we show how the finite-E_ψ orderings generalize in quasisymmetry, and we discuss applicability of these orderings to HSX. The model collision operator and the modifications to neoclassical transport are derived in section 5. The residual zonal flow is discussed in section 6, and we conclude in section 7.

2. Background

It was recently shown in [13,14,16,19] that a modest radial electric field can cause significant modifications to the radial ion heat flux, ion flow, and bootstrap current in a tokamak. These modifications become significant when

$$E_\psi \sim B_\mathrm{p} v_\mathrm{i} / c \quad (1)$$

where $v_\mathrm{i} = \sqrt{2T/m}$ is the ion thermal speed. Because $B_\mathrm{p} \ll B$ in a typical tokamak, the modifications to conventional neoclassical transport [20-22] become important when the $\mathbf{E}\times\mathbf{B}$ drift $\mathbf{v}_E = cB^{-2}\mathbf{E}\times\mathbf{B}$ is still much smaller than v_i. Physically, the electric field becomes important when it is as large as (1) because it then affects ion trapping [11,12,15], which can be understood as follows. An ion is trapped in a tokamak when its net poloidal motion, the sum of parallel and drift components, is small enough that the mirror force can stop the particle before it reaches the inboard midplane. For $E_\psi \sim B_\mathrm{p} v_\mathrm{i}/c$, this net poloidal motion receives contributions of comparable magnitude from the parallel motion and the $\mathbf{E}\times\mathbf{B}$ drift. It is therefore not ions of small $v_{||}$ which are trapped, but rather ions for which the two contributions nearly cancel. To restate this argument more quantitatively, note that an ion's net poloidal motion is given by

$$d\Theta/dt \approx \left(v_{||}\mathbf{b} + \mathbf{v}_E\right)\cdot\nabla\Theta = \left(v_{||} + u\right)\mathbf{b}\cdot\nabla\Theta \quad (2)$$

where Θ is a poloidal coordinate with periodicity 2π, $\mathbf{b} = \mathbf{B}/B$,

$$u = cIB^{-1}\partial\Phi/\partial\psi_\mathrm{p}, \quad (3)$$

and $I = RB_\mathrm{t}$ is the major radius times the toroidal magnetic field. A particle is trapped if its $d\Theta/dt$ can vanish, so we expect the trapped ions to be those localized in phase space near $v_{||} \approx -u$. The shift u becomes $O(v_\mathrm{i})$ when E_ψ is as large as (1).

References [13,14,16] show how this shift in the trapped-passing boundary causes changes to the neoclassical ion heat flux, ion flow, and bootstrap current in the banana collisionality regime. These calculations use a modified model operator for ion-ion collisions. In conventional calculations, only the pitch-angle scattering component of the collision operator is kept, together with an offset to conserve momentum. The justification for using this simplified operator, valid when $E_\psi \ll B_\mathrm{p} v_\mathrm{i} / c$, is that the distribution function is found to have a large derivative with respect to pitch angle. Loosely, this is the derivative normal to the trapped-passing boundary in velocity space. The modified collision operator employed by Kagan and Catto [13] instead keeps only derivatives normal to the *shifted* trapped-passing boundary, even as this boundary is shifted by the $\mathbf{E}\times\mathbf{B}$ drift as discussed above. The modified operator thereby captures the dominant velocity-space derivative of the distribution function when $E_\psi \sim B_\mathrm{p} v_\mathrm{i} / c$.

Besides neoclassical transport, another quantity which is modified when $E_\psi \sim B_\mathrm{p} v_\mathrm{i} / c$ is the residual zonal flow [11,12,15], a quantity introduced by Rosenbluth and Hinton in [23]-[24]. The "residual" summarizes the rate of zonal flow damping, sidestepping the more complicated analysis of the nonlinear turbulence which drives the flows. In the Rosenbluth-Hinton model, the nonlinear drive for zonal flow in the kinetic equation is effectively replaced by a delta function in time. After many ion bounce times, the ions' radial drift partially shields the initial potential perturbation $\Phi(t=0)$. The residual zonal flow is then defined as the ratio $\Phi(t\to\infty)/\Phi(t=0)$. Later authors have generalized the calculation to include additional effects [25] and nonaxisymmetric geometry [26,27]. Analytical expressions for the residual, obtained using a large-aspect-ratio approximation, can be used to validate gyrokinetic and gyrofluid turbulence codes. The residual also gives insight into the zonal flow amplitude which can be expected in the presence of turbulence.

One element of the calculations used to obtain the finite-E_ψ effects in [11-14,16,15,19] is a novel set of orderings. To understand the new orderings, first recall the conventional approaches. The standard ordering for transport calculations [20-22] is the "drift" or "low flow" ordering $\mathbf{v}_E \sim \delta v_\mathrm{i}$ where the small parameter $\delta = \rho / a$ is the ion gyroradius divided by a system scale-length. No distinction is typically made between radial and parallel scale-lengths. Also, all components of $\mathbf{B}$ are ordered the same, so $B_\mathrm{p} \sim B_\mathrm{t} \sim B$. The perpendicular guiding-center drifts are given to leading order by

$$\mathbf{v}_\mathrm{d} = \left(v_{||}/\Omega\right)\nabla\times\left(v_{||}\mathbf{b}\right). \quad (4)$$

Here and throughout this paper, derivatives hold $E = v^2 + Ze\Phi/m$ and $\mu = v_\perp^2/2B$ fixed, unless subscripts specify other quantities to be held fixed. The other conventional ordering [28-31] is the "large flow" or "MHD" ordering $v_E \sim v_\mathrm{i}$. Again, no distinction is typically made between radial and parallel scale-lengths or between different components of $\mathbf{B}$. In the large-flow ordering, the conserved magnetic moment is changed to $\left(\mathbf{v}-\mathbf{v}_E\right)_\perp^2/2B$, and other perpendicular drifts arise which are the same order as those in (4). In contrast, we will use "finite- E_ψ" to describe the orderings used by Kagan and Catto in [11-14,16,15,19]. In this approach, B_p/B is taken to be a small parameter, and the electric field is ordered using (1). As v_E is therefore $\ll v_\mathrm{i}$, then $\mu = v_\perp^2/2B$ is still conserved, the perpendicular drifts are still given by (4) to leading order, and the low-flow drift-kinetic or gyrokinetic equations are applicable. Also, the radial density scale-length is ordered $\left(B/B_\mathrm{p}\right)\rho$ rather than a. Whereas [28,29] give proofs that an electric field of magnitude $\sim B_p v_i/c$ implies a sonic toroidal flow under the large-flow ordering, these proofs do not apply in the finite-E_ψ ordering due to the larger magnitude of $\left(\partial f/\partial\psi_\mathrm{p}\right)_\mathbf{v}$. It is therefore permissible for the mean flow to be small compared to v_i, in agreement with measurements of flow in tokamak pedestals. In the finite- E_ψ ordering, $v_{||}\mathbf{b}\cdot\nabla\Theta$ and $\mathbf{v}_E\cdot\nabla\Theta$ are the same order, as in the high-flow ordering, but unlike the high-flow ordering, the leading-order Maxwellian is now taken to be stationary. Thus, analysis in the finite-E_ψ ordering differs from both of the conventional approaches.

To calculate neoclassical transport and the residual zonal flow in the new orderings, Kagan and Catto [10] introduce the following novel analytical technique. A change of variables is made in the kinetic equation, replacing ψ_p with the gyroaveraged canonical angular momentum $\Psi_* = \psi_\mathrm{p} - I v_{||}/\Omega$ as an independent variable. The drift-kinetic operator D gives zero when acting on Ψ_*, so the "radial" term $\left(D\Psi_*\right)\partial f/\partial\psi_*$ in the kinetic equation vanishes. Thus, the kinetic equation has the form $\left(D\Theta\right)\partial f/\partial\Theta = C\{f\}$, and the left-hand side can be annihilated by integrating in Θ after dividing by $D\Theta$, even when $v_{||}\mathbf{b}\cdot\nabla\Theta$ and $\mathbf{v}_E\cdot\nabla\Theta$ are the same order.

In nonaxisymmetric plasmas, canonical angular momentum is no longer conserved. It is not clear therefore whether the $\psi_\mathrm{p}\to\Psi_*$ change-of-variables technique described above can be generalized to a nonaxisymmetric plasma. However, it was shown in [1] that a quantity resembling Ψ_* is conserved by the guiding-center drift motion when the magnetic field is quasisymmetric. A field is defined to be quasisymmetric if B is independent of one of the Boozer angles, or if B depends on the Boozer angles only through a fixed linear combination [1-5]. Axisymmetric fields are quasisymmetric, but nearly quasisymmetric fields can be found which are far from being axisymmetric [6,7]. Since guiding-center drift motion can be expressed in terms of a Lagrangian in which only B (and not $\mathbf{B}$) appears, the symmetry in B gives rise to a conserved quantity through Noether's Theorem. By using this conserved quantity, we will show that the novel analytical methods used to find finite-E_ψ effects in tokamaks can be adapted for quasisymmetric devices. In doing so, we generalize all the results of Kagan and Catto [10-14,16] and Landreman and Catto [15] to this important class of stellarators.

3. Definitions and quasisymmetry isomorphism

We consider a scalar-pressure equilibrium with well defined flux surfaces and flow that is much smaller than the thermal speed. In this situation, poloidal and toroidal Boozer angles $\left(\theta,\ \zeta\right)$ can be defined such that

$$\mathbf{B} = q\nabla\psi_{\mathrm{p}} \times \nabla\theta + \nabla\zeta \times \nabla\psi_{\mathrm{p}} \quad (5)$$

$$= L\nabla\psi_{\mathrm{p}} + K\nabla\theta + I\nabla\zeta \quad (6)$$

where K and I are flux functions and $q = q\left(\psi_{\mathrm{p}}\right)$ is the safety factor. (Note that some references on quasisymmetry instead use I to denote the $\nabla\theta$ coefficient. We choose the new convention because RB_{t} in an axisymmetric plasma is often denoted by I, and I in (6) properly reduces to RB_{t} in axisymmetry. This can be seen by using Ampere's Law to show that both RB_{t} and the I in (6) equal $2/c$ times the current topologically linked outside a given flux surface.)

A quasisymmetric field is then defined by the property that B depends on the two Boozer angles only through a particular linear combination, that is, $B = B\left(\psi_{\mathrm{p}},\ \chi\right)$ where

$$\chi = M\theta - N\zeta \quad (7)$$

and M and N are fixed integers for a given device. In this situation, it can be shown [1,6,4] that $D\psi_* = 0$ where $D = \left(\partial/\partial t\right) + \left(v_{||}\mathbf{b} + \mathbf{v}_{\mathrm{d}}\right)\cdot\nabla + Zem^{-1}\left(\partial\Phi/\partial t\right)\left(\partial/\partial E\right)$ is the drift-kinetic operator (assuming there is no inductive electric field), $\mathbf{v}_{\mathrm{d}}$ is given by (4),

$$\psi_* = \psi_{\mathrm{h}} - I_{\mathrm{h}} v_{||}/\Omega, \quad (8)$$

$$I_{\mathrm{h}} = I + NK/M, \quad (9)$$

and

$$\psi_{\mathrm{h}} = \psi_{\mathrm{p}} - \left(N/M\right)\psi_{\mathrm{t}} \quad (10)$$

is a "helical" combination of the poloidal flux and the toroidal flux $2\pi\psi_{\mathrm{t}}$. We give a streamlined proof of $D\psi_* = 0$ in appendix A. The result closely resembles the result for an axisymmetric magnetic field that $D\Psi_* = 0$ where $\Psi_* = \psi_{\mathrm{p}} - Iv_{||}/\Omega$. While $D\Psi_* = 0$ reflects the conservation of canonical angular momentum in an axisymmetric field, $D\psi_* = 0$ reflects the conservation of ψ_* during drift motion in a quasisymmetric field (gyromotion must be neglected). As $\psi_{\mathrm{p}} - Iv_{||}/\Omega$ is conserved when $B = B\left(\psi_{\mathrm{p}},\ \Theta\right)$ while $\psi_{\mathrm{h}} - I_{\mathrm{h}}v_{||}/\Omega$ is conserved when $B = B\left(\psi_{\mathrm{h}},\ \chi\right)$, we might expect other tokamak formulae to be applicable to a quasisymmetric stellarator if we make the replacements

$$\left(\psi_{\mathrm{p}}, I, \Theta\right) \to \left(\psi_{\mathrm{h}}, I_{\mathrm{h}}, \chi\right). \quad (11)$$

We now sketch the proof that this isomorphism indeed holds for the conventional (low-flow) banana-regime neoclassical fluxes and flows. The analysis will be generalized to the finite-E_ψ case in sections 4-5. We begin with the drift-kinetic equation $Df = C$ for any particle species in a quasisymmetric plasma, using $\left(\psi_{\mathrm{h}},\ \chi,\ \zeta\right)$ as the spatial coordinates. (For $M = 0$ "quasi-poloidal" symmetry, χ and ζ are degenerate, so θ should be substituted for ζ as the third coordinate throughout.) We make an ansatz $\left(\partial f/\partial\zeta\right)_\chi = 0$, and the f we find will be consistent with this assumption. The leading order equation is taken to be $v_{\|}(\mathbf{b}\cdot\nabla\chi)\partial f_0/\partial\chi = C\{f_0\}$. The conventional entropy production argument then shows that f_0 is a Maxwellian and a flux function. The next order equation is then

$$v_{\|}(\mathbf{b}\cdot\nabla\chi)\frac{\partial f_1}{\partial\chi} + (\mathbf{v}_{\mathrm{d}}\cdot\nabla\psi_{\mathrm{h}})\frac{\partial f_0}{\partial\psi_{\mathrm{h}}} = C\{f_1\}. \quad (12)$$

Next, we apply the following identity (proven in appendix A):

$$\mathbf{v}_{\mathrm{d}}\cdot\nabla\psi_{\mathrm{h}} = v_{\|}\mathbf{b}\cdot\nabla\left(I_{\mathrm{h}}v_{\|}/\Omega\right) = v_{\|}(\mathbf{b}\cdot\nabla\chi)\frac{\partial}{\partial\chi}\left(I_{\mathrm{h}}v_{\|}/\Omega\right). \quad (13)$$

This result is what one would expect by naively applying the substitutions (11) to the corresponding identity for axisymmetry. We can then combine (12)-(13) as

$$v_{\|}(\mathbf{b}\cdot\nabla\chi)\frac{\partial g}{\partial\chi} = C\left\{g - \frac{I_{\mathrm{h}}v_{\|}}{\Omega}\frac{\partial f_0}{\partial\psi}\right\} \quad (14)$$

where

$$g = f_1 + I_{\mathrm{h}}v_{\|}\Omega^{-1}\partial f_0/\partial\psi_{\mathrm{h}}. \quad (15)$$

A subsidiary expansion $g = g^{(0)} + g^{(1)} + \ldots$ is then made in the smallness of the right side of (14) compared to the left. The leading order equation is $\partial g^{(0)}/\partial\chi = 0$. The $g^{(1)}$ term in the next order equation is then annihilated by a transit average to give the constraint

$$0 = \overline{C\left\{g^{(0)} - I_{\mathrm{h}}v_{\|}\Omega^{-1}\partial f_0/\partial\psi_{\mathrm{h}}\right\}} \quad (16)$$

which determines $g^{(0)}$, thereby determining f_1. Here, the transit average of any quantity Y is defined by

$$\overline{Y} = \frac{\oint d\chi\; Y/\left(v_{\|}\mathbf{b}\cdot\nabla\chi\right)}{\oint d\chi/\left(v_{\|}\mathbf{b}\cdot\nabla\chi\right)}. \quad (17)$$

For passing regions of $(E, \mu\;\; \psi_{\mathrm{h}})$-space (in which any χ is allowed), $\oint(\cdot)d\chi$ indicates $\int_0^{2\pi/M}(\cdot)d\chi$. For trapped regions (in which not all χ are allowed), $\oint(\cdot)d\chi$ denotes $\sum_{\varsigma}\varsigma\int_{\chi_{\min}}^{\chi_{\max}}(\cdot)d\chi$ where $\varsigma = \mathrm{sgn}(v_{\|})$.

To justify our assumption that $\partial f/\partial\zeta = 0$, we need to show that neither $\mathbf{b}\cdot\nabla\chi$ nor C introduce ζ-dependence in $g^{(0)}$ through (16)-(17). First, by forming the product of (5) with (6) we find $\mathbf{B}\cdot\nabla\theta = B^2/(qI+K)$, so $\mathbf{b}\cdot\nabla\chi = B^{-1}(M-Nq)\mathbf{B}\cdot\nabla\theta$ is independent of ζ. Second, as argued in the footnote of [32], the linearized and gyro-averaged collision operator only introduces spatial dependence through B, so no ζ-dependence is introduced. The pitch-angle scattering model operators have this same property. Thus, $g^{(0)}$ is independent of ζ, so f_1 is as well. The problem of finding f_1 in a 3D field has thereby become 2D if the field is quasisymmetric and (ψ_h, χ) variables are used.

Equations (15)-(16) can be obtained by naively applying the substitutions (11) to the corresponding tokamak expressions, so f_1 can be obtained by these same substitutions. Forming

$\int d^3v\ v_{||} f_1$, then the parallel flows and currents obey the isomorphism as well. Finally, as shown in appendix B, the moment equations used to obtain the particle and heat fluxes from f_1 also obey the isomorphism. Thus, all the banana-regime neoclassical fluxes and flows follow the isomorphism.

Table 1 summarizes the isomorphism rules (including the generalizations which will be derived in the next section). Care must be taken in two regards. First, whereas in axisymmetric plasmas it is common to apply $\mathbf{b}\cdot\nabla\Theta \approx (qR_0)^{-1}$, it is not generally true that $\mathbf{b}\cdot\nabla\chi \approx (qR_0)^{-1}$ in a quasisymmetric stellarator. Second, tokamak calculations use the model field magnitude $B = B_0\left[1+2\varepsilon\sin^2(\Theta/2)\right]$ with $\varepsilon = a/R_0$. In a stellarator, however, it will not generally be true that the relative field variation equals twice the inverse aspect ratio. We can use the expression $B = B_0\left[1+2\varepsilon\sin^2(\chi/2)\right]$ in stellarator calculations only if we understand the ε therein to be *defined* as $(B_{\max}-B_{\min})/(2B_{\min})$. Thus, the isomorphism substitutions must be made in tokamak expressions *before* either $\mathbf{b}\cdot\nabla\Theta \approx (qR_0)^{-1}$ or $\varepsilon = a/R_0$ are invoked.

4. Change of variables and generalized Kagan-Catto orderings

At a sufficiently large value of E_ψ, the contribution from the $\mathbf{E}\times\mathbf{B}$ drift to the $(\mathbf{v}_\mathrm{d}\cdot\nabla\chi)\partial f/\partial\chi$ term in the drift-kinetic equation $Df = C$ will no longer be negligible compared to the $v_{||}(\mathbf{b}\cdot\nabla\chi)\partial f/\partial\chi$ term. The presence of the extra term invalidates the steps (12)-(16), so we must use a different approach to find f. A weaker E_ψ is required to cause this problem for the ions than to cause it for electrons, so for the rest of this paper we assume all symbols are ion quantities unless specified otherwise.

We now make a change of variables which will permit a solution for f. We use ψ_* instead of ψ_h as an independent variable in the kinetic equation (along with χ and ζ), using the chain rule for changing to a new set of variables $\{Q_j\}$:

$$Df = \sum_j \left(DQ_j\right)\left(\partial f/\partial Q_j\right)_{Q_{k\neq j}}. \quad (18)$$

We make an ansatz $(\partial f/\partial\zeta)_{\psi_*} = 0$, and the $f = f(\psi_*, \chi\ E, \mu, t)$ we find will be consistent with this assumption. We have already shown $D\psi_* = 0$, so $Df = C$ becomes

$$\left(\frac{\partial f}{\partial t}\right)_{\psi_*} + \left(v_{||}\mathbf{b}+\mathbf{v}_\mathrm{d}\right)\cdot\nabla\chi\left(\frac{\partial f}{\partial\chi}\right)_{\psi_*} + \frac{Ze}{m}\left(\frac{\partial\Phi}{\partial t}\right)\left(\frac{\partial f}{\partial E}\right)_{\psi_*} = C \quad (19)$$

Note that unlike (12) there is now no "radial" derivative term.

As in the previous section, we neglect the contribution of the magnetic drifts to $\mathbf{v}_\mathrm{d}\cdot\nabla\chi$ in (19) compared to the adjacent $v_{||}\mathbf{b}\cdot\nabla\chi$. However, we now keep the contribution of the $\mathbf{E}\times\mathbf{B}$ drift to $\mathbf{v}_\mathrm{d}\cdot\nabla\chi$, giving

$$\left(v_{||}\mathbf{b}+\mathbf{v}_\mathrm{d}\right)\cdot\nabla\chi \approx \left(v_{||}+u\right)\left(\mathbf{b}\cdot\nabla\chi\right) \quad (20)$$

where

$$u = cI_{\mathrm{h}}\Phi' / B \quad (21)$$

and the prime denotes $\partial / \partial\psi_{\mathrm{h}}$. This is precisely the definition for u we would obtain by naively applying the isomorphism rules (11) to the u in (3) and [13,10,11,15].

To verify that E_ψ can be large enough in quasisymmetric stellarator experiments to make u comparable to $v_{||}$ in (20), we consider HSX [7], which has $N = 4$, $M = 1$, $K = 0$, and $q \approx 1$. Figures 4 and 5 in reference [33] give values of $\left\langle \left|2\pi\nabla\psi_{\mathrm{t}}\right|^2 \right\rangle$ and I for HSX. Taking $\left|\nabla\psi_{\mathrm{t}}\right| \sim \left\langle \left|\nabla\psi_{\mathrm{t}}\right|^2 \right\rangle^{1/2}$, noting $\nabla\psi_{\mathrm{h}} = \left(1 - Nq/M\right)\nabla\psi_{\mathrm{p}}$, and defining

$$\alpha = \frac{\left|\nabla\psi_{\mathrm{h}}\right|}{I_{\mathrm{h}}} = \frac{\left(1 - Nq/M\right)\left|\nabla\psi_{\mathrm{p}}\right|}{I + NK/M} \quad (22)$$

we find $|\alpha| \sim 0.3$ at the last closed flux surface, with α decreasing monotonically to zero at the magnetic axis. Then u is comparable to $v_{||}$ when the quantity

$$U = \frac{u}{v_{\mathrm{i}}} = 1.2\frac{\left(E_\psi / 400\ \mathrm{V/cm}\right)\sqrt{m_{\mathrm{i}} / m_{\mathrm{H}}}}{\left(\alpha / 0.3\right)\left(B / 1\ \mathrm{T}\right)\sqrt{T_{\mathrm{i}} / 60\ \mathrm{eV}}} \quad (23)$$

is $O(1)$. The normalization for each parameter above reflects a typical HSX magnitude [18]. The value $E_\psi \sim 400$ V/m above is not measured directly, but fields of this magnitude are predicted by calculations which solve for E_ψ using ambipolarity; the electron and ion particle fluxes are not automatically equal in these calculations because the departures of the real HSX field from perfect quasisymmetry are included. It is evident from (23) that U can be comparable to 1. In a tokamak, U is typically non-negligible only in an H-mode pedestal. However, several factors allow $U \sim 1$ in HSX even in the absence of a pedestal. First, the departure of the true magnetic field from perfect quasisymmetry, while small, is still sufficient to cause significant nonambipolar particle fluxes [8,18], leading to a large E_ψ. Second, the use of electron cyclotron heating leaves the ions relatively cold, and T_i enters the denominator of (23). That U can exceed 1 was also argued in [18], since the "resonant" electric field E_r^{res} discussed in that reference is defined such that $E_\psi > E_r^{\mathrm{res}}$ and $U > 1$ are equivalent conditions.

Next, observe that $\alpha \to B_{\mathrm{p}} / B$ in a tokamak. As $|\alpha| \ll 1$ throughout HSX, tokamak results which rely on the smallness of B_{p} / B will be relevant to HSX. We will show that ρ / α will play the role in a quasisymmetric plasma that the poloidal gyroradius $\left(B / B_{\mathrm{p}}\right)\rho$ plays in an axisymmetric plasma.

For the rest of this paper, we adopt the orderings $U \sim 1$ and $\alpha \ll 1$. We take the scale-length for magnetic quantities such as B and I_{h} to be a, with $a \gg \rho / \alpha$. We use the shorter scale-length ρ / α for the density and electrostatic potential. These "finite-E_ψ" orderings all reduce to Kagan and Catto's [10-14,16] in the limit of axisymmetry. Since $v_E = \alpha U v_{\mathrm{i}}$, then in order to use the form of the drifts (4) which is valid only for $v_E \ll v_{\mathrm{i}}$, we require $\alpha U \ll 1$.

5. Neoclassical transport

5.1. Expansion of the kinetic equation

We take the distribution function to be a stationary Maxwellian to leading order

$$f_{\mathrm{M}} = \eta\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mE}{T}\right) \quad (24)$$

where η and T are flux functions. Further motivation for this leading-order distribution is given in appendix C. We next define $g = f - F$ where $F(\psi_*, E)$ is obtained by replacing ψ_{h} with ψ_* in the arguments of η and T in f_{M}:

$$F = \eta(\psi_*)\left[\frac{m}{2\pi T(\psi_*)}\right]^{3/2} \exp\left[-\frac{mE}{T(\psi_*)}\right]. \quad (25)$$

Note that a Taylor-expansion of η and T about $\psi_* \approx \psi_{\mathrm{h}}$ in this definition gives $F \approx f_{\mathrm{M}} + F_1$ with

$$F_1 = -f_{\mathrm{M}} \frac{v_{||} I_{\mathrm{h}}}{\Omega}\left[\frac{p'}{p} + \frac{Ze\Phi'}{T} + \left(\frac{mv^2}{2T} - \frac{5}{2}\right)\frac{T'}{T}\right], \quad (26)$$

where primes again denote $\partial/\partial\psi_{\mathrm{h}}$, and p, Φ, and T are evaluated at ψ_{h} rather than ψ_*. Thus, the departure of f from the Maxwellian (24) has two parts: $f - f_{\mathrm{M}} \approx F_1 + g$, where both F_1 and g are small compared to f_{M}. Then, since time derivatives in the drift-kinetic equation (19)-(20) are small for the calculation of neoclassical transport,

$$\left(v_{||} + u\right)(\mathbf{b}\cdot\nabla\chi)(\partial g/\partial\chi)_{\psi_*} = C\{F + g\}. \quad (27)$$

We approximate the collision operator by the linearized ion-ion collision operator $C \approx C_{\mathrm{ii,l}}$. Using $C_{\mathrm{ii,l}}\{Xf_{\mathrm{M}}\} = 0$ for $X = 1$, $v_{||}$, or v^2, we can write $C\{F + g\} \approx C_{\mathrm{ii,l}}\{g - G\}$ where

$$G = f_{\mathrm{M}} \frac{\left(v_{||} + u\right) I_{\mathrm{h}}}{\Omega}\left[\frac{m\left(v^2 + u^2\right)}{2T} - z\right]\frac{T'}{T} \quad (28)$$

and z is independent of velocity. We will choose the value of z later to preserve momentum conservation by our model collision operator.

We next expand the kinetic equation and $g = g^{(0)} + g^{(1)} + \ldots$ for small collisionality. The leading order form of (27) is $\partial g^{(0)}/\partial\chi = 0$. The next order form is

$$\left(v_{||} + u\right)(\mathbf{b}\cdot\nabla\chi)\left(\partial g^{(1)}/\partial\chi\right)_{\psi_*} = C_{\mathrm{ii,l}}\left\{F + g^{(0)}\right\}. \quad (29)$$

5.2. Particle orbits and new transit average

To understand the proper operation for annihilating the $g^{(1)}$ term in this last equation, we analyze particle trapping in greater detail. In particular, we examine how $v_{||} + u$ varies with χ at fixed μ, E, and ψ_*, and what the periodicity requirement is on $g^{(1)}$ in the trapped part of phase space. The calculation of particle orbits proceeds much as in the tokamak case analyzed in [15], but making the substitutions (11). Therefore, we will only summarize the calculation and its main results here.

To allow for finite radial electric fields (i.e. nonzero u), we will take into account the changes in potential $\Phi(\psi_{\mathrm{h}})$ due to variation in a particle's radial coordinate ψ_{h} over its trajectory. However, we ignore the radial variation of all magnetic quantities, treating I_{h} as constant. We also ignore the effect of the radial drift on B, taking.

$$B(\psi_{\mathrm{h}},\chi) \approx B(\chi) = B_0 / h(\chi) \quad (30)$$

where the constant B_0 represents the minimum value of $|\mathbf{B}|$ over the particle's trajectory, so $h(\chi) \le 1$.

We define other quantities with a 0 subscript (u_0, $\psi_{\mathrm{h}0}$, etc.) to be the values when the particle crosses a minimum of B. This definition is unique for passing particles, which always have the same ψ_{h} when they cross through a B minimum, but not for trapped particles, for which ψ_{h} alternates between two values with each crossing. As long as all of the subscript 0 quantities for a given trajectory refer to the larger ψ_{h} crossing or all refer to the smaller ψ_{h} crossing, it is valid to choose either.

Next, the potential is Taylor-expanded to second order about $\psi_{\mathrm{h}0}$ to obtain

$$\Phi \approx \Phi_0 + \Delta\, \Phi_0' + \tfrac{1}{2}\Delta^2 \Phi_0'' \quad (31)$$

where $\Delta = \psi_{\mathrm{h}} - \psi_{\mathrm{h}0}$, and Φ_0, Φ_0', and Φ_0'' are respectively Φ, $d\Phi / d\psi_{\mathrm{h}}$, and $d^2\Phi / d\psi_{\mathrm{h}}^2$ evaluated at the reference flux surface $\psi_{\mathrm{h}} = \psi_{\mathrm{h}0}$. Using conservation of μ, E, and ψ_*, straightforward algebra yields

$$v_{\|} + u = \sigma\sqrt{\left(\frac{v_{\|0}}{h} + hu_0\right)^2 - \left(1 + S_0 h^2 - h^2\right)\left[v_{\|0}^2\left(\frac{1}{h^2} - 1\right) + 2\mu B_0\left(\frac{1}{h} - 1\right)\right]} \quad (32)$$

where $\sigma = \pm 1 = \operatorname{sgn}\left(v_{\|} + u\right)$, and

$$S_0 = 1 + \frac{cI_{\mathrm{h}}^2 \Phi_0''}{B_0 \Omega_0} \quad (33)$$

describes the electric field shear or orbit squeezing [34]. For certain values of $\left(v_{\|0}, u_0, S_0\right)$, corresponding to certain values of $\left(\psi_*, E, \mu\right)$, some range of χ is prohibited because the radicand in (32) becomes negative. This defines the trapped part of phase space.

In the usual phase-space coordinates (ψ,χ,ζ,E,μ), $v_{\|}$ is only fixed once $\varsigma = \operatorname{sgn}\left(v_{\|}\right)$ is specified. In contrast, (32) shows that in our new $(\psi_*,\chi,\zeta,E,\mu)$ phase-space variables, $v_{\|}$ (or equivalently $v_{\|} + u$) is only fixed once a different discrete degree of freedom $\sigma = \operatorname{sgn}\left(v_{\|} + u\right)$ is specified. When ψ_* is used as a coordinate, then the periodicity requirement in the trapped part of phase space is that f must be independent of σ at values of χ for which $v_{\|} + u = 0$. This constraint is the periodicity condition we need to annihilate the $g^{(1)}$ term in (29) for trapped ions. We thus introduce a new transit average operation, defined for any quantity Y by

$$\overline{Y} = \frac{\oint d\chi \, Y\left(v_{\|}+u\right)^{-1}\left(\mathbf{b}\cdot\nabla\chi\right)^{-1}}{\oint d\chi\left(v_{\|}+u\right)^{-1}\left(\mathbf{b}\cdot\nabla\chi\right)^{-1}}. \quad (34)$$

For regions of $(E,\ \mu\ \ \psi_*)$-space corresponding to passing particles, $\oint(\cdot)d\chi$ indicates $\int_0^{2\pi/M}(\cdot)d\chi$. For regions corresponding to trapped particles, $\oint(\cdot)d\chi$ means $\sum_\sigma \sigma\int_{\chi\mathrm{min}}^{\chi\mathrm{max}}(\cdot)d\chi$. It is important to notice that since ψ_* rather than ψ_h was taken as an independent variable in the kinetic equation (27), the integrations in the transit average hold ψ_* rather than ψ_h fixed. Applying the new transit average to (29) gives

$$\overline{C_{\mathrm{ii,l}}\left\{g^{(0)}-G\right\}} = 0. \quad (35)$$

As in the standard banana-regime analysis, the $g^{(0)}$ obtained from this equation can be used to find the radial heat flux and parallel flow. We henceforth drop the superscript on $g^{(0)}$ to simplify notation.

We consider a large-aspect-ratio model for the magnetic field well by taking $h(\chi)=1-2\varepsilon\sin^2(\chi/2)$ with $\varepsilon \ll 1$. (We are free to shift the coordinate χ such that $\chi=0$ aligns with B_0.) As stated earlier, in a stellarator, ε does not necessarily equal the geometric inverse aspect ratio, unlike the tokamak case. Following [15], $v_{\|}+u$ can vanish (i.e. a particle is trapped) only if $v_{\|0}+u_0 \sim \sqrt{\varepsilon}v_\mathrm{i}$. Thus, the trapped-passing boundary is shifted to $v_{\|}\approx -u$, that is, away from the center of the leading-order Maxwellian. We define "trapped and barely passing" ions to be those with $v_{\|0}+u_0 \sim v_{\|}+u \sim \sqrt{\varepsilon}v_\mathrm{i}$. These particles are found to have orbits of width $\sim\sqrt{\varepsilon}\rho/\alpha$. Most ions instead have $v_{\|0}+u_0 \sim v_{\|}+u \sim v_\mathrm{i}$. These "freely passing" ions have orbit widths $\sim \varepsilon\rho/\alpha$.

Recall from the discussion following (17) that $\mathbf{b}\cdot\nabla\chi = (M-Nq)B/(qI+K)$. Keeping the variation of B with χ in this definition would only give a $O(\varepsilon)$ correction to the new transit average (34), so we treat $\mathbf{b}\cdot\nabla\chi$ as constant in (34) for analytical calculations.

5.3. Model collision operator

In the conventional banana-regime analysis, the collision operator is replaced with the pitch angle scattering operator

$$C_{\mathrm{pas}} = \frac{v_\perp h v_{\|}}{2w}\frac{\partial}{\partial\lambda}v_{\|}\lambda\frac{\partial}{\partial\lambda} \quad (36)$$

where $w=v^2/2$, $\lambda = 2\mu B_0/v^2$, $v_\perp = v_\mathrm{B}3\sqrt{2\pi}\left[\mathrm{erf}(x)-\Psi(x)\right]\left(2x^3\right)^{-1}$, $x=v\sqrt{m/2T}$, $v_\mathrm{B}=4\sqrt{\pi}Z^4e^4n_\mathrm{i}\ln\Lambda_\mathrm{C}/\left(3\sqrt{m}T^{3/2}\right)$ is the Braginskii ion-ion collision frequency, $\ln\Lambda_\mathrm{C}$ is the Coulomb logarithm, $\mathrm{erf}(x)=\pi^{-1/2}2\int_0^x e^{-y^2}dy$ is the error function, and $\Psi(x)=\left(2x^2\right)^{-1}\left[\mathrm{erf}(x)-x\,\mathrm{erf}'(x)\right]$. Use of the model operator C_{pas} is justified by noting that for $1\varepsilon \ll$, the distribution function obtained using C_{pas} has a large ($O\left(\varepsilon^{-1}\right)$) λ derivative [35]. Since C_{pas} can be obtained by keeping only $\partial / \partial\lambda$ derivatives in the operator for collisions with a Maxwellian field (as we will show shortly), it is plausible that C_{pas} yields accurate results. The operator C_{pas} does not generally satisfy the momentum conservation property

$$\int d^3v\, v_{\|} C = 0 \quad (37)$$

that is satisfied by both the full Fokker-Planck ion-ion collision operator and the linearization thereof. However, (37) becomes true for a particular choice of the constant z in G, and so in conventional neoclassical calculations, z is selected to be this value [21].

We now review the reasoning used by Kagan and Catto to motivate the model ion-ion collision operator used in [13]. We seek an operator with several properties. First, the operator should give the same ion heat flux, flow, and bootstrap current as C_{pas} in the $E_\psi \to 0$ limit. Second, we will want to exchange the order of derivatives in the collision operator with the transit average integral in (35). To do so, the collision operator derivatives must be of the form $\partial / \partial X$ for some $X(\psi_*, E\ \mu)$ (independent of χ), holding other combinations of $(\chi, \psi_*\ E, \mu)$ fixed. Lastly, the operator should keep only velocity derivatives in a direction approximately normal to the modified trapped-passing boundary described by (32) and the discussion following it.

Here we consider only the case $S_0 = 1$, i.e. no electric field shear, $\Phi'' = 0$. The $S_0 \neq 1$ case is analyzed in [13,14,16]. By restricting our attention to the $\Phi'' = 0$ case, several expressions in the following discussion become much simplified. Also, Kagan and Catto [13,14,16] showed the most dramatic effect of E_ψ enters through the magnitude of E_ψ rather than through its derivative: $'\Phi'$ only affects the ion heat flux through an overall algebraic multiplier $\sqrt{S_0}$, and Φ'' does not affect the ion flow or bootstrap current at all.

The model collision operator is then derived from the linearized Fokker-Planck operator. The implicit field term dramatically complicates the analysis, so it is neglected [35]. The explicit test-particle term then gives the standard Rosenbluth potential for collisions with a Maxwellian. The resulting operator can then be written

$$C_{\text{M}}\left\{\hat{f}\right\} = \nabla_{\mathbf{v}} \cdot \left[f_{\text{M}} \overleftrightarrow{\mathbf{Q}} \cdot \nabla_{\mathbf{v}} \left(\hat{f} / f_{\text{M}} \right) \right] \quad (38)$$

where

$$\overleftrightarrow{\mathbf{Q}} = \left(\overleftrightarrow{\mathbf{I}} v^2 - \mathbf{vv} \right) \frac{\nu_\perp}{4} + \mathbf{vv} \frac{\nu_{\|}}{2} \quad (39)$$

and $\nu_{\|} = \nu_{\text{B}} 3\sqrt{2\pi}\,\Psi(x)\left(2x^3\right)^{-1}$.

We next cast (38) into a new set of velocity-space variables. The choice of variables is unusual outside of [13], so we motivate it with the following argument. Suppose we could find new variables W and Λ such that

$$v_{\|} + u = \pm\sqrt{2W}\sqrt{1 - \Lambda / h} \quad (40)$$

so as to closely resemble the expression

$$v_{\|} = \pm\sqrt{2w}\sqrt{1-\lambda/h} \quad (41)$$

which is used often in the conventional calculations, but with the same $v_{\|} \to v_{\|} + u$ replacement we have needed to make in the transit average. The parallelism between (40)-(41) will allow the finite-E_ψ calculations to then be done in much the same way as the conventional calculations, and allow the finite-E_ψ results to continuously reduce to the standard ones. Also, the shifted trapped-passing boundary will then be the curve $\Lambda = \min(h)$, just as the trapped-passing boundary in the $E_\psi \to 0$ case is the curve $\lambda = \min(h)$. Thus, keeping only $\partial/\partial\Lambda$ derivatives in the collision operator will capture the dominant velocity-space behavior for the finite-E_ψ regime, just as $\partial/\partial\lambda$ derivatives do in the $E_\psi \to 0$ case.

To construct the W and Λ variables, we rearrange (32) for $S_0 = 1$ to obtain

$$v_{\|} + u = \pm\sqrt{\left(v_{\|0} + u_0\right)^2 - \left(1-h^2\right)u_0^2 - \left(h^{-1}-1\right)2\mu B_0} \,. \quad (42)$$

We have used the result $hu_0 = u$ since $S_0 = 1$. Note that $v_{\|0}$, u_0, and μB_0 are all constants of the motion and/or adiabatic invariants. The χ dependence (i.e. the h dependence) in (40) is fundamentally different from that in (42), so there is no way to define a W and Λ to make (40) true exactly. However, to leading order in ε, $1-h^2 = -2(1-1/h) + O(\varepsilon^2)$ for $1/h = 1 + O(\varepsilon)$. Therefore (42) can be written

$$v_{\|} + u = \pm\sqrt{2}\sqrt{\frac{\left(v_{\|0} + u_0\right)^2}{2} + \left(\mu B_0 + u_0^2\right)\left(1 - \frac{1}{h}\right) + O\left(\varepsilon^2 v_{\mathrm{i}}^2\right)} \,. \quad (43)$$

In this form, it can be seen that to achieve the desired form (40), there is only one possible way to define W and Λ:

$$W = \frac{\left(v_{\|0} + u_0\right)^2}{2} + \mu B_0 + u_0^2 \,, \quad (44)$$

$$\Lambda = \frac{\mu B_0 + u_0^2}{W} \,. \quad (45)$$

Notice that as $E_\psi \to 0$, Λ reduces to λ, and $W \to w$ since then $v_{\|0}^2 + 2\mu B_0$ is conserved. We will use W and Λ along with gyrophase φ as the velocity space variables in most of the remaining calculation.

We will need to relate W and Λ to $v_{\|}$ and $v_\perp$, and therefore we must relate $v_{\|0}$ to $v_{\|}$ and $v_\perp$. To do so we combine (42) and (44) to obtain

$$2W = \left(v_{\|} + u\right)^2 + \left(3 - h^2\right)u_0^2 + v_\perp^2 \,. \quad (46)$$

Using (45) we then find

$$(1 - \Lambda/h)2W = \left(v_{\|} + u\right)^2 + \left(3 - h^2 - 2/h\right)u_0^2 \,. \quad (47)$$

Thus, instead of (40), the exact relationship is

$$v_{\|} + u = \pm\sqrt{(1-\Lambda/h)2W - \left(3-h^2-2/h\right)u_0^2} \,. \; (48)$$

For $\varepsilon \ll$, the $O(\varepsilon)$ terms in $3-h^2-2/h$ cancel, and so (40) is obtained within on overall $1+O(\varepsilon)$ multiplicative factor. A particle is trapped if and only if $v_{\|}+u$ can vanish, meaning the right hand side of (40) vanishes as h varies while Λ and W are fixed (since Λ and W are constants of the motion.) Therefore, to a very good approximation, a particle is trapped if and only if $\Lambda > \min(h)$.

Another useful property of the variables Λ and W is found by applying a Λ derivative to (47):

$$\left(\frac{\partial\left(v_{\|}+u\right)}{\partial\Lambda}\right)_{W,\chi} = \left(\frac{\partial v_{\|}}{\partial\Lambda}\right)_{W,\chi} = -\frac{W}{\left(v_{\|}+u\right)h} \,. \; (49)$$

This property is reminiscent of the result $\left(\partial v_{\|}/\partial\lambda\right)_w = -w/\left(v_{\|}h\right)$ which is used extensively in conventional neoclassical calculations. The equalities (49) are true regardless of whether ψ_{h} or ψ_* is held fixed in the partial derivatives, since $S_0 = 1$ and χ is fixed so u is constant.

Now consider the result of applying a velocity gradient to (46),

$$\nabla_v W = \left(v_{\|}+u\right)\mathbf{b} + \mathbf{v}_{\perp} \,. \; (50)$$

Applying a velocity gradient to (45) and using (50) we find

$$\nabla_v \Lambda = -\frac{\left(v_{\|}+u\right)\Lambda}{W}\mathbf{b} + \frac{(1-\Lambda/h)h}{W}\mathbf{v}_{\perp} \,. \; (51)$$

Then using $\nabla_v \varphi = v_{\perp}^{-2}\mathbf{b}\times\mathbf{v}$, we obtain the Jacobian

$$\mathcal{J} = \frac{1}{\nabla_v W \times \nabla_v \Lambda \cdot \nabla_v \varphi} = \frac{W}{\left(v_{\|}+u\right)h} \,. \; (52)$$

This expression closely resembles the Jacobian for the conventional variables

$$\frac{1}{\nabla_v w \times \nabla_v \lambda \cdot \nabla_v \varphi} = \frac{w}{v_{\|}h} \; (53)$$

with the same $v_{\|} \to v_{\|}+u$ replacement seen in (49).

Note that in contrast to [13], the small-ε approximation has not been used at all here to derive (49)-(52) (aside from motivating the definitions (44)-(45)).

For trapped and barely passing particles, the right hand side of (47) is $O\left(\varepsilon v_{\mathrm{i}}^2\right)$. Therefore $1-\Lambda/h$ must be $O(\varepsilon)$ for these particles. In light of (50) and (51), then $\left|\nabla_{\mathbf{v}} W\right| \sim v_{\mathrm{i}}$, $\left|\nabla_{\mathbf{v}}\Lambda\right| \sim \sqrt{\varepsilon}/v_{\mathrm{i}}$, and

$$\frac{\left|\nabla_{\mathbf{v}}\Lambda \cdot \nabla_{\mathbf{v}} W\right|}{\left|\nabla_{\mathbf{v}}\Lambda\right|\left|\nabla_{\mathbf{v}} W\right|} \sim \sqrt{\varepsilon} \,. \; (54)$$

Therefore, in the trapped and barely passing region of velocity space, the Λ and W coordinates are nearly orthogonal. Thus, $\left(\partial/\partial\Lambda\right)_W$ will act roughly normal to the shifted trapped-passing boundary, as desired.

To perform integrals later on, we will need to know the upper and lower bounds of W and Λ at given u_0 and χ. From (46), W can be arbitrarily large, and the lower bound is $\left(3-h^2\right)u_0^2/2 \approx u_0^2\left[1+O(\varepsilon)\right]$. To find the bounds on Λ, we can combine (46) and (47) to write

$$\Lambda = \frac{hv_\perp^2 + 2u_0^2}{\left(v_\| + u\right)^2 + \left(3-h^2\right)u_0^2 + v_\perp^2}. \quad (55)$$

It follows that at given u_0 and χ, the minimum of Λ is exactly 0 (which occurs when $v_\| \to \pm\infty$) and the maximum allowed Λ is precisely $2/\left(3-h^2\right)$ (which occurs when $v_\| = -u$ and $v_\perp = 0$.) For $\varepsilon \ll 1$, this upper bound equals $h + O\left(\varepsilon^2\right)$.

Next, we use the general formula for the divergence in an arbitrary coordinate system to write (38) as

$$C_\mathrm{M}\left\{\hat{f}\right\} = \frac{1}{\mathcal{J}}\sum_{X,Y}\frac{\partial}{\partial X}\left[f_\mathrm{M}\mathcal{J}\left(\nabla_\mathbf{v}X\right)\cdot\overleftrightarrow{\mathbf{Q}}\cdot\left(\nabla_\mathbf{v}Y\right)\frac{\partial}{\partial Y}\left(\frac{\hat{f}}{f_\mathrm{M}}\right)\right] \quad (56)$$

where X and Y range over the set $\{\Lambda, W \ \varphi\}$. The partial derivatives in (56) hold fixed the remaining elements of this set, along with ψ and χ. Recall that the collision operator appearing in the drift-kinetic equation (and therefore in (35)) has been gyroaveraged. If we gyroaverage (56), the $X=\varphi$ terms vanish since the quantity in square brackets is periodic in φ. Then $\nabla_\mathbf{v}W\cdot\overleftrightarrow{\mathbf{Q}}\cdot\nabla_\mathbf{v}\varphi = 0$ from (50) and $\nabla_\mathbf{v}\Lambda\cdot\overleftrightarrow{\mathbf{Q}}\cdot\nabla_\mathbf{v}\varphi=0$ from (51), so the gyroaveraged $C_\mathrm{M}\left\{\hat{f}\right\}$ is given by the right hand side of (56) with $X,\ Y\in\{\Lambda \ \ W\}$.

In analogy to the weak-E_ψ case, we now drop the $\partial/\partial W$ derivatives in (56) in order to obtain a tractable model operator. For $u=0$, the result of this simplification is precisely C_pas as defined in (36). For the general $u\neq 0$ case, the distribution function we obtain using our final model operator has a large Λ derivative, making it plausible that discarding the $\partial/\partial W$ derivatives will not dramatically affect the calculations for $\varepsilon \ll 1$.

To evaluate (56) we must compute $\left(\nabla_\mathbf{v}\Lambda\right)\cdot\overleftrightarrow{\mathbf{Q}}\cdot\left(\nabla_\mathbf{v}\Lambda\right)$. The algebra becomes intractable unless we use $1-\Lambda/h\sim O(\varepsilon)$ and $\left(v_\|+u\right)\sim O\left(\sqrt{\varepsilon}v_\mathrm{i}\right)$ to discard terms which are small for trapped and barely passing particles. We thereby neglect the $\mathbf{v}_\perp$ term in (51) to obtain

$$\left(\nabla_\mathbf{v}\Lambda\right)\cdot\overleftrightarrow{\mathbf{Q}}\cdot\left(\nabla_\mathbf{v}\Lambda\right) = \left(v_\|+u\right)^2\frac{\Lambda^2}{W^2}\left\{\frac{\nu_\perp}{4}v_\perp^2 + \frac{\nu_\|}{2}v_\|^2 + O\left(\sqrt{\varepsilon}\nu v_\mathrm{i}^2\right)\right\}. \quad (57)$$

We approximate $v_\|^2 \approx u_0^2$, and using (44), we approximate $v_\perp^2 \approx 2\left(W-u_0^2\right)$. Our model operator becomes

$$C\left\{\hat{f}\right\} = \frac{\left(v_\|+u\right)}{2W^2}\left(\frac{\partial}{\partial\Lambda}\right)_{W,\psi}\left[\left(v_\|+u\right)\Lambda f_\mathrm{M}\left[W\nu_\perp + u_0^2\left(\nu_\|-\nu_\perp\right)\right]\left(\frac{\partial}{\partial\Lambda}\right)_{W,\psi}\left(\frac{\hat{f}}{f_\mathrm{M}}\right)\right]. \quad (58)$$

Notice that (58) has a similar form to C_pas (in (36)). (To obtain (58) we have made the replacement $\Lambda^2 \to \Lambda$, which is permissible since $\Lambda \approx 1$. Kagan and Catto make a different replacement $\Lambda^2 \to 1$ at this point in deriving the model operator of [13]. All results will be independent of the exponent on Λ because the identity (76) is independent of this exponent.)

Where v appears inside $\nu_\perp$, $\nu_{||}$, and $f_{\rm M}$ in the operator, we make the approximation

$$v \approx \sqrt{2W - u^2} \; . \quad (59)$$

The quantities $\nu_\perp$, $\nu_{||}$, and $f_{\rm M}$ are then all constant with respect to the Λ derivative. We now apply the chain rule, so as to hold ψ_* rather than $\psi_{\rm h}$ fixed in the partial derivatives. For any quantity ξ,

$$\left(\frac{\partial \xi}{\partial \Lambda}\right)_{\psi,\chi,W} = \left(\frac{\partial \xi}{\partial \Lambda}\right)_{\psi_*,\chi,W} + \left(\frac{\partial \psi_*}{\partial \Lambda}\right)_{\psi,\chi,W}\left(\frac{\partial \xi}{\partial \psi_*}\right)_{\Lambda,\chi,W} . \quad (60)$$

To obtain a tractable model collision operator, the last term is dropped for both of the partial derivatives in (58). Then defining

$$\nu_{\rm K} = \nu_\perp + \left(\nu_{||} - \nu_\perp\right) u_0^2 / W \quad (61)$$

we have

$$C = \frac{\left(v_{||}+u\right)}{2W} f_{\rm M} \nu_{\rm K} \left(\frac{\partial}{\partial \Lambda}\right)_{W,\psi_*}\left[\left(v_{||}+u\right)\Lambda\left(\frac{\partial}{\partial \Lambda}\right)_{W,\psi_*}\left(\frac{\hat{f}}{f_{\rm M}}\right)\right] . \quad (62)$$

We may now plug in $\hat{f} \to g - G$ from (35). In G we use (59) and $\Omega \approx \Omega_0$. Thus

$$C_{\rm K} = \frac{\left(v_{||}+u\right)}{2W} f_{\rm M} \nu_{\rm K} \left(\frac{\partial}{\partial \Lambda}\right)_{W,\psi_*}\left[\left(v_{||}+u\right)\Lambda\left(\frac{\partial}{\partial \Lambda}\right)_{W,\psi_*}\left(\frac{g}{f_{\rm M}} - \frac{\left(v_{||}+u\right) I_{\rm h} T'}{\Omega_0 T}\left[\frac{mW}{T} - z\right]\right)\right] . \quad (63)$$

This operator is the one employed by Kagan and Catto [13] with $I \to I_{\rm h}$ and $\psi_{\rm p} \to \psi_{\rm h}$. Since T, T', $I_{\rm h}$, and Ω_0 do not vary significantly over an orbit width, these quantities are all treated as constant with respect to derivatives and integrals at constant ψ_*.

We have demonstrates that many expressions for the new W, Λ variables are identical to the conventional results in the w, λ variables, but with the replacement $v_{||} \to v_{||} + u$. This pattern can be seen in the form of the model operator, the derivative (49), and the Jacobian (52). Due to the correspondence between the new expressions and the conventional ones, the steps used to calculate neoclassical quantities with the new collision operator will mirror steps in the conventional calculations. However, the replacement $\nu_\perp(w) \to \nu_{\rm K}(W)$ in the new collision operator is a significant change, for now energy diffusion as well as pitch-angle scattering is retained. This change to the effective collision frequency will cause finite- E_ψ modifications to the ion heat flux, ion flow, and bootstrap current.

5.4. Banana constraint

We must now find the g piece of the distribution function by solving (35). First consider the trapped particles, for which this equation becomes

$$0=\sum_{\sigma}\sigma\int_{\chi_{\min}}^{\chi_{\max}}d\chi\frac{1}{\mathbf{b}\cdot\nabla\chi}\frac{\partial}{\partial\Lambda}\left\{\left(v_{\|}+u\right)\Lambda\frac{\partial}{\partial\Lambda}\left[\frac{g}{f_{\mathrm{M}}}-\left(v_{\|}+u\right)\left(\frac{mW}{T}-z\right)\frac{I_{\mathrm{h}}T'}{\Omega_0 T}\right]\right\}. \quad (64)$$

The T' drive term vanishes due to the σ sum. Therefore $g=0$ is a solution for trapped particles, as in the standard banana-regime calculation.

Next we consider passing particles, for which (35) becomes

$$0=\int_{0}^{2\pi M}d\chi\frac{\partial}{\partial\Lambda}\left\{\left(v_{\|}+u\right)\Lambda\frac{\partial}{\partial\Lambda}\left[\frac{g}{f_{\mathrm{M}}}-\left(v_{\|}+u\right)\left(\frac{mW}{T}-z\right)\frac{I_{\mathrm{h}}T'}{\Omega_0 T}\right]\right\}. \quad (65)$$

It is permissible to switch the order of the integral and the first $\partial/\partial\Lambda$ derivative because we have constructed Λ and W to be functions of $\left(\psi_*, \mu\ E\right)$, We integrate in Λ from $\Lambda=0$ and apply (49) to find

$$\frac{\partial}{\partial\Lambda}\left(\frac{g}{f_{\mathrm{M}}}\right)=-\frac{HW}{\left\langle v_{\|}+u\right\rangle_*}\left(\frac{mW}{T}-z\right)\frac{I_{\mathrm{h}}T'}{\Omega_0 T} \quad (66)$$

where

$$\left\langle\xi\right\rangle_*=\frac{M}{2\pi}\int_{0}^{2\pi M}\xi\ d\chi\Big|_{\psi_*,\mu,E} \quad (67)$$

and $H=H\left(h_{\min}-\Lambda\right)$ is a Heavyside step function which is 1 for passing particles and 0 for trapped particles.

5.5. Momentum conservation

We choose the parameter z by requiring

$$\int d^3v\left(v_{\|}+u\right)C_{\mathrm{K}}=0, \quad (68)$$

a combination of the particle and momentum conservation properties of our ion-ion collision operator. Using a parity argument as in appendix D, it can be shown that number conservation ($\int d^3v\ C_{\mathrm{K}}=0$) and energy conservation ($\int d^3v\ v^2 C_{\mathrm{K}}=0$) are both satisfied to leading order regardless of z.

To evaluate velocity integrals such as (68) we need to write d^3v in $\left(W,\Lambda\right)$ variables. Notice from (48) that for given W, Λ, and χ, there are two allowed values for $v_{\|}+u$. Therefore at given χ, equations (48) and (46) give a 1-to-1 map between $\left(W, \Lambda\ \varphi, \sigma\right)$ and $\mathbf{v}$. The proper way to integrate a quantity ξ over velocity space in our new variables is therefore

$$\begin{aligned}\int d^3v\ X &= \sum_{\sigma}\int dW\int d\Lambda\int d\varphi\ \left|\mathcal{J}\right|\xi \\ &= \frac{1}{h}\sum_{\sigma}\sigma\int dW\int d\Lambda\int d\varphi\frac{W\xi}{\left(v_{\|}+u\right)}\end{aligned} \quad (69)$$

with the Jacobian $\mathcal{J}$ given by (52).

Combining (68)-(69) with our model operator (63) and the distribution function (66), we therefore require

$$0=\int dW\, f_{\mathrm{M}} W^{3/2} \nu_{\mathrm{K}} \left(\frac{mW}{T}-z\right) \int d\Lambda \sqrt{1-\Lambda/h}\, \frac{\partial}{\partial\Lambda}\left[\Lambda-\frac{H\Lambda\sqrt{1-\Lambda/h}}{\left\langle\sqrt{1-\Lambda/h}\right\rangle_*}\right] \quad (70)$$

The Λ integral is independent of W so we divide it out of the equation. We then change variables from W to $y=\left(W-u_0^2\right)m/T$. From the earlier discussion of the lower bound on W, the lower bound on y is $O(\varepsilon)$ so effectively zero. Therefore

$$z=\frac{\int_0^\infty dy\, e^{-y}\left(y+U^2\right)^{3/2}\left(y\nu_\perp+2U^2\nu_\parallel\right)}{\int_0^\infty dy\, e^{-y}\sqrt{y+U^2}\left(y\nu_\perp+2U^2\nu_\parallel\right)} \quad (71)$$

where we have slightly redefined U to be the flux function $cI_{\mathrm{h}}\Phi'/\left(v_{\mathrm{i}}B_0\right)$, a definition which differs from the original one (23) only by a factor of $h\approx 1$. Using $x=\sqrt{y+U^2}$ in the definitions of $\nu_\perp$ and $\nu_\parallel$, we can now evaluate z for any given U. For $U=0$, (71) gives $z=1.33$, in agreement with conventional neoclassical theory.

5.6. Neoclassical ion heat flux

In appendix B we derive the following equation to relate the radial ion heat flux to an integral of the collision operator:

$$\left\langle \mathbf{q}\cdot\nabla\psi_{\mathrm{h}}\right\rangle=-\frac{I_{\mathrm{h}}}{\Omega_0}\left\langle h\int d^3v\, v_\parallel \frac{mv^2}{2}C\right\rangle. \quad (72)$$

Although this equation was derived directly from the full Fokker-Planck equation using only the quasisymmetry condition $B=B\left(\psi_{\mathrm{h}},\chi\right)$, the same equation would result if the isomorphism substitutions (11) were naively applied to the analogous equation for tokamaks. Using the number, momentum, and energy conservation properties of the collision operator, as well as (59), then (72) is equivalent to

$$\left\langle \mathbf{q}\cdot\nabla\psi_{\mathrm{h}}\right\rangle=-\frac{mI_{\mathrm{h}}}{\Omega_0}\left\langle h\int d^3v\left(v_\parallel+u\right)WC\right\rangle. \quad (73)$$

Substituting in the collision operator and distribution function, we then have

$$\begin{aligned}\left\langle \mathbf{q}\cdot\nabla\psi_{\mathrm{h}}\right\rangle=&-\frac{2\pi m I_{\mathrm{h}}^2 T'}{\Omega_0^2 T}\int dW\, W^2 f_{\mathrm{M}}\nu_{\mathrm{K}}\left(\frac{mW}{T}-z\right)\\ &\times\left\langle\int_0^h d\Lambda\left(v_\parallel+u\right)\frac{\partial}{\partial\Lambda}\left[\Lambda-\frac{\left(v_\parallel+u\right)H\Lambda}{\left\langle v_\parallel+u\right\rangle_*}\right]\right\rangle.\end{aligned} \quad (74)$$

We next integrate by parts in Λ, noting there is no contribution from the boundary. Applying (40) then results in

$$\langle \mathbf{q}\cdot\nabla\psi_{\rm h}\rangle = -\frac{\sqrt{2}\pi m I_{\rm h}^2 T'}{\Omega_0^2 T}\int dW\, W^{5/2} f_{\rm M}\nu_{\rm K}\left(\frac{mW}{T}-z\right)$$
$$\times\left\langle \int_0^h d\Lambda\,\Lambda\left(\frac{1}{\sqrt{1-\Lambda/h}}-\frac{H}{\left\langle\sqrt{1-\Lambda/h}\right\rangle_*}\right)\right\rangle. \quad (75)$$

The Λ integral can then be performed using the method in appendix B of [35]. In general,

$$\int_0^h d\Lambda\,\Lambda^{\gamma}\left(\frac{1}{\sqrt{1-\Lambda/h}}-\frac{H}{\left\langle\sqrt{1-\Lambda/h}\right\rangle_*}\right) = 95\sqrt{\varepsilon}+O(\varepsilon) \quad (76)$$

for any $\gamma > -1$ where

$$1.95 = 2\sqrt{2}\left\{1+\int_0^1\frac{d\kappa}{\kappa^2}\left[1-\frac{\pi}{2E\left(\kappa^2\right)}\right]\right\} \quad (77)$$

and $E\left(\kappa^2\right)$ is the complete elliptic integral of the second kind. Again changing to the variable $y=\left(W-u_0^2\right)m/T$, then

$$\langle \mathbf{q}\cdot\nabla\psi_{\rm h}\rangle = -1.95\sqrt{\varepsilon}\,\frac{nI_{\rm h}^2 TT'}{2\sqrt{\pi}m\Omega_0^2}e^{-U^2}$$
$$\times\int_0^\infty dy\, e^{-y}\left(y+U^2\right)^{3/2}\left(y+U^2-z\right)\left(y\nu_\perp+2U^2\nu_\parallel\right). \quad (78)$$

Plugging in the collision frequencies,

$$\langle \mathbf{q}\cdot\nabla\psi_{\rm h}\rangle = -1.35\sqrt{\varepsilon}\,\frac{\nu_{\rm B} nI_{\rm h}^2 TT'}{m\Omega_0^2}Q(U) \quad (79)$$

where

$$Q(U) = 1.53e^{-U^2}\int_0^\infty dy\, e^{-y}\left(y+U^2\right)^{3/2}\left(y+U^2\right)^{-3/2}\left(y+2U^2-z\right)$$
$$\times\left[y\,\mathrm{erf}\left(\sqrt{y+U^2}\right)+\left(2U^2-y\right)\Psi\left(\sqrt{y+U^2}\right)\right]. \quad (80)$$

This function is plotted in figure 1a. At $U=0$, $Q=1$ and (79) recovers the conventional heat flux. Multiplying the right-hand side of (79) by $\sqrt{S}$ accounts for orbit squeezing effects [13], where

$$S(\psi_{\rm h}) = 1+\frac{cI_{\rm h}^2\Phi''}{B_0\Omega_0}. \quad (81)$$

5.7. Ion Flow

The parallel flow is obtained by forming the integral

$$V_\parallel = \int d^3v\, v_\parallel f \approx \int d^3v\, v_\parallel\left(F_1+g\right)$$
$$= -\frac{pI_{\rm h}}{m\Omega}\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}\right)+\int d^3v\, v_\parallel g. \quad (82)$$

We then write the remaining integral as

$$\int d^3v\, v_{\|} g = \int d^3v\, v_{\|} G + \int d^3v \left(v_{\|} + u\right)(g - G) - u \int d^3v (g - G). \quad (83)$$

Using (28), the first integral on the right-hand side gives

$$\int d^3v\, v_{\|} G = \left(\frac{5}{2} + U^2 - z\right) \frac{p I_{\mathrm{h}} T'}{m \Omega T}. \quad (84)$$

The second integral on the right-hand side of (83) can be evaluated in the same manner as the integral (75) for the heat flux, and the result is $\sqrt{\varepsilon}$ smaller than (84). Similarly, the last integral in (83) is also $\sqrt{\varepsilon}$ smaller than (84). This integral is discussed further in appendix D. Thus, the final integral in (82) is approximately given by the right-hand side of (84). Defining

$$A(U) = \frac{1}{1.17}\left(\frac{5}{2} + U^2 - z\right) \quad (85)$$

then the parallel flow can be written

$$V_{\|} \approx -\frac{p I_{\mathrm{h}}}{n m \Omega}\left[\frac{p'}{p} + \frac{Ze\Phi'}{T} - 1.17\, A(U) \frac{T'}{T}\right] \quad (86)$$

The function $A(U)$ is plotted in figure 1b and agrees with the corrected result from [13,14]. Note that $A(0) = 1$, and so (86) recovers the conventional result for $U = 0$.

5.8. Bootstrap current

The bootstrap current calculation for the finite- E_ψ regime in a quasisymmetric stellarator proceeds exactly as for the low-flow regime in a tokamak (e.g. as shown in [21]), but with two modifications. First, the electron kinetic equation is written in $(\psi_{\mathrm{h}},\ \chi\ \ \zeta)$ variables and analyzed as in (12)-(17). Second, the parallel ion velocity (86) is used. This latter change affects the electron-ion collision operator, but otherwise the conventional model electron collision operator is used. After solving for the electron distribution function in the banana regime in the usual way, taking a velocity moment to obtain the parallel current, using the Spitzer function f_s as in [21], and approximating f_s with two Sonine polynomials [19], the bootstrap current for arbitrary Z is found to be

$$\begin{aligned} j_{\|}^{\mathrm{bs}} \approx -1.46\sqrt{\varepsilon}\, \frac{c I_{\mathrm{h}} B}{\left\langle B^2 \right\rangle} \left[\frac{Z^2 + 2.21Z + 0.75}{Z\left(Z + \sqrt{2}\right)}\right] & \\ \times \left[\frac{dp}{d\psi_{\mathrm{h}}} - \frac{(2.07Z + 0.88) n_{\mathrm{e}}}{\left(Z^2 + 2.21Z + 0.75\right)} \frac{dT_{\mathrm{e}}}{d\psi_{\mathrm{h}}} - 1.17\, A(U) \frac{n_{\mathrm{e}}}{Z} \frac{dT_{\mathrm{i}}}{d\psi_{\mathrm{h}}}\right] & \quad (87) \end{aligned}$$

where $p = p_{\mathrm{e}} + p_{\mathrm{i}}$. For $Z = 1$, (87) becomes

$$j_{\|}^{\mathrm{bs}} \approx -2.42\sqrt{\varepsilon}\, \frac{c I_{\mathrm{h}} B}{\left\langle B^2 \right\rangle} \left[\frac{dp}{d\psi_{\mathrm{h}}} - 0.75 n \frac{dT_{\mathrm{e}}}{d\psi_{\mathrm{h}}} - 1.17\, A(U) n \frac{dT_{\mathrm{i}}}{d\psi_{\mathrm{h}}}\right]. \quad (88)$$

As expected, these expressions can be found from the large- E_ψ tokamak results [16] if the isomorphism substitutions are made. Also, setting $A(U)=1$ in (88) we recover equation (37) from [1], Boozer's low-flow-regime result for a quasisymmetric stellarator.

6. Residual zonal flow

We now briefly discuss the residual zonal flow in a quasisymmetric stellarator. Much of the analysis is identical to the tokamak analysis in [15] if the isomorphism substitutions (11) are applied, so here we merely summarize the framework and results of the calculation.

We assume the potential Φ can be decomposed into an equilibrium component $\phi(\psi)$, which is constant in time on the timescale of interest, and a perturbation $\delta\phi(\psi,\, t)$. We further assume that $|\nabla\phi| \gg |\nabla\delta\phi|$, so the electric field used to calculate u and U in (21) and (23) is only $-\nabla\phi$. Unlike the neoclassical transport calculation, here we allow $\Phi'' \neq 0$ so orbit squeezing effects are included.

We again use the kinetic equation (19) in which ψ_* is used as an independent variable. In contrast to the neoclassical transport analysis, collisions are dropped but time variation is kept. Note that in (19), the $\partial\Phi/\partial t = \partial\delta\phi/\partial t$ derivative is performed at constant ψ_h, not ψ_*. A change of variables can be made using

$$\begin{aligned}\left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_\mathrm{h}} &= \left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_*} + \left(\frac{\partial\psi_*}{\partial t}\right)_{\psi_\mathrm{h}}\left(\frac{\partial\delta\phi}{\partial\psi_*}\right) \\ &= \left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_*} + \frac{cI_\mathrm{h}}{v_\|B}\left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_\mathrm{h}}\left(\frac{\partial\delta\phi}{\partial\psi_\mathrm{h}}\right)\left(\frac{\partial\psi_\mathrm{h}}{\partial\psi_*}\right).\end{aligned} \quad (89)$$

The last term can be evaluated using the remarkable identity

$$\frac{\partial\psi_*}{\partial\psi_\mathrm{h}} = \frac{\left(v_\|\mathbf{b}+\mathbf{v}_\mathrm{d}\right)\cdot\nabla\chi}{v_\|\mathbf{b}\cdot\nabla\chi}, \quad (90)$$

which can be shown with a few lines of algebra. Therefore $\partial\psi_\mathrm{h}/\partial\psi_* \approx v_\|/\left(v_\|+u\right)$. We have already assumed $\partial\delta\phi/\partial\psi_\mathrm{h} \ll \partial\Phi/\partial\psi_\mathrm{h}$, and so the last term in (89) is small compared to the left-hand side. Thus, $\left(\partial\Phi/\partial t\right)_{\psi_\mathrm{h}} \approx \left(\partial\delta\phi/\partial t\right)_{\psi_*}$ to leading order, giving the kinetic equation

$$\left(\frac{\partial f}{\partial t}\right)_{\psi_*} + \left(v_\|+u\right)\left(\mathbf{b}\cdot\nabla\chi\right)\left(\frac{\partial f}{\partial\chi}\right)_{\psi_*} + \frac{Ze}{m}\left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_*}\left(\frac{\partial f}{\partial E}\right)_{\psi_*} = 0\,. \quad (91)$$

As in the neoclassical transport analysis, we take the distribution function to be a stationary Maxwellian f_M to leading order, and we again define $g = f - F$ with $F(\psi_*,\, E)$ in (25). The kinetic equation (91) is linearized by approximating $\partial\Phi/\partial\psi_\mathrm{h} \approx \partial\phi/\partial\psi_\mathrm{h}$ in the definition of u and by taking $\left(\partial f/\partial E\right)_{\psi_*} \approx \left(\partial F/\partial E\right)_{\psi_*}$, giving

$$\left(\frac{\partial g}{\partial t}\right)_{\psi_*} + \left(v_\|+u\right)\left(\mathbf{b}\cdot\nabla\chi\right)\left(\frac{\partial g}{\partial\chi}\right)_{\psi_*} - \frac{Ze}{T(\psi_*)}\left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_*} = 0\,. \quad (92)$$

We then consider the dynamics of the system over a timescale τ which is long compared to the thermal bounce time $\left(v_{\|}+u\right)^{-1}\left(\mathbf{b}\cdot\nabla\chi\right)^{-1}$. We therefore expand g as a series in the small parameter $\left(v_{\|}+u\right)\left(\mathbf{b}\cdot\nabla\chi\right)/\tau$, writing $g=g^{(0)}+g^{(1)}+\ldots$. The leading order equation gives $\left(\partial g^{(0)}/\partial\chi\right)_{\psi_*}=0$. To next order,

$$\left(\frac{\partial g^{(0)}}{\partial t}\right)_{\psi_*}+\left(v_{\|}+u\right)\left(\mathbf{b}\cdot\nabla\chi\right)\left(\frac{\partial g^{(1)}}{\partial\chi}\right)_{\psi_*}-\frac{Ze}{T(\psi_*)}\left(\frac{\partial\delta\phi}{\partial t}\right)_{\psi_*}=0\,. \quad (93)$$

The $g^{(1)}$ term can be annihilated by applying the transit average (34). Following [15], we assume the potential has an eikonal form $\delta\phi(\psi_{\mathrm{h}},t)=\widehat{\delta\phi}(t)\exp\left[i\xi(\psi_{\mathrm{h}})\right]$ and we Taylor expand $\xi(\psi_{\mathrm{h}})$ about $\psi_{\mathrm{h}0}$. Then integrating the transit-averaged (93) in time from 0^- to any positive *t*, we obtain

$$g^{(0)}=\frac{Ze}{T(\psi_*)}Fe^{-iP}\overline{e^{iP}}\delta\phi \quad (94)$$

where

$$P=\frac{\xi' I_{\mathrm{h}}}{\Omega_0}\left(S_0-1+\frac{1}{h^2}\right)^{-1}\left(v_{\|}+u-\frac{v_{\|0}}{h^2}-u_0\right) \quad (95)$$

and we have taken the system state for $t<0$ to be $g=0$ and $\delta\phi=0$. We then assume that the radial electric field does not modify the weak-E_ψ result [25]

$$\frac{\Phi(t\to\infty)}{\Phi(t=0^+)}=\frac{n\left\langle k_\perp^2\rho^2\right\rangle/2}{n\left\langle k_\perp^2\rho^2\right\rangle/2+\left\langle\int d^3v\,\left[f(t\to\infty)-f_{\mathrm{M}}\right]\right\rangle\;\Phi(t\to\infty)} \quad (96)$$

if we substitute $\Phi\to\delta\phi$ and evaluate f_{M} and n at the unperturbed energy $v^2/2+Ze\phi/m$. Writing $f=F+g$ and Taylor-expanding $F(\psi_*,E)$ in both arguments about f_{M}, we find $\int d^3v[f-f_{\mathrm{M}}]\approx Ze\;\delta\phi\;T(\psi)^{-1}\int d^3v\,f_{\mathrm{M}}\left(e^{-iP}\overline{e^{iP}}-1\right)$ in (96). For $|P|\ll 1$ we then obtain

$$\frac{\delta\phi(t\to\infty)}{\delta\phi(t=0^+)}=\frac{1}{1+\Re} \quad (97)$$

where

$$\Re=\frac{2}{n\left\langle k_\perp^2\rho_{\mathrm{i}}^2\right\rangle}\left\langle\int d^3v\;f_{\mathrm{M}}\left(iP-i\overline{P}+\frac{P^2-2P\overline{P}+\overline{P^2}}{2}\right)\right\rangle. \quad (98)$$

Upon performing the integrations as described in [15], the residual zonal flow can be expressed as

$$\Re=\Re_{\mathrm{RH}}\left[\frac{\Upsilon(U)}{\sqrt{S}}+i\frac{\Xi(U,\,S)}{k_\perp\rho/\alpha}\right] \quad (99)$$

where $\Re_{\mathrm{RH}}=1.64\varepsilon^{3/2}/\alpha^2$ is the Rosenbluth-Hinton (weak-E_ψ) result, $k_\perp=\xi'\alpha I_{\mathrm{h}}$, S is defined in (81), $U=cI_{\mathrm{h}}\Phi'/(v_{\mathrm{i}}B_0)$ as described following (71),

$$\begin{aligned}\Upsilon(U) &= \frac{4e^{-U^2}}{3\sqrt{\pi}}\int_0^\infty e^{-y}\left(y+2U^2\right)^{3/2} dy \\ &= |U| e^{-U^2}\sqrt{\frac{2}{\pi}}\left(2+\frac{8}{3}U^2\right)+e^{U^2}\left[1-\operatorname{erf}\left(\sqrt{2}|U|\right)\right]\end{aligned} \tag{100}$$

gives the real part of $\Re$,

$$\begin{aligned}\Xi(U,\,S) &= \frac{2U}{\sqrt{S}}\left[S\Upsilon(U)+\left(\;-S\right)\frac{4e^{-U^2}}{\sqrt{\pi}}\int_0^\infty e^{-y}\sqrt{y+2U^2}\,dy\right] \\ &= \frac{2U}{\sqrt{S}}\left[S\Upsilon(U)+2(1-S)\left\{2|U|e^{-U^2}\sqrt{\frac{2}{\pi}}+e^{U^2}\left[1-\operatorname{erf}\left(\sqrt{2}|U|\right)\right]\right\}\right]\end{aligned} \tag{101}$$

describes the imaginary part. The functions Υ and Ξ are plotted in [15] (with Ξ denoted there by Λ).

It can be seen as follows that the weak- E_ψ result $\Re \to \Re_{\mathrm{RH}}$ agrees with [27]. Due to (13), the parameter $\bar{\omega}_r$ in [27] vanishes, and so $\Lambda_2 = 0$. Consequently, there are no oscillations of the potential. If we choose $s = \psi_{\mathrm{h}}$, then $G = I_{\mathrm{h}} v_{\|} / \Omega$. By applying the magnetic field model $B = B_0\left[1+2\varepsilon\sin^2\left(\chi/2\right)\right]$, recognizing $\oint d\alpha = 2\pi$ and $\sum_n = M$, and carrying out the integrals, we recover (97) with $\Re = \Re_{\mathrm{RH}}$.

7. Discussion and conclusions

In the preceding sections we have shown how to calculate finite- E_ψ effects in a quasisymmetric stellarator. Kagan and Catto's technique of changing from the radial variable ψ_{p} to the canonical angular momentum Ψ_* in the kinetic equation can be generalized because a similar conserved quantity ψ_* exists in a quasisymmetric field. The conservation of ψ_* allows an analytical treatment of the particle orbits, which is not possible in a more general stellarator field. To define the finite- E_ψ regime in a quasisymmetric field, the geometric factor $\alpha = |\nabla\psi_{\mathrm{h}}| / I_{\mathrm{h}}$ plays the role that B_{p}/B does in a tokamak, so we order $\alpha \ll 1$. We allow strong density and potential variation, $\nabla n \sim n\alpha/\rho$ and $E_\psi \sim B v_{\mathrm{i}}/(\alpha c)$. Present estimates of E_ψ in the HSX stellarator suggest it may indeed be as large as in this ordering, making finite- E_ψ effects important. However, it is not clear whether our small ordering of the density scale-length, or our assumption that flows are much less than v_{i}, are appropriate for that experiment.

Generalizing the tokamak procedures to a quasisymmetric stellarator, we have calculated the finite- E_ψ modifications to the neoclassical ion heat flux, ion flow, bootstrap current, and residual zonal flow. We find these expressions match those which would be obtained by applying Boozer's isomorphism substitutions to the tokamak results. Physically, the isomorphism holds for these finite- E_ψ effects because these processes result from guiding-center drift dynamics and not from additional physics such as the gyromotion. The isomorphism relates the guiding-center drifts but not the gyromotion, which is why neoclassical transport obeys the isomorphism but classical transport does not.

The modifications to neoclassical transport are obtained by generalizing the modified model collision operator proposed in [13]. Our derivation emphasizes that the (W, Λ) variables employed by Kagan and Catto are unique, in that they are the only possible way to generalize the conventional relation $|v_{||}| = \sqrt{2w}\sqrt{1-\lambda/h}$ to the form $|v_{||}+u| = \sqrt{2W}\sqrt{1-\Lambda/h}$. The finite- E_ψ modifications to neoclassical transport are in part due to the replacement of the deflection frequency $\nu_\perp$ in the usual pitch-angle scattering operator by a new frequency ν_{K} in the new collision operator. The frequency ν_{K} accounts for energy scatter across the modified trapped-passing boundary when this boundary is shifted due to E_ψ.

The discussion of finite- E_ψ effects in earlier tokamak references is also applicable to the quasisymmetry case. For example, the trapped-passing boundary shifts from $v_{||} \approx 0$ to $v_{||} \approx -u$, but in the finite- E_ψ ordering it is consistent for the ion flow to be subsonic so the leading-order distribution remains centered at $v_{||} = 0$. The trapped fraction therefore diminishes as $\exp(-U^2)$. Therefore the heat flux becomes exponentially small, and the residual zonal flow $\delta\phi(t \to \infty)/\delta\phi(t=0)$ approaches 1. As noted in [11], this latter effect creates a positive feedback loop. If a weak transport barrier develops, the associated E_ψ would reduce zonal flow damping, strengthening the transport barrier.

The parallel ion flow is mostly carried by passing particles, so for a strong radial electric field ($U \sim 1$) the flow does not become exponentially small, though it is substantially modified. The bootstrap current depends on the ion flow, so it is modified as well. The coefficient of the ion temperature gradient in the parallel ion flow and bootstrap current reverses sign when U exceeds 1.2. Importantly, the bootstrap current grows stronger as E_ψ is increased.

Appendix A. Proof of ψ_* conservation

We now derive the identity (13) and the conservation of ψ_*. Proofs of the latter have been given previously in references including [1], [6], and [4].

Both relations require that the potential, if it varies at all on a flux surface, have the same helicity as B: $\Phi = \Phi(\psi_{\mathrm{h}}, \chi\ t)$. In this case, since $v_{||}^2 = 2(E - \mu B - Ze\Phi/m)$, then $\partial(v_{||}/B)/\partial\zeta = -(N/M)\partial(v_{||}/B)/\partial\theta$. This result, together with the Boozer representations for $\mathbf{B}$ in (5)-(6), gives

$$v_{||}\mathbf{b}\cdot\nabla\left(\frac{I_{\mathrm{h}}v_{||}}{\Omega}\right) = \frac{I_{\mathrm{h}}v_{||}}{\Omega}\left(1-\frac{Nq}{M}\right)\left(\nabla\psi_p\cdot\nabla\theta\times\nabla\zeta\right)\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right). \quad (A.1)$$

Also, using $\mathbf{v}_{\mathrm{d}} = (v_{||}/\Omega)\nabla\times(v_{||}\mathbf{b})$, we can similarly show

$$\mathbf{v}_{\mathrm{d}}\cdot\nabla\psi_{\mathrm{p}} = \frac{I_{\mathrm{h}}v_{||}}{\Omega}\left(\nabla\psi_{\mathrm{p}}\cdot\nabla\theta\times\nabla\zeta\right)\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right). \quad (A.2)$$

The identity (13) immediately follows.

Before completing the proof that $D\psi_* = 0$, we first prove a lemma: B depends on θ and ζ only through the combination $M\theta - N\zeta$ (i.e. the field is quasisymmetric) if and only if L has this same property. We begin by casting the equilibrium condition $(\nabla\times\mathbf{B})\times\mathbf{B}/4\pi = \nabla p$ into Boozer coordinates. Then applying $\nabla\psi_{\mathrm{p}}\cdot\nabla\theta\times\nabla\zeta = B^2/(qI+K)$ (which follows from the scalar product of (5) with (6)) we obtain

$$\frac{\partial L}{\partial\theta} + q\frac{\partial L}{\partial\zeta} - \frac{dK}{d\psi_{\mathrm{p}}} - q\frac{dI}{d\psi_{\mathrm{p}}} = \frac{4\pi(qI+K)}{B^2}\frac{dp}{d\psi_{\mathrm{p}}}. \quad \text{(A.3)}$$

Note that the only quantities in this equation which vary in θ or ζ are B and L. By expanding (A.3) in Fourier series in θ and ζ, it follows that L depends on θ and ζ only through the combination $M\theta - N\zeta$ if and only if B does the same, proving the lemma.

In a quasisymmetric field therefore $\partial L/\partial\zeta = -(N/M)\partial L/\partial\theta$. Using (5) and (6) we can then show $\mathbf{v}_{\mathrm{d}}\cdot\nabla(I_{\mathrm{h}}v_{\|}/\Omega) = 0$. Combining this result with (13), we obtain $(v_{\|}\mathbf{b}+\mathbf{v}_{\mathrm{d}})\cdot\nabla\psi_* = 0$. It quickly follows that $D\psi_* = 0$.

Appendix B: Moment equations for the radial particle and heat fluxes

We now derive the result (72) which relates the radial heat flux to a moment of the collision operator in a quasisymmetric stellarator. Along the way, we will also derive an analogous relation for the particle flux. We first note the identity

$$\mathbf{B}\times\nabla\psi_{\mathrm{h}}\cdot\nabla B = I_{\mathrm{h}}\mathbf{B}\cdot\nabla B, \quad \text{(B.1)}$$

obtained by writing $\mathbf{B}$ in the Boozer representations (5) and (6) and using $\partial B/\partial\zeta = -(N/M)\partial B/\partial\theta$.

Next, we follow [8,36] and define the vector

$$\mathbf{y} = \frac{1}{B^2}\mathbf{B}\times\nabla\psi_{\mathrm{h}} - \frac{I_{\mathrm{h}}}{B^2}\mathbf{B}. \quad \text{(B.2)}$$

Using (B.1) and $\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right]\times\nabla\psi_h = 0$ we find the useful properties

$$\nabla\cdot\mathbf{y} = 0 \quad \text{and} \quad \mathbf{b}\cdot(\nabla\mathbf{y})\cdot\mathbf{b} = 0. \quad \text{(B.3)}$$

In the axisymmetric limit, $\mathbf{y}\to -R^2\nabla\zeta_{\mathrm{t}}$ where ζ_{t} is the conventional toroidal angle in cylindrical $(R,\ \zeta_{\mathrm{t}}\ \ Z)$ coordinates.

Now take the full Fokker-Planck equation, multiply it by any function $X(\mathbf{r},\mathbf{v})$, integrate over velocity, and apply a flux surface average. The result can be written

$$\left\langle\int d^3v\, f\dot{X}\right\rangle = \left\langle\frac{\partial}{\partial t}\int d^3v\, Xf\right\rangle + \frac{1}{V'}\frac{d}{d\psi_{\mathrm{h}}}\left(V'\left\langle\int d^3v\, Xf\mathbf{v}\cdot\nabla\psi_{\mathrm{h}}\right\rangle\right) - \left\langle\int d^3v\, XC\right\rangle \quad \text{(B.4)}$$

where the overdot indicates the Vlasov operator $\partial_t + \mathbf{v}\cdot\nabla + Zem^{-1}\left(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}}$ with $V' = (1-Nq/M)^{-1}\oint d\theta\oint d\zeta\,(\mathbf{B}\cdot\nabla\theta)^{-1}$. Consider the choice $X = v^2\mathbf{v}\cdot\mathbf{y}$. Assuming $\mathbf{E} = -\nabla\Phi$ with Φ a flux function, then

$$\dot{X} = v^2\mathbf{v}\cdot(\nabla\mathbf{y})\cdot\mathbf{v} - 2\frac{Ze\Phi'}{m}\mathbf{y}\cdot\mathbf{vv}\cdot\nabla\psi_\mathrm{h} - \frac{Ze}{mc}v^2\mathbf{v}\cdot\nabla\psi_\mathrm{h} \quad \text{(B.5)}$$

where a prime denotes $d/d\psi_\mathrm{h}$ as usual.

We now proceed to order the various terms in (B.4) using the conventional drift orderings rather than the finite-E_ψ orderings. We use the small parameter $\delta = \rho/a$ with $\rho = v_\mathrm{i}/\Omega$ and a a macroscopic scale length. We expand the full distribution function as $f = \sum_j f_j$ with $f_j \sim \delta^j f_0$ and f_0 the Maxwellian of (24). We order $\partial_t \sim \delta^2\Omega$, $\nu \sim \delta\Omega$, $T/|\nabla T| \sim \eta/|\nabla\eta| \sim B^{-1}|\nabla\psi_\mathrm{h}| \sim \mathbf{y} \sim a$, and $|\mathbf{E}| \sim \delta B v_\mathrm{i}/c$.

We define $\langle\cdot\rangle_\varphi$ to be a gyroaverage holding ψ, θ, ζ, $\mu = v_\perp^2/2B$, and E fixed. We then define $\tilde{f} = f - \langle f\rangle_\varphi$. By the standard drift-kinetic procedure [37],

$$\tilde{f}_1 = \frac{s}{\Omega}f_0\mathbf{b}\times\nabla\psi_\mathrm{h}\cdot\mathbf{v} \quad \text{where} \quad s(v,\psi_h) = \frac{p'}{p} + \frac{Ze\Phi'}{T} + \left(\frac{mv^2}{2T} - \frac{5}{2}\right)\frac{T'}{T}. \quad \text{(B.6)}$$

In (B.4), the time derivative term is $O\left(\delta^2 v_\mathrm{i}^4 n\right)$ and therefore negligible compared to the collision term, which is $O\left(\delta v_\mathrm{i}^4 n\right)$. In the V' term, the contribution from $\langle f\rangle_\varphi$ is proportional to $\int d^3v\ \langle f\rangle_\varphi v^2 s(v)\mathbf{vv} = \int d^3v\ \langle f\rangle_\varphi v^2 s(v)\langle\mathbf{vv}\rangle_\varphi$. Due to

$$\langle\mathbf{vv}\rangle_\varphi = v_\parallel^2\mathbf{bb} + \left(v_\perp^2/2\right)\left(\overleftrightarrow{\mathbf{I}} - \mathbf{bb}\right) \quad \text{(B.7)}$$

then $\mathbf{y}\cdot\langle\mathbf{vv}\rangle_\varphi\cdot\nabla\psi_\mathrm{h} = 0$, so $\langle f\rangle_\varphi$ does not contribute to the V' term. The contribution to the V' term from $\tilde{f}_1$ is proportional to $\int d^3v\ f_0 v^2 s(v)\mathbf{vvv} = 0$. Thus, the largest contribution to the V' term in (B.4) comes from $\tilde{f}_2$, making the V' term $O\left(\delta^2 v_\mathrm{i}^4 n\right)$ and negligible. From (B.4) we therefore have

$$\begin{aligned}\frac{Ze}{mc}\left\langle\int d^3v\ fv^2\mathbf{v}\cdot\nabla\psi_\mathrm{h}\right\rangle &= \left\langle\int d^3v\ v^2\mathbf{v}\cdot\mathbf{y}C\right\rangle + \left\langle\int d^3v\ fv^2\mathbf{v}\cdot(\nabla\mathbf{y})\cdot\mathbf{v}\right\rangle \\ &\quad -2\frac{Ze\Phi'}{m}\left\langle\mathbf{y}\cdot\int d^3v\ f\mathbf{vv}\cdot\nabla\psi_\mathrm{h}\right\rangle + O\left(\delta^2 v_\mathrm{i}^4 n\right)\end{aligned} \quad \text{(B.8)}$$

Due to (B.7) and (B.3), $\langle f\rangle_\varphi$ does not contribute to the $\nabla\mathbf{y}$ term in (B.8). The contribution to the $\nabla\mathbf{y}$ term from $\tilde{f}_1$ vanishes since $\int d^3v\ f_0 v^2 s(v)\mathbf{vvv} = 0$. Thus, the largest contribution to the $\nabla\mathbf{y}$ term in (B.8) comes from $\tilde{f}_2$, making the term $O\left(\delta^2 v_\mathrm{i}^4 n\right)$ and therefore negligible compared to the collision term.

Now consider the Φ' term in (B.8). Noting $\int d^3v\langle f\rangle_\varphi\mathbf{vv} = \int d^3v\langle f\rangle_\varphi\langle\mathbf{vv}\rangle_\varphi$ and (B.7), then $\langle f\rangle_\varphi$ does not contribute to this term. The contribution from $\tilde{f}_1$ is proportional to $\int d^3v\ f_0 s(v)\mathbf{vvv} = 0$. The largest contribution to the Φ' term in (B.8) therefore comes from $\tilde{f}_2$. This term is $O\left(\delta^2 v_\mathrm{i}^4 n\right)$ and therefore negligible (though it would remain if $\mathbf{E}$ were ordered larger.)

To restrict our attention to neoclassical transport and exclude classical transport, we keep the parallel component of $\mathbf{y}$ in the collision term, but drop the perpendicular component. This leaves

$$\left\langle \int d^3v\, fv^2 \mathbf{v}\cdot\nabla\psi_{\mathrm{h}} \right\rangle \approx -\frac{I_{\mathrm{h}}}{\Omega_0}\left\langle h\int d^3v\; v^2 v_{\|} C\right\rangle . \text{ (B.9)}$$

The particle flux can be found by repeating the preceding argument using $X=\mathbf{v}\cdot\mathbf{y}$ in (B.4). We find

$$\dot{X} = \mathbf{v}\cdot(\nabla\mathbf{y})\cdot\mathbf{v} - \frac{Ze}{mc}\mathbf{v}\cdot\nabla\psi_{\mathrm{h}} . \text{ (B.10)}$$

The ordering of terms in (B.4) proceeds as before, and so

$$\left\langle \int d^3v\, f\mathbf{v}\cdot\nabla\psi_{\mathrm{h}} \right\rangle = \frac{mc}{Ze}\left\langle \mathbf{y}\cdot\int d^3v\; \mathbf{v}C\right\rangle + O\left(\delta^2 v_{\mathrm{i}}^2 \frac{nmc}{Ze}\right). \text{ (B.11)}$$

For a plasma with a single species of ions, $C\approx C_{\mathrm{ii}}$ in the ion kinetic equation, and so the collision term in (B.11) vanishes to leading order in $\sqrt{m_{\mathrm{e}}/m_{\mathrm{i}}}$.

In light of this result and (B.9), the ion heat flux can be written as

$$\left\langle \mathbf{q}\cdot\nabla\psi_{\mathrm{h}}\right\rangle = \left\langle \int d^3v\; f\left(\frac{mv^2}{2}-\frac{5T}{2}\right)\mathbf{v}\cdot\nabla\psi_{\mathrm{h}}\right\rangle \approx \left\langle \int d^3v f\,\frac{mv^2}{2}\mathbf{v}\cdot\nabla\psi_{\mathrm{h}}\right\rangle, \text{ (B.12)}$$

which when combined with (B.9) gives (72) as desired.

Appendix C: Exact and leading-order solutions of the drift-kinetic equation

We first prove a theorem regarding the time-independent drift-kinetic equation

$$\left(v_{\|}\mathbf{b}+\mathbf{v}_{\mathrm{d}}\right)\cdot\nabla f = C\{f\} \text{ (C.1)}$$

where $C\{f\}$ is the (nonlinear) Fokker-Planck operator for self-collisions, and the magnetic field is quasisymmetric. We look for solutions f which are independent of ζ at fixed χ. Casting into $(\psi_*,\ \chi\ \ \zeta, E, \mu)$ variables as in (18) we obtain $(D\chi)(\partial f/\partial\chi)_{\psi_*} = C\{f\}$ where $D\chi = \left(v_{\|}\mathbf{b}+\mathbf{v}_{\mathrm{d}}\right)\cdot\nabla\chi$. We multiply both sides by $(\ln f)/D\chi$ and recognize a perfect derivative:

$$\left(\frac{\partial}{\partial\chi}\right)_{\psi_*}(f\ln f - f) = \frac{C\{f\}\ln f}{D\chi} . \text{ (C.2)}$$

Now multiply by $\sigma=\mathrm{sgn}(D\chi)$, integrate over all allowed χ, and sum over σ and (in the case of trapped particles) all helical wells. These operations annihilate the left-hand side. Next, integrate over all allowed ψ_*, μ, and E , so we have integrated over all of position- and velocity-space (except for the unimportant ζ coordinate). This leaves

$$0=\sum_{\sigma}\int d\psi_*\; d\chi\; d\mu\; dE\left|D\chi\right|^{-1} C\{f\}\ln f \text{ . (C.3)}$$

We now change from ψ_* to ψ_{h} as an integration variable. Using the Jacobian mentioned previously in (90), then (C.3) becomes

$$0=\sum_{\varsigma}\int d\mu\; dE\; d\psi_{\mathrm{h}}\; d\chi\left|v_{\|}\mathbf{b}\cdot\nabla\chi\right|^{-1} C\{f\}\ln f \text{ (C.4)}$$

where $\varsigma=\mathrm{sgn}\left(v_{\|}\right)$. This can be rewritten as

$$0=\int d\psi_{\mathrm{h}}\int(\mathbf{B}\cdot\nabla\chi)^{-1} d\chi\int d^3v\; C\{f\}\ \mathrm{n}\, f \text{ . (C.5)}$$

Using the Landau form of the operator for self-collisions, the Cauchy-Schwartz inequality as usual implies $\int d^3v\, C\{f\} \ln f \le 0$ for any f. Thus, (C.5) implies that $\int d^3v\, C\{f\} \ln f = 0$ at all positions, so f must be Maxwellian

$$f = \eta\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m}{T}\left(E - \mathbf{v}\cdot\mathbf{V} + \frac{V^2}{2}\right)\right), \text{(C.6)}$$

where, for now, η, T, and $\mathbf{V}$ may depend on position. Since f must be independent of gyrophase, the mean flow $\mathbf{V}$ must be parallel to $\mathbf{B}$, so we write $\mathbf{V} = V\mathbf{b}$. Next, (C.1) becomes $\left(\partial f / \partial \chi\right)_{\psi_*} = 0$, so f can vary only through ψ_*, μ, and E. Therefore, η, T, and V must be position-independent. Forming the velocity moment of (C.6) gives

$$\int d^3v\, f\mathbf{v} = V\eta \exp\left(-\frac{Ze\Phi}{T}\right)\mathbf{b}. \text{(C.7)}$$

The divergence of (C.7) must vanish to satisfy number conservation, implying V must be zero.

Thus, we have proven that the only ζ-independent exact solutions of the equilibrium drift-kinetic equation (C.1) in a quasisymmetric field are stationary Maxwellians as in (24) but with no gradients in temperature or pseudo-density η.

We now consider the related problem of finding the leading-order distribution function for the neoclassical transport or residual zonal flow analysis. Suppose the leading-order kinetic equation is taken to be $\left(v_{||} + u\right)\left(\mathbf{b}\cdot\nabla\chi\right)\partial f_0 / \partial\chi = C\{f_0\}$. Adding the small magnetic drifts to the left-hand side results in (C.1), so the proof following (C.1) applies and f_0 must be Maxwellian. Although T' and η' are nonzero in a realistic plasma, we interpret the proof as indication these gradients are weak. If we instead expand the kinetic equation for small collisionality, to leading order $Df \approx 0$, so f must be a function of the constants of the motion $\left(\psi_*, \mu, E\right)$. Since we want f to also be nearly Maxwellian, we therefore must take $f \approx F\left(\psi_*, E\right)$ with F given by (25), and we demand that F be Maxwellian to leading order. A Taylor-expansion of η and T in F about $\psi_* \approx \psi_{\mathrm{h}}$ gives $F \approx f_{\mathrm{M}} + F_1$ where

$$F_1 = -f_{\mathrm{M}} \frac{v_{||} I_{\mathrm{h}}}{\Omega}\left[\frac{\eta'}{\eta} + \left(\frac{mE}{T} - \frac{3}{2}\right)\frac{T'}{T}\right] \text{(C.8)}$$

(equivalent to (26)) with η and T evaluated at ψ_{h} rather than ψ_*. Therefore $F_1 / f_{\mathrm{M}} \sim \rho / \left(\alpha r_{\eta T}\right)$ where $r_{\eta T}$ is the shorter of the scale-lengths of η and T. For f to remain Maxwellian to leading order, T and η can vary only on a scale length which is long compared to ρ / α. It is still possible that the true density n and the potential Φ vary on the length scale ρ / α as long as their combination in η varies more slowly.

Appendix D: Integral for the parallel flow

Here we argue that

$$\int d^3v\left(g-G\right)\sim\sqrt{\varepsilon}\,\frac{nI_{\mathrm{h}}T'}{v_{\mathrm{i}}m\Omega}\quad\text{(D.9)}$$

and therefore that the last integral in (83) can be dropped. We begin by writing the integral in terms of $\left(W,\Lambda\right)$ variables:

$$\int d^3v\left(g-G\right)=\frac{2\pi}{h}\sum_{\sigma}\int dW\int d\Lambda\frac{\left(g-G\right)W}{\left|v_{\|}+u\right|}=-2\pi\sum_{\sigma}\int dW\int d\Lambda\left(g-G\right)\frac{\partial}{\partial\Lambda}\left|v_{\|}+u\right|\quad\text{(D.10)}$$

We are free to add a constant behind the derivative, so

$$\int d^3v\left(g-G\right)=2\pi\sum_{\sigma}\int dW\sqrt{2W}\int d\Lambda\left(g-G\right)\frac{\partial}{\partial\Lambda}\left(1-\sqrt{1-\Lambda/h}\right).\quad\text{(D.11)}$$

We next integrate by parts in Λ. There is no contribution from the lower boundary $\Lambda=0$ because the last quantity in parentheses vanishes there. There is also no contribution from the upper boundary since $G=0$ there and $g=0$ in this trapped region to leading order. Thus,

$$\int d^3v\left(g-G\right)=2\pi\sum_{\sigma}\int dW\sqrt{2W}\int d\Lambda\left(\sqrt{1-\Lambda/h}-1\right)\left[f_{\mathrm{M}}\frac{\partial}{\partial\Lambda}\frac{\left(g-G\right)}{f_{\mathrm{M}}}+\frac{\left(g-G\right)}{f_{\mathrm{M}}}\frac{\partial f_{\mathrm{M}}}{\partial\Lambda}\right].\quad\text{(D.12)}$$

To leading order in $\sqrt{\varepsilon}$, f_{M} is independent of Λ, so the $\partial f_{\mathrm{M}}/\partial\Lambda$ term vanishes. Also, from (66), $f_{\mathrm{M}}\partial\left[\left(g-G\right)/f_{\mathrm{M}}\right]/\partial\Lambda$ is odd in σ to leading order, and so it vanishes in the σ sum. To properly calculate the leading nonvanishing contribution to the integral above, we need not only the next correction to f_{M} in the $\sqrt{\varepsilon}$ expansion, but also the next correction to g, which is not feasible. In any event, since the right-hand side of (D.12) vanishes in leading order, the estimate (D.9) is adequate.

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Table 1: Quasisymmetry-axisymmetry isomorphism

	Axisymmetry	**Quasisymmetry**
Symmetry of B	$B = B(\psi_{\mathrm{p}},\ \Theta)$	$B = B(\psi_{\mathrm{h}},\ \chi)$
Poloidal	angle Θ Helical	angle $\chi = M\theta - N\zeta$
Radial coordinate	$\psi_{\mathrm{p}} =$ poloidal flux $/(2\pi)$ Helical	flux $\psi_{\mathrm{h}} = \psi_{\mathrm{p}} - N\psi_{\mathrm{t}} / M$
	$I = RB_{\mathrm{t}}$	$I_{\mathrm{h}} = I - NK / M$
Conserved quantity	$\Psi_* = \psi_{\mathrm{p}} - Iv_{\parallel} / \Omega$	$\psi_* = \psi_{\mathrm{h}} - I_{\mathrm{h}} v_{\parallel} / \Omega$
	$\mathbf{b} \cdot \nabla\Theta \approx 1/(qR)$	$\mathbf{b} \cdot \nabla\chi = \dfrac{[M - Nq]B}{qI + K}$
Relative B variation	$\varepsilon = a / R$	$\varepsilon = (B_{\max} - B_0)/(2B_0)$
Small geometrical factor	$\lvert\nabla\psi_{\mathrm{p}}\rvert / I \approx B_{\mathrm{p}} / B$	$\lvert\nabla\psi_{\mathrm{h}}\rvert / I_{\mathrm{h}} = \alpha$
Normalized electric field	$U = cI(v_{\mathrm{i}} B_0)^{-1} d\Phi / d\psi_{\mathrm{p}}$	$U = cI_{\mathrm{h}}(v_{\mathrm{i}} B_0)^{-1} d\Phi / d\psi_{\mathrm{h}}$

Figure captions

1. Numerical functions which appear in (a) the heat flux and (b) the parallel flow and bootstrap current.

Figure 1