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An Analytic Determination of Beta Poloidal and Internal Inductance in an Elongated Tokamak from Magnetic Probe Measurements

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by

Joseph Mark Sorci

Submitted to the Department of Nuclear Engineering
in Partial Fulfillment of the Requirements for the Degrees of

Bachelor of Science in Nuclear Engineering

and

Master of Science in Nuclear Engineering

at the

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Abstract

Analytic calculations of the magnetic fields available to magnetic diagnostics are performed for tokamaks with circular and elliptical cross sections. The explicit dependence of the magnetic fields on the poloidal beta and internal inductances is sought.

For tokamaks with circular cross sections, Shafranov's results are reproduced and extended. To first order in the inverse aspect ratio expansion of the magnetic fields, only a specific combination of beta poloidal and internal inductance is found to be measurable. To second order in the expansion, the measurements of beta poloidal and the internal inductance are demonstrated to be separable but excessively sensitive to experimental error.

For tokamaks with elliptical cross sections, magnetic measurements are found to determine beta poloidal and the internal inductance separately. A second harmonic component of the zeroth order field in combination with the dc harmonic of the zeroth order field specifies the internal inductance. The internal inductance in hand, measurement of the first order, first harmonic component of the magnetic field then determines beta poloidal. The degeneracy implicit in Shafranov's result (i.e. that only a combination of beta poloidal and internal inductance is measurable for a circular plasma cross section) reasserts itself as the elliptic results are collapsed to their circular limits.

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Chapter 1
Introduction

1.1 Background

The realization of controlled thermonuclear fusion is one of the Holy Grails of modern physics and engineering. Promising clean, practically limitless energy, fusion is one of the principal hopefuls for future energy development. To this end, fusion research is being conducted worldwide.

A favorite scheme for realization of controlled thermonuclear fusion is the tokamak, a toroidal confinement device pioneered by Soviet scientists. Briefly, a tokamak consists of a toroidal vacuum chamber that loops through powerful magnets, called toroidal field or TF magnets. The TF magnets create a strong magnetic field in the toroidal direction. In modern, high field experiments such as Alcator C-Mod, the toroidal field can be as high as 10 T. In addition to the applied toroidal field, a tokamak realizes plasma confinement by means of a self-generated poloidal field. A powerful transformer commonly referred to as the ohmic transformer is pulsed to initiate tokamak operation. The magnetic flux created by the ohmic transformer links the plasma that is being created simultaneously in the vacuum chamber. The resultant electric field drives a current through the plasma in the toroidal direction creating a magnetic field in the poloidal direction. This current is called the plasma current, henceforth denoted as I_p. $I_p = 3$ MA in Alcator C-Mod. The poloidal field created by I_p combines vectorially with the applied toroidal field to create a rotational transform or "screw pinch" equilibrium that has proved remarkably efficient in confining fusion plasmas for brief periods of time.

The difficulty in achieving breakeven, much less appreciable gain, in a fusion experiments lies in confining plasma that is hot enough, long enough so that the necessary number of fuel nuclei overcome their mutual Coulomb repulsion and fuse. Typically, modern fusion experiments have $T_e \approx 8$ keV, $n_e \approx 5 \times 10^{20} m^{-3}$, and energy confinement times

(τ_E) on the order of 500 msec. Lawson formulated a criterion for achieving breakeven in a deuterium-tritium plasma that is summarized below.

$$n_e \tau_{E\ DT} \simeq 10^{20} m^{-3} s \tag{1.1}$$

To date, fusion experiments have improved dramatically, by a factor of 10^4 from initial devices, but still fall short of achieving ignition. Research is ongoing and progress is being made. Several new concepts are being explored in the newer experiments including elongated plasma cross sections, divertors, pellet-fueling, neutral beam and rf heating to name a few.

Crucial in gauging the performance of a given tokamak experiment are two parameters β_p and ℓ_i. β_p, known as beta poloidal, is the ratio of plasma kinetic pressure to poloidal magnetic field pressure. β_p has several definitions depending on which convention is employed. For the present calculation the following two definitions of β_p will be employed where appropriate.

$$\beta_p = \frac{\langle p \rangle 2\mu_0}{B^2_{p_{edge}}} \tag{1.2}$$

$$\beta_p = \frac{\langle p \rangle 8\pi}{\mu_0 I_p^2} A_{plasma} \tag{1.3}$$

$\langle p \rangle$ is the volume averaged plasma kinetic pressure. $B^2_{p_{edge}}/2\mu_0$ can be thought of as the poloidal magnetic field pressure at the edge of the plasma. Equation (1.3) reduces to Eq. (1.2) if the plasma has a circular cross section.

β_p is a measure of how much plasma is being confined for a given edge value of poloidal field. In some sense high β_p means better overall plasma confinement and tokamak performance. However, it can be demonstrated that if β_p becomes too high, that is reaches a certain limit, plasma equilibrium is no longer possible. For the case of a tokamak of circular cross section the β_p limit can be expressed thus

$$\epsilon \beta_p \leq 1 \tag{1.4}$$

where $\epsilon = a/R_0$ is the inverse aspect ratio of the tokamak.

ℓ_i is the internal inductance of the plasma per unit length normalized to $\mu_0/4\pi$. It is a measure of the width of the current profile which has direct bearing on the **stability** of a given equilibrium.

One way in which experimentalists have sought to determine these two important operational parameters is with magnetic diagnostics. For a comprehensive overview of the most commonly employed magnetic diagnostics including Rogowski coils, flux loops, and field coils, see Hutchinson (1987).

Several numerical studies have been undertaken to determine how and under what conditions ℓ_i, β_p, and I_p can be measured with magnetic diagnostics (Brahms [1990]). Luxon and Brown (1982) while working on Doublet IIa and Doublet III employed a scheme whereby the Grad-Shafranov equation was solved for a particular set of profiles and simulated measurements were computed for the 24 one-turn loops and 12 partial Rogowski coils actually monitoring the experiments. These simulated measurements were then compared to the actual data and the differences minimized. Lau et. al. (1985) performed a similar analysis on Doublet III adding 11 local magnetic probes to the diagnostics listed above. Both groups found that the differences between actual and simulated measurements had well-defined minima for non-circular cross sections and that I_p, β_p, and ℓ_i could be determined separately with some measure of confidence. For circular cross sections, only I_p and $\beta_p + \ell_i/2$ could be determined. In a later work, Lao et. al. (1985) demonstrated that in the circular case β_p and ℓ_i could be separated by appealing to a diamagnetic flux measurement in addition to the other measurements cited above. The validity of such an approach is in doubt however as diamagnetic flux measurements are subject to substantial errors because of a large toroidal field offset.

Numerical work on JET pursued by Brusati, et al (1984), Blum et al (1981, 1985), and Lazarro and Mantica (1988) proceeded along the same lines. Their conclusions were nearly identical with the Doublet III groups'. From magnetic measurements alone, I_p and the combination $\beta_p + \ell_i/2$ could be determined for low β_p in near circular plasmas and I_p, β_p, and ℓ_i for non-circular plasmas. A critical elongation of 1.25 was calculated. For plasma with elongations $\kappa \geq 1.25$, the measurements were separable.

Much analytic work has been done by Shafranov (1962, 1966) and Mukhovatov and Shafranov (1971). Shafranov demonstrated that to first order in the inverse aspect ratio, $\epsilon = a/R_0$, the radial and azimuthal components of the poloidal field outside the plasma can be expressed in the following forms.

$$B_\theta(r,\theta) \simeq -\frac{\mu_0 I_p}{2\pi r} - \frac{\mu_0 I_p}{4\pi R_0}\left[(1+\frac{a^2}{r^2})(\beta_p + \frac{\ell_i - 1}{2}) + \ln\frac{r}{a} - 1 + \frac{2R_0\Delta_a}{r^2}\right]\cos\theta \qquad (1.5)$$

$$B_r(r,\theta) \simeq -\frac{\mu_0 I_p}{4\pi R_0}\left[(1-\frac{a^2}{r^2})(\beta_p + \frac{\ell_i - 1}{2}) + \ln\frac{r}{a} - \frac{2R_0\Delta_a}{r^2}\right]\sin\theta \qquad (1.6)$$

a is the minor radius of the tokamak, R_0 the major radius. Δ_a is the famous Shafranov shift which represents the distance the plasma has shifted outward in order to reach an equilibrium that creates a toroidal force balance. One can determine I_p from the steady component of B_θ and Δ_a and the combination $\beta_p + \ell_i/2$ from the first harmonics of B_r and B_θ. Shafranov's model and the studies cited agree.

Wind (1972, 1984) and Brahms et al (1986) applied function parameterizations to the magnetic data analysis on the ASDEX experiment. The goal was to obtain a simple functional form for intrinsic physical parameters of a tokamak in terms of the values of measurements. Again, only $\beta_p + \ell_i/2$ was determined with good accuracy in the presence of realistic measurement errors in a near circular geometry.

The objective of the present work is to demonstrate analytically what has been heretofore known only computationally. Namely, magnetic measurements are sufficient to determine β_p and ℓ_i independently only if the plasma is sufficiently elongated. What follows in the present chapter is a short review of the ideal MHD model. Chapter 2 reproduces Shafranov's results in the circular limit and extends the model further demonstrating how although second order, second harmonic field measurements allow one to separate β_p and ℓ_i, the measurements are too sensitive to determine them with any confidence. Chapter 3 addresses the elliptic problem in which (for profiles fundamentally identical to those used in the Shafranov model), the Grad-Shafranov equation is solved explicitly to first order. The resultant magnetic fields available to a hypothetical set of magnetic probes are then calculated explicitly.

1.2 Ideal MHD

For a comprehensive overview of the subject, refer to Freidberg (1987). A few salient points are summarized here.

Ideal MHD treats a plasma as a single, electrically neutral fluid capable of supporting large electric currents. The currents are modeled as being carried by massless electrons while the fluid's inertia lies with the ions. A reduction of the two-fluid equations for electrons and ions to a single fluid equation with these approximations in mind yields the famous force balance equation shown below.

$$\mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} p \tag{1.7}$$

Ideal MHD models a plasma as having no resistivity. Therefore, Ohm's law can be cast in the following form.

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \tag{1.8}$$

In combination with Maxwell's laws and an equation of state (1.7) and (1.8) can be used to solve for a very wide range of MHD equilibria.

In fusion configurations with confined plasmas, the magnetic lines lie on a set of nested toroidal surfaces called flux surfaces.

Taking the **B** component of Eq. (1.7) reveals that flux surfaces must also be surfaces of constant pressure.

$$\mathbf{B} \cdot \boldsymbol{\nabla} p = 0 \tag{1.9}$$

It is also worthwhile to note that taking the **J** component of Eq. (1.7) demonstrates that the current flows along flux surfaces and never across them.

Consider the following two Maxwell's equations where the displacement current has been ignored.

$$\nabla \cdot \mathbf{B} = 0 \tag{1.10}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1.11}$$

Define the magnetic field $\mathbf{B}$ in the following manner.

$$\mathbf{B} = B_\phi \hat{\mathbf{e}}_\phi + \mathbf{B}_p \tag{1.12}$$

$$\mathbf{B}_p = \frac{1}{R} \nabla\psi \times \hat{\mathbf{e}}_\phi \tag{1.13}$$

ψ is the flux function. Flux surfaces are surfaces of constant ψ. Combining Eqs. (1.7) with (1.10-1.13) one can derive the famous Grad-Shafranov equation.

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} \tag{1.14}$$

The elliptic operator $\Delta^* = R^2\nabla \cdot \left(\frac{\nabla}{R^2}\right)$. $F = RB_\phi$ and can be shown to be a free function of flux only.

$$F = F(\psi) \tag{1.15}$$

Likewise for the pressure p.

$$p = p(\psi) \tag{1.16}$$

Equation (1.14), the Grad-Shafranov equation, describes tokamak equilibrium in terms of the flux function ψ. Solving the Grad-Shafranov equation for certain prescribed, ideal profiles p and F, one can thus calculate $\mathbf{B}_p$ explicitly from ψ. This is exactly the approach taken in Chapters 2 and 3. In both cases an inverse aspect ratio (ϵ) expansion is performed. For the circular case the expansion must be carried out to second order in ϵ. For the elliptical case only first order is required, but the zeroth order solutions are much more complicated. From these solutions, it is possible to deduce the desired information concerning β_p, ℓ_i, and the magnetic diagnostics.

1.3 Notation

A brief word about notation. Throughout the work, whenever a "caret" appears above any quantity except a unit vector, that quantity is understood to be defined outside the plasma. For example, $\hat{\psi}$ denotes the flux function outside the plasma while ψ denotes the flux function inside the plasma. Also, magnetic fields are labeled with subscripted direction, order, and angular harmonic. For example, $\mathbf{B}_{\theta_{11}}$ denotes the first order, first harmoinc magnetic field in the $\hat{\theta}$ direction.

Chapter 2
The Circular Limit

2.1 Introduction

In this chapter the Grad-Shafranov equation will be solved to second order in the ohmic tokamak expansion. See Shajii et al (1992). Then having explicit formulas for the flux functions $\hat{\psi}_0, \hat{\psi}_1$, and $\hat{\psi}_2$, the magnetic fields available to an idealized set of probes are calculated. The dependence of these field amplitudes on β_p and ℓ_i are sought.

2.2 The Ohmic Tokamak Expansion of the Grad-Shafranov Equation

Consider a circular tokamak as illustrated in Fig. 2.1. The plasma of radius a is surrounded by magnetic probes conveniently located on a concentric circle of radius b. These magnetic probes sample the radial and azimuthal fields during the flattop portion of tokamak operation. Assume that the signals are Fourier analyzed to yield the following information.

$$B_r(\theta, b) = B_{r1}(b)\sin\theta + B_{r2}(b)\sin 2\theta \tag{2.1}$$

$$B_\theta(\theta, b) = B_{\theta 0}(b) + B_{\theta 1}(b)\cos\theta + B_{\theta 2}(b)\cos 2\theta \tag{2.2}$$

- B_{r1} is the first order radial field.
- B_{r2} is the second order radial field.
- $B_{\theta 0}$ is the zeroth order tangential field.
- $B_{\theta 1}$ is the first order tangential field.
- $B_{\theta 2}$ is the second order tangential field.

The field amplitudes are ordered with respect to the inverse aspect ratio $\epsilon \equiv \frac{a}{R_0} \ll 1$. That is

$$\frac{B_{\theta 1}}{B_{\theta 0}} \propto \frac{B_{\theta 2}}{B_{\theta 1}} \propto \frac{B_{r2}}{B_{r1}} \propto \epsilon \tag{2.3}$$

The data yields five pieces of information with which it should be possible to obtain the following five plasma parameters:

- I_p total plasma current.
- Δ_a the Shafranov shift.
- β_p the poloidal β.
- ℓ_i normalized internal inductance.
- κ the plasma elongation.

To obtain analytic expressions for the field amplitudes in terms of the desired parameters one proceeds as follows.

The MHD equilibrium of the plasma is described by the Grad-Shafranov equation developed in Chapter 1.

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} \tag{2.4}$$

Again $p = p(\psi)$ and $F = F(\psi)$, free functions of flux that describe the pressure and toroidal field profiles respectively. For this particular problem assume an ohmic regime of tokamak operation as opposed to the high beta or flux conserving regimes. The regime of operation gives the ordering and appropriate parameters in which to asymptotically expand the Grad-Shafranov equation in order to obtain ψ to the desired accuracy.

Ohmic tokamak operation is characterized by low β, paramagnetic plasma behavior, and $q \sim 1$ for stability. q is the safety factor where $q(r) = \frac{rB_\phi(r)}{R_0 B_\theta(r)}$. Ohmic operation assumes that plasma kinetic pressure is confined mainly by a poloidal field generated by ohmic current and **not** by any magnetic well in the toroidal field.

Expand the Grad-Shafranov in the parameter ϵ, the inverse aspect ratio, where $\epsilon \equiv \frac{a}{R_0} \ll 1$. The ohmically heated tokamak expansion is given by

$$\frac{B_p}{B_\phi} \sim \epsilon \tag{2.5}$$

$$q \sim 1 \tag{2.6}$$

$$\beta_t \sim \frac{2\mu_0 p}{B_\phi^2} \sim \epsilon^2 \tag{2.7}$$

$$\beta_p \sim \frac{2\mu_0 p}{B_p^2} \sim 1 \tag{2.8}$$

$$\psi(r,\theta) = \psi_0(r) + \psi_1(r)\cos\theta + \psi_2(r,\theta) + \ldots \tag{2.9}$$

$$\frac{\psi_1}{\psi_0} \sim \epsilon \tag{2.10}$$

$$\frac{\psi_2}{\psi_1} \sim \epsilon \tag{2.11}$$

$$\psi_0 \sim rR_0B_\theta \tag{2.12}$$

Choose $F^2(\psi)$ and $p(\psi)$ most conveniently and Taylor expand these free functions about ψ_0.

$$F^2 \simeq R_0^2(B_0^2 + 2B_0B_2(\psi)) \tag{2.13}$$

$$p(\psi) \simeq p(\psi_0) + \frac{dp}{d\psi_0}(\psi_1 + \psi_2) + \frac{1}{2}\frac{d^2p}{d\psi_0^2}(\psi_1 + \psi_2)^2 + \ldots \tag{2.15}$$

Rewrite the Grad-Shafranov in toroidal coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = -\mu_0(R_0 + r\cos\theta)^2\frac{dp}{d\psi} - \frac{d}{d\psi}\frac{F^2}{2} + \frac{1}{R}\left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right) \tag{2.16}$$

By substituting the expansions for $\psi, p(\psi), F(\psi)$ and collecting terms of the same order in ϵ, three inter-related equations are obtained.

$$\epsilon^0: \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_0}{\partial r}\right) = -R_0^2B_0\frac{dB_2}{d\psi_0} - \mu_0R_0^2\frac{dp}{d\psi_0} \tag{2.17}$$

$$\epsilon^1: \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right) - \frac{\psi_1}{r^2} = \frac{1}{R_0}\frac{\partial\psi_0}{\partial r} - R_0^2B_0\psi_1\frac{d^2B_2}{d\psi_0^2} - \mu_0R_0^2\left(\psi_1\frac{d^2p}{d\psi_0^2} + \frac{2r}{R_0}\frac{dp}{d\psi_0}\right) \tag{2.18}$$

$$\begin{aligned}\epsilon^2: \quad \nabla^2\psi_2 = {} & \frac{1}{R_0}\left(\frac{\partial\psi_1}{\partial r} - \frac{r}{R_0}\frac{\partial\psi_0}{\partial r}\right)\cos^2\theta + \frac{\psi_1}{rR_0}\sin^2\theta - R_0^2B_0\frac{d^2B_2}{d\psi_0}\psi_2 - \frac{R_0^2B_0}{2}\frac{d^3B_2}{d\psi_0^3}\psi_1^2\cos^2\theta \\ & - \mu_0R_0^2\psi_2\frac{d^2p}{d\psi_0^2} - \left(\frac{\psi_1^2}{2}\frac{d^3p}{d\psi_0^3} + \frac{2r\psi_1}{R_0}\frac{d^2p}{d\psi_0^2} + \frac{r^2}{R_0^2}\frac{dp}{d\psi_0}\right)\cos^2\theta\mu_0R_0^2\end{aligned} \tag{2.19}$$

Equations 2.17–2.19 shall henceforth be referred to as the zeroth, first and second order equations respectively. The zeroth order equation is a statement of radial pressure balance and the zeroth order poloidal field is given by

$$B_\theta = \frac{1}{R_0}\frac{d\psi_0}{dr} \tag{2.20}$$

Rearranging terms in the zeroth order equation, it is a simple matter to show that $\frac{d}{d\psi_0} = \frac{1}{R_0 B_\theta}\frac{d}{dr}$. The first order equation can then be simplified and written in the following form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_1}{dr}\right) - \left[\frac{1}{r^2} + \frac{\mu_0}{B_\theta}\frac{dJ}{dr}\right]\psi_1 = B_\theta - \frac{2\mu_0 r}{B_\theta}\frac{dp}{dr} \tag{2.21}$$

Likewise, the second order equation can be cast in a more tractable form

$$\nabla^2\psi_2 - \frac{\mu_0}{B_\theta}\frac{dJ}{dr}\psi_2 = \overline{R}(r) + S(r)\cos 2\theta \tag{2.22}$$

$$\overline{R}(r) = \frac{1}{2R_0}\left\{\frac{d\psi_1}{dr} + \frac{\psi_1}{r} - rB_\theta - \frac{\mu_0 r^2}{B_\theta}\frac{dp}{dr} - \frac{2\mu_0 r}{B_\theta}\psi_1\frac{d}{dr}\left(\frac{1}{B_\theta}\frac{dp}{dr}\right) + \frac{\psi_1^2}{2B_\theta}\frac{d}{dr}\left(\frac{\mu_0}{B_\theta}\frac{dJ}{dr}\right)\right\}$$

$$S(r) + \frac{1}{2R_0}\left\{\frac{d\psi_1}{dr} - \frac{\psi_1}{r} - rB_\theta - \frac{\mu_0 r^2}{B_\theta}\frac{dp}{dr} - \frac{2\mu_0 r}{B_\theta}\psi_1\frac{d}{dr}\left(\frac{1}{B_\theta}\frac{dp}{dr}\right) + \frac{\psi_1^2}{2B_\theta}\frac{d}{dr}\left(\frac{\mu_0}{B_\theta}\frac{dJ}{dr}\right)\right\}$$

It is important to note that, as the complexity of each equation increases in proportion to its order so does information content. In fact, Eq. (2.22) contains more information than is required to derive the field amplitudes of the particular harmonics being sampled. Since $B_{\theta 2}(r)\cos 2\theta \propto d\psi_2(r,\theta)/dr$ only, the $S(r)\cos 2\theta$ term on the right hand side of (2.22) will be needed. For all practical purposes $\overline{R}(r)$ can be ignored for the remainder of the calculation.

To specify the problem completely, the boundary conditions on $\psi_1(a)$ and $\psi_2(a,\theta)$ must be imposed. Before turning to the detailed behavior of ψ on the plasma boundary $r = a$ it is worthwhile to mention that ψ must be regular at the origin. Whatever the functional form given by the solutions of 2.21 and 2.22, an infinite flux at $r = 0$ is unphysical and the coefficients of any terms that diverge as $r \to 0$ must be set to zero in the region $r < a$.

It was mentioned earlier that the boundary of the plasma is circular. That is only true to zeroth order. Let the surface of the plasma be circular with small ellipticity. Assume the surface of the plasma is described by $r(\theta)$ where

$$r = a\left[1 + \frac{\kappa - 1}{2}(1 - \cos 2\theta)\right] \tag{2.23}$$

The ellipticity is second order in ϵ.

$$\kappa - 1 \sim \epsilon^2 \tag{2.24}$$

Here it is implicitly assumed that the equilibrium has been so arranged to set the Shafranov shift $\Delta_a = 0$. This is not a necessary condition and has only been assumed for the sake of simplicity.

The surface of the plasma is also a flux surface; that is, $\psi(a, \theta) = \text{const}$. Therefore, we can Taylor expand $\psi_0(r)$ at the boundary, add the first and second order contributions to ψ, and set the entire sum equal to a conveniently chosen constant.

$$\psi_0 + a\frac{d\psi_0}{dr}\left[\frac{\kappa - 1}{2}(1 - \cos 2\theta)\right] + \psi_1 \cos\theta + \psi_2 = 0 \tag{2.25}$$

Immediately, it becomes apparent that in order to satisfy the condition that $\psi(r_a(\theta), 0)$

$$\psi_1(a) = 0 \tag{2.26}$$

$$\psi_2(a, \theta) = -a\frac{R_0 B_{\theta a}}{2}(\kappa - 1)(1 - \cos 2\theta) \tag{2.27}$$

To carry out this calculation analytically it is necessary to use very simple profiles for $p(r), J(r)$ and $B_\theta(r)$. The following profiles are used to solve (2.21) and (2.22).

$$p = p_0(1 - \frac{r^2}{c^2}) \qquad r < c \tag{2.28}$$

$$p = 0 \qquad r \geq c \tag{2.29}$$

$$B_\theta = B_{\theta c}\frac{r}{c} \qquad r < c \tag{2.30}$$

$$B_\theta = B_{\theta c}\frac{c}{r} \qquad r \geq c \tag{2.31}$$

$$J = J_0 \qquad r < c \tag{2.32}$$

$$J = 0 \qquad r \geq c \tag{2.33}$$

See Fig. 2.2 for a depiction of these elementary profiles. This very simple model is intended to replicate the behavior of plasmas with dense, current carrying cores, the ratio of c/a being a measure of the peakedness of the actual smooth profiles that are measured in experimental plasmas.

Before substituting these profiles into the first and second order equations, they are used to calculate ℓ_i and β_p quantities which depend only on zeroth order quantities.

Here, let $\beta_p \equiv \langle p \rangle \frac{2\mu_0}{B_{\theta a}^2}$ where $\langle p \rangle$ is the volume averaged kinetic pressure and $\frac{B_{\theta a}^2}{2\mu_0}$ is the edge value of poloidal magnetic pressure. Given the profiles outlined above β_p is simple to calculate.

$$\beta_p = \frac{\mu_0 p_0}{B_{\theta c}^2} \tag{2.34}$$

Now calculate ℓ_i, the internal inductance of the plasma per unit length normalized to $\mu_0/4\pi$. Actually the determination of ℓ_i is merely a statement of the conservation of zeroth order magnetic energy.

$$\frac{1}{2} L_i I_p^2 = \int \frac{B_\theta^2(r)}{2\mu_0} d^3 V_{plasma} \tag{2.35}$$

$$\ell_i = \frac{L_i}{2\pi R_0} \Big/ \frac{\mu_0}{4\pi} \tag{2.36}$$

Breaking up the volume integral into two regions $r < c$ and $r \geq c$, and substituting Eqs. (2.30) and (2.31) in the appropriate regions, ℓ_i is obtained.

$$\ell_i = \frac{1}{2} - 2 \ln \alpha \tag{2.37}$$

$$\alpha = \frac{c}{a}$$

The dimensionless ratio α is a measure of how peaked actual, smoothly varying profiles such as these realized in experiments might be. As $\alpha \rightarrow 1$ the profiles become flat and as $\alpha \rightarrow 0$ the profiles become highly peaked. Intermediate values of α can be chosen to approximate a given experimental situation.

Expressions for β_p and ℓ_i in hand, one is in a position to solve Eqs. (2.21) and (2.22) and obtain expressions for the field amplitudes to be measured.

2.3 The First Order Solution

Upon substitution of the given profiles into (2.21) in the region $0 < r < c$ the ψ_1 equation becomes

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_1}{dr} - \frac{\psi_1}{r^2} = \frac{B_{\theta c}}{c}(1 + 4\beta_p)r \tag{2.38}$$

Setting the coefficient of any terms that diverge as $r \to 0$ to zero, the solution of (2.38) can be expressed in the form below

$$\psi_1(r) = \frac{B_{\theta c}}{8c}(1 + 4\beta_p)r^3 + c_1 r \tag{2.39}$$

Repeat this procedure for (2.21) in the region $r > c$ keeping in mind that the decaying solutions must now be kept in the form of the solution. Equation (2.21) becomes

$$\frac{1}{r}\frac{d}{dr}(r\frac{d\hat{\psi}_1}{dr}) - \frac{\hat{\psi}_1}{r^2} = B_{\theta c}\frac{c}{r} \tag{2.40}$$

Since $\psi_1(a) = 0$ the solution to (2.40) can be expressed as

$$\hat{\psi}_1 = \frac{cB_{\theta c}}{2}r\ln\frac{r}{a} + c_2(r - \frac{a^2}{r}) \tag{2.41}$$

At this point in the calculation there are two undetermined coefficients, c_1 and c_2. Application of the and the jump conditions at $r = c$ determine these constants.

$J(r)$ is a step function. $\frac{dJ(r)}{dr}$ is a delta function at $r = c$. ψ_1 must be continuous at $r = c$.

$$\frac{dJ(r)}{dr} = \frac{1}{\mu_0 r}\frac{d}{dr}rB_\theta(r)\delta(r - c)$$

$$\frac{dJ(r)}{dr} = \frac{1}{\mu_0 r}\frac{d}{dr}r^2\frac{B_{\theta c}}{c}\delta(r - c)$$

$$\frac{dJ(r)}{dr} = \frac{2B_{\theta c}}{\mu_0 c}\delta(r - c) \tag{2.42}$$

$$\hat{\psi}_1(c) - \psi_1(c) = 0 \tag{2.43}$$

Now examine Eq. (2.21) again integrating over the jump from $r = c-$ to $r = c+$.

$$\int_{c-}^{c+} \frac{1}{r}\frac{d}{dr}(r\frac{d\psi_1}{dr})dr - \int_{c-}^{c+} \frac{\psi_1}{r^2}dr + \int_{c-}^{c+} \frac{2}{c}\psi_1(r)\delta(r-c)dr = \int_{c-}^{c+} (B_\theta - \frac{2\mu_0 r}{B_\theta}\frac{dp}{dr})dr \quad (2.44)$$

The term on the right hand side of (2.44) is continuous. The first term on the left hand side of (2.44) can be integrated by parts twice.

$$\left.\frac{d\psi_1}{dr}\right|_{c-}^{c+} + \left.\frac{\psi_1}{r}\right|_{c-}^{c+} + \int_{c-}^{c+} \frac{\psi_1}{r^2}dr - \int_{c-}^{c^+} \frac{\psi_1}{r^2}dr + \frac{2\psi_1(c)}{c} = 0 \quad (2.45)$$

Applying the continuity of ψ_1 at $r = c$, it becomes evident that the delta function in $\frac{dJ(r)}{dr}$ requires there to be a step in $\frac{d\psi_1}{dr}$ at $r = c$.

$$\frac{d\hat{\psi}_1}{dr} - \frac{d\psi_1}{dr} = -\frac{2\psi_1(c)}{c} \quad (2.46)$$

Now apply the jump conditions to the solutions ψ_1 and $\hat{\psi}_1$. This gives two equations in two unknowns.

$$\frac{cB_{\theta c}}{2}c\ln\alpha + c_2(c - \frac{a^2}{c}) - \frac{B_{\theta c}}{8}(1 + 4\beta_p)c^2 - c_1 c = 0 \quad (2.47)$$

$$c_1 - c_2(1 - \frac{a^2}{c^2}) = \frac{cB_{\theta c}}{2}\left[\ln\alpha - \frac{1}{4}(1 + 4\beta_c)\right] \quad (2.48)$$

The algebra is sufficiently simple that the steps are omitted.

$$c_1 = -\frac{B_{\theta c}c}{2\alpha^2}\left[\beta_p + \frac{\ell_i}{2} - \frac{1}{2}(1 - \alpha^2)\right] \quad (2.49)$$

$$c_2 = \frac{B_{\theta c}c}{2}\left[\beta_p + \frac{\ell_i}{2} - \frac{1}{2}\right] \quad (2.50)$$

ψ_1 is now completely determined.

$$\psi_1(r) = \frac{B_{\theta c}}{8c}(1 + 4\beta_p)r^3 - \frac{cB_{\theta c}}{2\alpha^2}\left[\beta_p + \frac{\ell_i}{2} - \frac{1}{2}(1 - \alpha^2)\right]r \quad (2.51)$$

$$\hat{\psi}_1(r) = \frac{cB_{\theta c}c}{2}r\ln\frac{r}{a} + \frac{cB_{\theta c}}{2}\left[\beta_p + \frac{\ell_i}{2} - \frac{1}{2}\right](r - \frac{a^2}{r}) \quad (2.52)$$

Having ψ_1 and $\hat{\psi}_1$, it is now possible to obtain the first order magnetic fields measured by the probes at $r = b$. After the discussion in Chapter 1, the poloidal magnetic field is exactly $B_p = \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi$. Taylor expanding the $1/R$, substituting the perturbed solution for $\psi = \psi_0 + \psi_1 + \psi_2 + \ldots$ and collecting terms of comparable order, the first order fields are given below.

$$\mathbf{B}_{1r} = \frac{1}{R_0}\frac{1}{r}\frac{\partial\psi_1(r,\theta)}{\partial\theta}$$

$$\mathbf{B}_{1r} = -\frac{1}{R_0}\frac{\psi_1}{r}\sin\theta \tag{2.53}$$

$$\mathbf{B}_\theta = \frac{1}{R}\frac{\partial\psi}{\partial r} = \frac{1}{R_0}\frac{\partial\psi_0}{\partial r} + \left(\frac{1}{R_0}\frac{\partial\psi_1}{\partial r} - \frac{1}{R_0^2}r\frac{\partial\psi_0}{\partial r}\right)\cos\theta$$

$$\mathbf{B}_{1\theta} = \left[\frac{1}{R_0}\frac{\partial\psi_1}{\partial r} - \frac{r}{R_0}B_\theta(r)\right]\cos\theta \tag{2.54}$$

Substitute (2.52) and (2.31) into (2.53) and (2.54). Evaluate the expressions at $r = b$.

$$|\hat{B}_{1r}| = \frac{\mu_0 I_p}{4\pi R_0}\left[(\beta_p + \frac{\ell_i}{2} - \frac{1}{2})(1 - \frac{a^2}{b^2}) + \ln\frac{b}{a}\right] \tag{2.55}$$

$$|\hat{B}_{1\theta}| = \frac{\mu_0 I_p}{4\pi R_0}\left[(\beta_p + \frac{\ell_i}{2} - \frac{1}{2})(1 + \frac{a^2}{b^2}) + \ln\frac{b}{a} - 1\right] \tag{2.56}$$

Note that both field amplitudes only specify the combination $\beta_p + \frac{\ell_i}{2}$ uniquely. They give the same information. It can be shown that taking the combination $\hat{B}_1 \equiv |\hat{B}_{\theta 1}(b)| - |\hat{B}_{r1}(b)|$ subtracts out any shift information wrapped up in the first order fields. This combination is included here for reference only as the Shafranov shift Δ_a has already been set to zero for convenience.

$$\hat{B}_1 = -\frac{\mu_0 I_p}{4\pi R_0}\left[1 - \frac{2a^2}{b^2}\left(\beta_p + \frac{\ell_i - 1}{2}\right)\right] \tag{2.57}$$

Again note that only the **combination** $\beta_p + \frac{\ell_i}{2}$ can be found from the data. The first order field measurements do not specify β_p and ℓ_i separately. Although it would seem that having determined the plasma current I_p from zeroth order measurements and knowing the geometry, the two field measurements, $\hat{B}_{1r}$ and $\hat{B}_{1\theta}$, are sufficient to determine β_p and ℓ_i separately, they are not. β_p and ℓ_i relate to the first order measurements in a linearly **dependent** fashion.

2.4 The Second Order Solution

Next, turn to Eq. (2.22) for ψ_2. Perhaps the second order field measurements can supply the additional information necessary to find β_p and ℓ_i. Focus on $S(r)$ in the region $r < c$. Substitute the expression for ψ_1 in that region and the given profiles.

$$S(r) = \frac{1}{2R_0}\left\{\frac{d\psi_1}{dr} - \frac{\psi_1}{r} - rB_\theta(r) - \frac{\mu_0 r^2}{B_\theta(r)}\frac{dp}{dr} - \frac{2\mu_0 r\psi_1}{B_\theta}\frac{d}{dr}\left(\frac{1}{B_\theta}\frac{dp}{dr}\right) + \frac{\psi_1^2}{2B_\theta}\frac{d}{dr}\left(\frac{\mu_0}{B_\theta}\frac{dJ}{dr}\right)\right\}$$

Since $\frac{dJ}{dr} = 0$, the last term on the right hand side vanishes.

$$S(r) = \frac{1}{2R_0}\left\{\frac{3B_{\theta c}}{8c}(1+4\beta_p)r^2 + c_1 - \frac{B_{\theta c}}{8c}(1+4\beta_p)r^2 - c_1 - \frac{B_{\theta c}}{c}r^2 - \mu_0 r^2\frac{c}{B_{\theta c}r}\left(-\frac{2p_0 r}{c^2}\right)\right.$$

$$S(r) = \frac{1}{2R_0}\left\{\frac{B_{\theta c}}{4c}(1+4\beta_p)r^2 - \frac{B_{\theta c}}{c}r^2 + \frac{2\mu_0 p}{B_{\theta c}c}r^2\right\}$$

$$S(r) = \frac{B_{\theta c}}{2R_0 c}\left\{\frac{1}{4} + \beta_p - 1 + 2\frac{\mu_0 p}{B_{\theta c}^2}\right\}r^2$$

$$S(r) = \frac{B_{\theta c}}{2R_0 c}\left\{3\beta_p - \frac{3}{4}\right\}r^2$$

$$S(r) = \frac{3B_{\theta c}}{2R_0 c}\left\{\beta_p - \frac{1}{4}\right\}r^2 \qquad r < c \tag{2.58}$$

For the purpose of calculating the amplitude of the second harmonic that appears in second order, Eq. (2.22) becomes

$$\nabla^2\psi_2 = \frac{3B_{\theta c}}{2R_0 c}\left(\beta_p - \frac{1}{4}\right)r^2\cos 2\theta \tag{2.59}$$

A solution of the form $\psi_2(r,\theta) = \psi_2(r)\cos 2\theta$ is sought. Substituting this form of the solution into (2.59) converts a second order partial differential equation into a second order linear ordinary differential equation which is trivial to solve.

$$\frac{d^2\psi_2}{dr^2} + \frac{d\psi_2}{dr} - \frac{4}{r^2}\psi_2 = \frac{3}{2}\frac{B_{\theta c}}{R_0 c}\left(\beta_p - \frac{1}{4}\right)r^2 \tag{2.60}$$

Immediately one can write down the solution in the following convenient form

$$\psi_2(r) = \frac{B_{\theta c}}{8R_0 c}\left(\beta_p - \frac{1}{4}\right)r^4 + b_1 r^2 \tag{2.61}$$

The same procedure is followed in order to find $\hat{\psi}_2(r,\theta)$. This time $\hat{\psi}_1(r)$ and the appropriate profiles for $r > c$ must be used to compute $S(r)$ in this region.

$$S(r) = \frac{1}{2R_0}\left\{\frac{d\hat{\psi}_1}{dr} - \frac{\hat{\psi}_1}{r} - rB_\theta(r) - \frac{\mu_0 r^2}{B_\theta(r)}\frac{dp}{dr} - \frac{2\mu_0 r\hat{\psi}_1}{B_\theta(r)}\frac{d}{dr}\left(\frac{1}{B_\theta}\frac{dp}{dr}\right) + \frac{\hat{\psi}_1^2}{2B_\theta}\frac{d}{dr}\left(\frac{\mu_0}{B_\theta(r)}\frac{dJ}{dr}\right)\right\}$$

Again the last term on the right hand side vanishes because $J(r) = 0$ for $r = c$. Also the pressure $p(r)$ as well as its derivative $\frac{dp(r)}{dr}$ are zero in this region. $S(r)$ simplifies greatly.

$$S(r) = \frac{1}{2R_0}\left\{\frac{d\hat{\psi}_1}{dr} - \frac{\hat{\psi}_1}{r} - rB_\theta(r)\right\}$$

$$S(r) = \frac{1}{2R_0}\left\{\frac{cB_{\theta c}}{2}\frac{r}{a}\frac{a}{r} + \frac{cB_{\theta c}}{2}\ln\frac{r}{a} + c_2 + c_2\frac{a^2}{r^2} - \frac{cB_{\theta c}}{2}\ln\frac{r}{a} - c_2 + c_2\frac{a^2}{r^2} - r\frac{B_{\theta c}c}{r}\right\}$$

$$S(r) = \frac{1}{2R_0}\left\{2c_2\frac{a^2}{r^2} - \frac{cB_{\theta c}}{2}\right\}$$

$$S(r) = \frac{1}{2R_0}\left\{\frac{2B_{\theta c}c}{2}\left[\beta_p + \frac{\ell_i - 1}{2}\right]\frac{a^2}{r^2} - \frac{B_{\theta c}c}{2}\right\}$$

$$S(r) = \frac{B_{\theta c}c}{2R_0}\left\{\left[\beta_p + \frac{\ell_i - 1}{2}\right]\frac{a^2}{r^2} - \frac{1}{2}\right\} \qquad r > c \tag{2.62}$$

For the purpose of calculating the amplitude of the second harmonic that appears in second order in the region $r > c$, Eq. (2.22) becomes

$$\nabla^2\hat{\psi}_2 = \frac{B_{\theta c}c}{2R_0}\left\{\left[\beta_p + \frac{\ell_i - 1}{2}\right]\frac{a^2}{r^2} - \frac{1}{2}\right\} \tag{2.63a}$$

A solution of the form $\hat{\psi}_2(r,\theta) = \hat{\psi}_2(r)\cos 2\theta$ is sought. Substituting this form of the solution into (2.59) converts a second order partial differential equation into a second order linear ordinary differential equation.

$$\frac{d^2\hat{\psi}_2}{dr^2} + \frac{1}{r}\frac{d\hat{\psi}_2}{dr} - \frac{4\hat{\psi}_2}{r^2} = \frac{B_{\theta c}c}{2R_0}\left[\left(\beta_p + \frac{\ell_i - 1}{2}\right)\frac{a^2}{r^2} - \frac{1}{2}\right] \tag{2.63b}$$

The solution is expressed most conveniently below.

$$\hat{\psi}_2(r) = -\frac{B_{\theta c}c}{8R_0}\left[\left(\beta_p + \frac{\ell_i - 1}{2}\right)a^2 + \frac{r^2}{2}\ln\frac{r}{a}\right] + b_2 r^2 + \frac{b_3}{r^2} \tag{2.64a}$$

At this point in the calculation of ψ, there are three undetermined coefficients b_1, b_2 and b_3. Application of the jump conditions at $r = c$ and the boundary conditions at $r = a$ will fix these three coefficients. Up to this point in the analysis, the microscopic details of the calculations have been omitted as they were for the most part trivial. From this point on however the algebra becomes both subtle and cumbrous and therefore it is worthwhile to include each step.

First, determine the jump conditions on ψ_2 and $\hat{\psi}_2$ across $r = c$. Again, return to Eq. (2.22) and rewrite it in the following form.

$$\frac{d^2\psi_2}{dr^2} + \frac{1}{r}\frac{d\psi_2}{dr} - \frac{4}{r^2}\psi_2 - \frac{\mu_0}{B_\theta(r)}\frac{dJ}{dr}\psi_2 =$$

$$\frac{1}{2R_0}\left[\frac{d\psi_1}{dr} - \frac{1}{r}\psi_1 - rB_\theta(r) - \mu_0 r^2\frac{dP}{dr} - \frac{2\mu_0 r\psi_1}{B_\theta(r)}\frac{d}{dr}\frac{1}{B_\theta(r)}\frac{dp}{dr} + \frac{\psi_1^2}{2B_\theta(r)}\frac{d}{dr}\left(\frac{\mu_0}{B_\theta(r)}\frac{dJ}{dr}\right)\right] \tag{2.64b}$$

Considering the step behavior of $J(r)$ and $\frac{dp(r)}{dr}$, one finds that the jump conditions can be determined quickly if $\psi_2(r)$ near $r = c$ is expressed in the following form

$$\psi_2(r) = AJ(r) + Bp(r) + \hat{\psi}_2(c^+) + \text{ smooth functions } \quad \rightarrow 0 \text{ as } r \rightarrow c \tag{2.64c}$$

Substitute this form of $\psi_2(r)$ into (2.64) in order to determine the constants A and B.

$$A\frac{d^2J}{dr^2} + A\frac{1}{r}\frac{dJ}{dr} - A\frac{\mu_0}{B_\theta(r)}J\frac{dJ}{dr} + B\frac{d^2p}{dr^2} - B\frac{\mu_0}{B_\theta(r)}P\frac{dJ}{dr} - \frac{\mu_0}{B_\theta(r)}\hat{\psi}_2(c^+)\frac{dJ}{dr} =$$

$$\frac{1}{2R_0}\left[-\frac{2\mu_0 r\psi_1}{B_\theta^2(r)}\frac{d^2p}{dr^2} + \frac{\mu_0\psi_1^2}{2B_\theta(r)}\left(\frac{1}{B_\theta(r)}\frac{d^2J}{dr^2} - \frac{1}{B_\theta^2(r)}\frac{dJ}{dr}\frac{dB_\theta}{dr}\right)\right] \tag{2.65}$$

$$A\left[\frac{d^2J}{dr^2} + \frac{1}{r}\frac{dJ}{dr} - \frac{\mu_0}{B_\theta}J\frac{dJ}{dr}\right] + B\left[\frac{d^2p}{dr^2} - \frac{\mu_0}{B_\theta}\frac{dJ}{dr}P\right] = \frac{\mu_0}{B_\theta}\frac{dJ}{dr}\hat{\psi}_2(c^+)$$

$$-\frac{\mu_0 r\psi_1}{R_0 B_\theta^2}\frac{d^2p}{dr^2} + \frac{\mu_0\psi_1^2}{4R_0 B_\theta^2}\left[\frac{d^2J}{dr^2} + \frac{1}{r}\frac{dJ}{dr} - \frac{\mu_o}{B_\theta}J\frac{dJ}{dr}\right]$$

Find A and B in terms of $\hat{\psi}_2$, $B_{\theta c}$, P_0 and the normalized first order flux $\overline{\psi}_1 \equiv \frac{\psi_1(c)}{c^2 B_{\theta c}^2}$.

$$A = \frac{\mu_0 c^4}{4R_0}\overline{\psi}_1^2 \tag{2.66}$$

$$B\frac{d^2p}{dr^2} = \frac{\mu_0}{B_{\theta c}}\frac{dJ}{dr}\hat{\psi}_2 - \frac{\mu_0 c\psi_1}{R_0 B_{\theta c}^2}\frac{d^2p}{dr^2}$$

Recall $J = J_0\theta(c-r)$ and $\frac{dp}{dr} = -\frac{2p_0}{c}\theta(c-r)$ where $\theta(c-r)$ is the heaviside step function. Therefore, $\frac{dJ}{dr}$ can be written in terms of $\frac{d^2p}{dr^2}$.

$$\frac{dJ}{dr} = -\frac{cJ_0}{2p_0}\frac{d^2p}{dr^2} \tag{2.67}$$

$$B\frac{d^2p}{dr^2} = \frac{\mu_0}{B_{\theta c}}\left(-\frac{cJ_0}{2p_0}\right)\frac{d^2p}{dr^2}\hat{\psi}_2 - \frac{\mu_0 c}{R_0 B_{\theta c}^2}\psi_1\frac{d^2p}{dr^2}$$

$$B = -\frac{\mu_0 c J_0}{2B_{\theta c}p_0}\hat{\psi}_2 - \frac{\mu_0 c^3\overline{\psi}_1}{R_0 B_{\theta c}}$$

Rewrite B evaluating $B_{\theta c}$ with Ampere's law around a circular contour at $r = c$.

$$\mu_0 J = \frac{2B_{\theta c}}{c} \tag{2.68}$$

$$B = -\frac{\hat{\psi}_2}{p_0} - \frac{\mu_0 c^3\overline{\psi}_1}{R_0 B_{\theta c}} \tag{2.69}$$

The reason for writing $\psi_2(r)$ in the form of Eq. (2.64b) now becomes transparent.

$$\hat{\psi}_2 - \psi_2 = -AJ_0 \tag{2.70}$$

$$\frac{d\hat{\psi}_2}{dr} - \frac{d\psi_2}{dr} = -B\frac{dp}{dr} \tag{2.71}$$

$$\hat{\psi}_2 - \psi_2 = -\frac{2B_{\theta c}}{\mu_0 c}\frac{\mu_0 c^4\overline{\psi}_1^2}{4R_0}$$

$$\hat{\psi}_2 - \psi_2 = -\frac{c^3 B_{\theta c}}{2R_0}\overline{\psi}_1^2 \tag{2.72}$$

$$\frac{d\hat{\psi}_2}{dr} - \frac{d\psi_2}{dr} = \frac{2p_o}{c}\left(-\frac{\hat{\psi}_2}{p_0} - \frac{\mu_0 c^3\overline{\psi}_1}{R_0 B_{\theta c}}\right)$$

$$\frac{d\hat{\psi}_2}{dr} - \frac{d\psi_2}{dr} = -\frac{2\hat{\psi}_2}{c} - \frac{2c^2}{R_0}B_{\theta c}\beta_p\overline{\psi}_1 \tag{2.73}$$

At this point in the calculation the jump conditions at $r = c$ given by Eqs. (2.72) and (2.73) and the boundary conditions on ψ_2 at $r = a$ given by Eq. (2.27) completely determine the three unknown coefficients b_1, b_2, and b_3. Equations (2.72), (2.73) and (2.27) can be written as follows.

$$\lambda \equiv \beta_p + \frac{\ell_i - 1}{2} \tag{2.74}$$

$$b_2c^2 + \frac{b_3}{c^2} - \frac{cB_{\theta c}}{8R_0}\left(\lambda a^2 + \frac{c^2}{2}\ln\alpha\right) - b_1c^2 - \frac{B_{\theta c}}{8R_0}(\beta_p - \frac{1}{4})c^3 = -\frac{c^3B_{\theta c}}{2R_0}\overline{\psi}_1^2$$

$$2b_2c - 2\frac{b_3}{c^3} - \frac{cB_{\theta c}}{8R_0}\left(c\ln\alpha + \frac{c}{2}\right) - 2b_1c - \frac{B_{\theta c}c^2}{2R_0}(\beta_p - \frac{1}{4}) =$$

$$-\frac{2c^2B_{\theta c}}{R_0}\beta_p\overline{\psi}_1 - \frac{2}{c}\left[(b_2c^2 + \frac{b_3}{c^2}) - \frac{cB_{\theta c}}{8R_0}\left(\lambda a^2 + \frac{c^2}{2}\ln\alpha\right)\right]$$

$$b_2a^2 + \frac{b_3}{a^2} - \frac{cB_{\theta c}}{8R_0}\lambda a^2 = \frac{cB_{\theta c}}{2}R_0(\kappa - 1)$$

Write these equations in matrix form.

$$b_2c^2 + \frac{b^3}{c^2} - b_1c^2 = c_1$$

$$2b_2c^2 - b_1c^2 = c_2$$

$$b_2a^2 + \frac{b_3}{a^2} = c_3$$

$$c_2 = -\frac{2c^2B_{\theta c}}{R_0}\beta_p\overline{\psi}_1\frac{c}{2} + \frac{cB_{\theta c}}{8R_0}(c\ln\alpha + \frac{c}{2})\frac{c}{2} + \frac{B_{\theta c}c^2}{2R_0}(\beta_p - \frac{1}{4})\frac{c}{2}$$

$$+\frac{cB_{\theta c}}{8R_0}(\lambda a^2 + \frac{c^2}{2}\ln\alpha)$$

$$c_2 = \frac{c^3B_{\theta c}}{2R_0}\left[-2\beta_p\overline{\psi}_1 + \frac{1}{8}(\ln\alpha + \frac{1}{2}) + \frac{1}{2}(\beta_p - \frac{1}{4}) + \frac{\lambda}{4}\frac{a^2}{c^2} + \frac{1}{8}\ln\alpha\right]$$

$$c_2 = \frac{c^3B_{\theta c}}{2R_0}\left[-2\beta_p\overline{\psi}_1 + \frac{1}{4}\ln\alpha + \frac{1}{16} + \frac{1}{2}(\beta_p - \frac{1}{4}) + \frac{\lambda}{4}\frac{a^2}{c^2}\right] \tag{2.75}$$

$$c_1 = \frac{c^3B_{\theta c}}{2R_0}\left[-\overline{\psi}_1^2 + \frac{\lambda}{4}\frac{a^2}{c^2} + \frac{1}{8}\ln\alpha + \frac{1}{4}(\beta_p - \frac{1}{4})\right] \tag{2.76}$$

$$c_3 = \frac{c^3B_{\theta c}}{2R_0}\left[\frac{R_0^2}{c^2}(\kappa - 1) + \frac{1}{4}\frac{a^2}{c^2}\lambda\right] \tag{2.77}$$

Solve for $b_2c^2, b_3/c^2, b_1c^2$. That is, write the system of equations developed above as a single matrix equation.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ a^2/c & c^2/a^2 & 0 \end{bmatrix} \begin{bmatrix} b_2c^2 \\ b_3/c^2 \\ b_1c^2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{2.78}$$

c_1, c_2c_3 are known and given by (2.75–2.77). Repeated application of Kramer's rule to (2.78) wll solve for the column vector on the left hand side. However, since the present calculation is aimed at determining $\hat{B}_{2\theta}(b)$ and $\hat{B}_{2r}(b)$, it will only be necessary to solve for b_2 and b_3. The second order magnetic fields are uniquely determined inside the plasma but are not of interest here.

It is a simple matter to calculate the determinant of the 3×3 matrix on the left hand side of (2.78). Then, two applications of Kramer's rule give b_2 and b_3, completely specifying $\hat{\psi}_2$.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ a^2/c^2 & c^2/a^2 & 0 \end{vmatrix} = -\frac{(a^4 + c^4)}{a^2 c^2} \tag{2.79}$$

$$Db_2c^2 = \begin{vmatrix} c_1 & 1 & -1 \\ c_2 & 0 & -1 \\ c_3 & c^2/a^2 & 0 \end{vmatrix} = -[c_3 + \frac{c^2}{a^2}(c_2 - c_1)] \tag{2.80}$$

$$D\frac{b_3}{c^2} = \begin{vmatrix} 1 & c_1 & -1 \\ 2 & c_2 & -1 \\ a^2/c^2 & c_3 & 0 \end{vmatrix} = -[c_3 - \frac{a^2}{c^2}(c_2 - c_1)] \tag{2.81}$$

Note the combination $c_2 - c_1$ appears in both (2.80) and (2.81).

$$c_2 - c_1 = \frac{c^3 B_{\theta c}}{2R_0}\Big[-2\beta_p\overline{\psi}_1 + \frac{1}{4}\ln\alpha + \frac{1}{16} + \frac{1}{2}(\beta_p - \frac{1}{4}) + \frac{\lambda}{4}\frac{a^2}{c^2}$$
$$+\overline{\psi}_1^2 - \frac{\lambda}{4}\frac{a^2}{c^2} - \frac{1}{8}\ln\alpha - \frac{1}{4}(\beta_p - \frac{1}{4})\Big]$$

$$c_2 - c_1 = \frac{c^3 B_{\theta c}}{2R_0}\left[\overline{\psi}_1^2 - 2\beta_p\overline{\psi}_1 + \frac{1}{8}\ln\alpha + \frac{1}{4}\beta_p\right] \tag{2.82}$$

At this point in the calculation it is possible to obtain analytical expressions for the second order field amplitudes measured by the probes. Again the poloidal field can be expressed exactly $\mathbf{B}_p = \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi$. As before, substitute the perturbed solution for $\psi = \psi_0 + \psi_1 + \psi_2 \ldots$ and Taylor expand the $1/R$

$$\hat{B}_r = -\frac{1}{rR}\frac{\partial\hat{\psi}}{\partial\theta}$$

$$\hat{B}_r = \frac{1}{bR_0}\left[\hat{\psi}_1 \sin\theta + 2\hat{\psi}_2 \sin 2\theta\right]\left[1 - \frac{b}{R_0}\cos\theta + \ldots\right]$$

Keep only second order terms with $\sin 2\theta$ dependence.

$$\hat{B}_{r2}(b) = \frac{1}{bR_0}\left[2\hat{\psi}_2(b) - \frac{b\hat{\psi}_1(b)}{2R_0}\right] \tag{2.83}$$

Repeat the same procedure to find $\hat{B}_{\theta 2}(b)$.

$$\hat{B}_{\theta 2} = \frac{1}{R}\frac{\partial \hat{\psi}}{\partial r}$$

$$\hat{B}_{\theta 2} = \frac{1}{R}\left[\frac{\partial \hat{\psi}_0}{\partial r} + \frac{\partial \hat{\psi}_1}{\partial r}\cos\theta + \frac{\partial \hat{\psi}_2}{\partial r}\cos 2\theta\right]\left[1 - \frac{b}{R}\cos\theta + \frac{b^2}{R_0^2}\cos^2\theta\right]$$

$$\hat{B}_{\theta 2}(b) = \frac{1}{R_0}\left[\frac{\partial \hat{\psi}_2}{\partial r} - \frac{b}{2R_0}\frac{\partial \psi_1}{\partial r} + \frac{b^2}{2R_0^2}\frac{\partial \hat{\psi}_0}{\partial r}\right] \tag{2.84}$$

Turn to Eq. (2.83) and evaluate each term.

$$-\frac{1}{2R_0^2}\hat{\psi}_1(b) = -\frac{1}{2R_0^2}\left[\frac{cB_{\theta c}}{2} b\ln\frac{b}{a} + b(1-\frac{a^2}{b^2})\frac{cB_{\theta c}}{2}\lambda\right]$$

$$-\frac{1}{2R_0^2}\hat{\psi}_1(b) = -\frac{cbB_{\theta c}}{4R_0^2}\left[\ln\frac{b}{a} + (1-\frac{a^2}{b^2})\lambda\right]$$

$$-\frac{1}{2R_0^2}\hat{\psi}_1(b) = -\frac{\mu_0 I_p}{8\pi R_0}\frac{b}{R_0}\left[\ln\frac{b}{a} + (1-\frac{a^2}{b^2})\lambda\right] \tag{2.85}$$

$$\frac{2}{bR_0}\hat{\psi}_2(b) = \frac{2}{bR_0}\left[b_2c^2\left(\frac{b^2}{c^2}\right) + \frac{b_3}{c^2}\left(\frac{c^2}{b^2}\right) - \frac{cB_{\theta c}}{8R_0}(\lambda a^2 + \frac{b^2}{2}\ln\frac{b}{a})\right]$$

$$\frac{2}{bR_0}\hat{\psi}_2(b) = T_a + T_b \tag{2.86}$$

where

$$T_a \equiv \frac{2}{bR_0}\left[\frac{b^2}{c^2}b_2c^2 + \frac{c^2}{b^2}\frac{b_3}{c^2}\right] \tag{2.87}$$

$$T_b \equiv \frac{-2}{bR_0}\frac{cB_{\theta c}}{8R_0}b^2\left[\frac{1}{2}\ln\frac{b}{a} + \lambda\frac{a^2}{b^2}\right] \tag{2.88}$$

Evaluate T_a, then T_b.

$$T_a = \frac{2}{DbR_0}\left[\frac{b^2}{c^2}(-c_3 - \frac{c^2}{a^2}(c_2 - c_1)) + \frac{c^2}{b^2}(-c_3 + \frac{a^2}{c^2}(c_2 - c_1))\right]$$

$$T_a = \frac{2}{DbR_0}\left[-\frac{b^4+c^4}{b^2c^2}c_3 - \left(\frac{b^2}{a^2} - \frac{a^2}{b^2}\right)(c_2 - c_1)\right]$$

$$T_a = \frac{-2}{DbR_0}\left[\frac{b^4+c^4}{b^2c^2}c_3 + \frac{b^4-a^4}{a^2b^2}(c_2 - c_1)\right]$$

$$T_a = \frac{-2}{DbR_0}\left[\frac{b^4+c^4}{b^2c^2}\frac{c^3B_{\theta c}}{2R_0}\left(\frac{R_0^2}{c^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{c^2}\lambda\right)\right.$$

$$\left.+\frac{b^4-a^4}{a^2b^2}\frac{c^3B_{\theta c}}{2R_0}\left(\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p\right)\right]$$

$$T_a = \frac{\mu_0 I}{2\pi R_0}\frac{b}{R_0}\frac{a^2c^2}{a^4+c^4}\frac{c^2}{b^2}\left[\frac{b^4+c^4}{b^2c^2}\left(\frac{R_0^2}{c^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{c^2}\lambda\right)\right.$$

$$\left.+\frac{b^4-a^4}{a^2b^2}\left(\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p\right)\right] \tag{2.89}$$

$$T_b = \frac{-\mu_0 I}{8\pi R_0}\frac{b}{R_0}\left(\frac{1}{2}\ln\frac{b}{a}+\lambda\frac{a^2}{b^2}\right) \tag{2.90}$$

Recall from Eq. (2.66) that $\overline{\psi}_1 = \frac{\hat{\psi}_1(c)}{c^2B_{\theta c}}$.

$$\overline{\psi}_1 = \frac{1}{2}\ln\alpha+\frac{1}{2}(1-\frac{a^2}{c^2})\lambda \tag{2.91}$$

$$\hat{B}_{r2}(b) = -\frac{1}{2R_0^2}\hat{\psi}_1(b)+T_a+T_b$$

Define the dimensionless quantity b_{r2}.

$$b_{r2} = \frac{\hat{B}_{r2}(b)}{\mu_o I_p/8\pi R_0}\frac{b}{R_0} \tag{2.92}$$

$$b_{r2} = -\ln\frac{b}{a}-(1-\frac{a^2}{b^2})\lambda-\frac{1}{2}\ln\frac{b}{a}-\lambda\frac{a^2}{b^2}+\frac{4a^2c^2}{a^4+c^4}\frac{c^2}{b^2}\frac{b^4+c^4}{b^2c^2}\left[\frac{R_0^2}{c^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{c^2}\lambda\right]$$

$$+\frac{4a^2c^2}{a^4+c^4}\frac{c^2}{b^2}\frac{b^4-a^4}{a^2b^2}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p\right] \tag{2.93}$$

$$b_{r2} = -\frac{3}{2}\ln\frac{b}{a}-\lambda+4\frac{b^4+c^4}{a^4+c^4}\frac{a^2}{b^2}\left[\frac{R_0^2}{b^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{b^2}\lambda\right]+4\frac{b^4-a^4}{a^4+c^4}\frac{c^4}{b^4}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p\right] \tag{2.94}$$

For the interesting case when $b = a$, that is when the probes are on the plasma surface, b_{r2} reduces to the following simple form.

$$b_{r2} = 4\frac{R_0^2}{a^2}(\kappa-1) \qquad b = a \tag{2.95}$$

It is also useful to examine the opposite limit $b \gg a$, the case when the probes are located far away from the plasma surface.

$$b_{r2} \approx -\frac{3}{2}\ln\frac{b}{a} - \lambda + \frac{4b^4a^2}{(a^4+c^4)b^2}\frac{1}{b^2}\left[R_0^2(\kappa-1)+\frac{1}{4}a^2\lambda\right] + \frac{4c^4}{a^4+c^4}\left[\overline{\psi}_1^2 - 2\beta_p\overline{\psi}_1 + \frac{1}{8}\ln\alpha + \frac{1}{4}\beta_p\right]$$

$$b_{r2} \approx -\frac{3}{2}\ln\frac{b}{a} + \frac{4a^4}{a^4+c^4}\frac{R_0^2(\kappa-1)}{a^2} + \frac{4c^4}{a^4+c^4}\left[\overline{\psi}_1^2 - 2\beta_p\overline{\psi}_1 + \frac{1}{8}\ln\alpha + \frac{\beta_p}{4} - \frac{\lambda}{4}\right] \qquad b \gg a \tag{2.96}$$

Now unfold the algebra in Eq. (2.84) in order to obtain analytic expressions for the second order second harmonic tangential field.

$$\hat{B}_{2\theta}(b) = \frac{1}{R_0}\left[\frac{\partial\hat{\psi}_2}{\partial r} - \frac{b}{2R_0}\frac{\partial\hat{\psi}_1}{\partial r} + \frac{b^2}{2R_0^2}\frac{\partial\hat{\psi}_0}{\partial r}\right]\Bigg|_b$$

$$T_1 \equiv \frac{b^2}{2R_0^3}\frac{\partial\hat{\psi}_0}{\partial r}\Bigg|_b \tag{2.97}$$

$$T_2 \equiv -\frac{b}{2R_0^2}\frac{\partial\hat{\psi}_1}{\partial r}\Bigg|_b \tag{2.98}$$

$$T_3 = \frac{1}{R_0}\frac{\partial\hat{\psi}_2}{\partial r}\Bigg|_b \tag{2.99}$$

$$T_1 = \frac{\mu_0 I}{4\pi R_0}\frac{b}{R_0} \tag{2.100}$$

$$T_2 = -\frac{b}{2R_0^2}\left[\frac{cB_{\theta c}}{2}\left(\ln\frac{b}{a}+1\right) + \left(1+\frac{a^2}{b^2}\right)\frac{cB_{\theta c}}{2}\lambda\right]$$

$$T_2 - \frac{b}{2R_0^2}\frac{cB_{\theta c}}{2}\left[\ln\frac{b}{a} + 1 + (1+\frac{a^2}{b^2})\lambda\right] \tag{2.101}$$

$$T_3 = \frac{1}{R_0}\left[-\frac{cB_{\theta c}}{8R_0}(b\ln\frac{b}{a}+\frac{b}{2}) + 2b_2b - \frac{2b_3}{b^3}\right]$$

$$T_3 = -\frac{\mu_0 I}{16\pi R_0}\frac{b}{R_0}(\ln\frac{b}{a}+\frac{1}{2}) + \frac{2}{R_0}\left[b_2c^2\frac{b}{c^2} - \frac{b_3}{c^2}\frac{c^2}{b^3}\right]$$

$$T_3 = -\frac{\mu_0 I}{16\pi R_0}\frac{b}{R_0}(\ln\frac{b}{a}+\frac{1}{2})$$
$$-\frac{2}{DbR_0}\left[\frac{b^2}{c^2}(c_3 - \frac{c^2}{a^2}(c_2-c_1)) + \frac{c^2}{b^2}(-c_3 + \frac{a^2}{c^2}(c_2-c_1))\right]$$

$$T_3 = -\frac{\mu_0 I}{16\pi R_0}\frac{b}{R_0}(\ln\frac{b}{a}+\frac{1}{2})$$

$$-\frac{2}{DbR_0}\left[(\frac{b^2}{c^2}-\frac{c^2}{b^2})c_3+(\frac{b^2}{a^2}+\frac{a^2}{b^2})(c_2-c_1)\right]$$

$$T_3 = -\frac{\mu_o I}{16\pi R_0}\frac{b}{R_0}(\ln\frac{b}{a}+\frac{1}{2})$$

$$-\frac{2}{DbR_0}\left[\frac{b^4-c^4}{c^2b^2}\frac{c^3B_{\theta c}}{2R_0}\left(\frac{R_0^2}{c^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{c^2}\lambda\right)+\frac{b^4+a^4}{a^2b^2}\frac{c^3B_{\theta c}}{2R_0}(\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p)\right]$$

$$T_3 = -\frac{\mu_0 I}{16\pi R_0}\frac{b}{R_0}(\ln\frac{b}{a}+\frac{1}{2})$$

$$+\frac{\mu_0 I}{2\pi R_0}\frac{b}{R_0}\left[\frac{a^2}{b^2}\frac{c^4}{a^4+c^4}\frac{b^4-c^4}{c^2b^2}\left(\frac{R_0^2(\kappa-1)}{c^2}+\frac{1}{4}\frac{a^2}{c^2}\lambda\right)\right.$$

$$\left.+\frac{a^2}{b^2}\frac{c^4}{a^4+c^4}\frac{b^4+a^4}{a^2b^2}(\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p)\right]$$

$$T_3 = -\frac{\mu_0 I}{16\pi R_0}\frac{b}{R_0}(\ln\frac{b}{a}+\frac{1}{2})$$

$$+\frac{\mu_0 I}{2\pi R_0}\frac{b}{R_0}\left[\frac{a^2}{b^2}\frac{b^4-c^4}{a^4+c^4}\left(\frac{R_0^2}{b^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{b^2}\lambda\right)\right.$$

$$\left.+\frac{c^4}{b^4}\frac{b^4+a^4}{a^4+c^4}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{1}{4}\beta_p\right]\right] \qquad (2.102)$$

$$\hat{B}_{2\theta}(b) = T_1+T_2+T_3$$

Define the dimensionless quantity $b_{\theta 2}$.

$$b_{\theta 2} = \frac{\hat{B}_{\theta 2}(b)}{\mu_0 I/8\pi R_0}\frac{b}{R_0} \qquad (2.103)$$

Combining terms and normalizing properly $b_{\theta 2}$ can be written in the following form.

$$b_{\theta 2} = \frac{3}{4}-\frac{3}{2}\ln\frac{b}{a}-(1+\frac{a^2}{b^2})\lambda+\frac{4a^2}{b^2}\frac{b^4-c^4}{a^4+c^4}\left(\frac{R_0^2}{b^2}(\kappa-1)+\frac{1}{4}\frac{a^2}{b^2}\lambda\right) \qquad (2.104)$$

$$+\frac{4c^4}{b^4}\frac{b^4+a^4}{a^4+c^4}(\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{\beta_p}{4})$$

Before examining $b_{\theta 2}$ in the limits $b=a$ and $b \gg a$, it is necessary to simplify b_{r2} and $b_{\theta 2}$ once more.

Combine λ terms in (2.94) and (2.104).

$$b_{r2}: \quad \lambda\left[-1+\frac{a^4}{b^4}\frac{b^4+c^4}{a^4+c^4}\right]=\lambda\left[-\frac{(b^4-a^4)\,c^4}{(a^4-c^4)\,b^4}\right] \tag{2.105}$$

$$b_{\theta 2}: \quad \lambda\left[-\frac{a^2}{b^2}-1+\frac{a^4}{b^4}\frac{b^4-c^4}{a^4c^4}\right]=\lambda\left[-\frac{a^2}{b^2}-\frac{(b^4+a^4)\,c^4}{(a^4+c^4)\,b^4}\right] \tag{2.106}$$

Simplify λ.

$$\lambda=\beta_p+\ell_i-\frac{1}{2}=\beta_p-\ln\alpha-\frac{1}{4} \tag{2.107}$$

Also

$$\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{1}{8}\ln\alpha+\frac{\beta_p}{4}+\frac{\lambda}{4}=\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{3}{8}\ln\alpha+\frac{1}{16} \tag{2.108}$$

$\overline{\psi}_1$ can also be simplified.

$$\overline{\psi}_1=\frac{1}{2}\left[\left(1-\frac{1}{\alpha^2}\right)\left(\beta_p-\frac{1}{4}\right)+\frac{1}{\alpha^2}\ln\alpha\right] \tag{2.109}$$

Rewrite b_{r2} and $b_{\theta 2}$ using (2.105–2.109).

$$\epsilon\equiv\frac{a}{R_0}$$

$$b_{r2}=-\frac{3}{2}\ln\frac{b}{a}+4\left(\frac{b^4+a^4}{a^4+c^4}\right)\frac{a^4}{b^4}\left(\frac{\kappa-1}{\epsilon^2}\right)+4\left(\frac{b^4-a^4}{a^4+c^4}\right)\frac{c^4}{b^4}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{3}{8}\ln\alpha+\frac{1}{16}\right] \tag{2.110}$$

$$b_{\theta 2}=-\frac{3}{2}\ln\frac{b}{a}+\frac{3}{4}-\frac{a^2}{b^2}\lambda+4\left(\frac{b^4-c^4}{a^4+c^4}\right)\frac{a^4}{b^4}\left(\frac{\kappa-1}{\epsilon^2}\right)+4\left(\frac{b^4+a^4}{a^4+c^4}\right)\frac{c^4}{b^4}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{3}{8}\ln\alpha+\frac{1}{16}\right] \tag{2.111}$$

Consider $b \gg a$, that is when the measurements surface is far from the edge of the plasma.

$$b_{r2}\approx-\frac{3}{2}\ln\frac{b}{a}+\frac{4}{1+\alpha^2}\left(\frac{\kappa-1}{\epsilon^2}\right)+\frac{4\alpha^2}{1+\alpha^4}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{3}{8}\ln\alpha+\frac{1}{16}\right] \tag{2.112}$$

$$b_{\theta 2}\approx-\frac{3}{2}\ln\frac{b}{a}+\frac{3}{4}+\frac{4}{1+\alpha^2}\left(\frac{\kappa-1}{\epsilon^2}\right)+\frac{4\alpha^2}{1+\alpha^4}\left[\overline{\psi}_1^2-2\beta_p\overline{\psi}_1+\frac{3}{8}\ln\alpha+\frac{1}{16}\right] \tag{2.113}$$

Notice that b_{r2} and $b_{\theta 2}$ have exactly the same dependence on κ (also unknown) β_p, and α. Hence, profile effects cannot be separated from the ellipticity for measurements of $\hat{B}_{r2}$ and $\hat{B}_{\theta 2}$ far away from the plasma edge. β_p and ℓ_i are still not uniquely determined.

Consider $b \rightarrow a$, the opposite limit, when the magnetic probes are placed close to the plasma edge.

$$b_{r2} = 4\left(\frac{\kappa - 1}{\epsilon^2}\right) \tag{2.114}$$

$$b_{\theta 2} = \frac{3}{4} - \lambda + 4\left(\frac{1-\alpha^4}{1+\alpha^4}\right)\left(\frac{\kappa-1}{\epsilon^2}\right) + \frac{8\alpha^4}{1+\alpha^4}\left[\overline{\psi}_1^2 - 2\beta_p\overline{\psi}_1 + \frac{3}{8}\ln\alpha + \frac{1}{16}\right] \tag{2.115}$$

Consider (2.114) and (2.115) in the limit $\alpha \approx 1$ flat profiles. Let $\alpha = 1 - \delta, \delta \ll 1$.

$$\overline{\psi}_1 \approx \frac{1}{2}\left[-\delta + \lambda(1 - 1 + 2\delta)\right] = \frac{1}{2}(2\lambda - 1)\delta \tag{2.116}$$

Rewrite (2.114) and (2.115) in this limit.

$$b_{r2} = \frac{4(\kappa - 1)}{\epsilon^2} \tag{2.117}$$

$$b_{\theta 2} = \frac{3}{4} - \lambda + \frac{16\delta}{2}\left(\frac{\kappa-1}{\epsilon^2}\right) + 4\left[\frac{1}{16} + \frac{3}{8}(-\delta) - 2\beta_p\frac{1}{2}(2\lambda-1)\delta\right]$$

$$b_{\theta 2} = 1 - \lambda + \delta\left[8\left(\frac{\kappa-1}{\epsilon^2}\right) - \frac{3}{2} - 4(2\lambda-1)(\lambda - \frac{1}{4})\right] \tag{2.118}$$

For flat profiles, $\alpha \approx 1$, b_{r2} gives no information about α and hence ℓ_i and $b_{\theta 2}$ gives information but is is a small correction of order δ compared to $1 - \lambda$, already a small quantity. This will be difficult to measure in practice.

Hence, even resor ting to second order magnetic field measurements does not uniquely specify ℓ_i and β_p in the circular limit.

2.5 Summary

The equations of interest are summarized below.

$$\hat{B}_1 = -\frac{\mu_0 I_p}{4\pi R_0}\left[1 - \frac{2a^2}{b^2}\left(\beta_p + \frac{\ell_i - 1}{2}\right)\right] \tag{2.57}$$

For $b \gg a$

$$b_{r2} \approx -\frac{3}{2}\ln\frac{b}{a} + \frac{4}{1+\alpha^2}\left(\frac{\kappa-1}{\epsilon^2}\right) + \frac{4\alpha^2}{1+\alpha^4}\left[\overline{\psi}_1^2 - 2\beta_p\overline{\psi}_1 + \frac{3}{8}\ln\alpha + \frac{1}{16}\right] \tag{2.112}$$

$$b_{\theta 2} \approx -\frac{3}{2}\ln\frac{b}{a} + \frac{3}{4} + \frac{4}{1+\alpha^2}\left(\frac{\kappa-1}{\epsilon^2}\right) + \frac{4\alpha^2}{1+\alpha^4}\left[\overline{\psi}_1^2 - 2\beta_p\overline{\psi}_1 + \frac{3}{8}\ln\alpha + \frac{1}{16}\right] \quad (2.113)$$

For $b \to a$

$$b_{r2} = \frac{4(\kappa-1)}{\epsilon^2} \quad (2.117)$$

$$b_{\theta 2} = 1 - \lambda + \delta\left[8\left(\frac{\kappa-1}{\epsilon^2}\right) - \frac{3}{2} - 4(2\lambda-1)(\lambda - \frac{1}{4})\right] \quad (2.118)$$

It has been shown that for plasmas with circular cross sections with small, second order ellipticities, first order, first harmonic field measurements determine only the combination $\beta_p + \ell_i/2$. Second order, second harmonic field measurements when taken far away from the plasma, cannot separate the ellipticity from the profile effects. The same measurements if taken close to the plasma edge depend so sensitively on already small quantities that experimental errors invalidate them. Therefore, we conclude that even appealing to second order, only the combination $\beta_p + \ell_i/2$ is available to practical magnetic diagnostics for near circular cross sections.

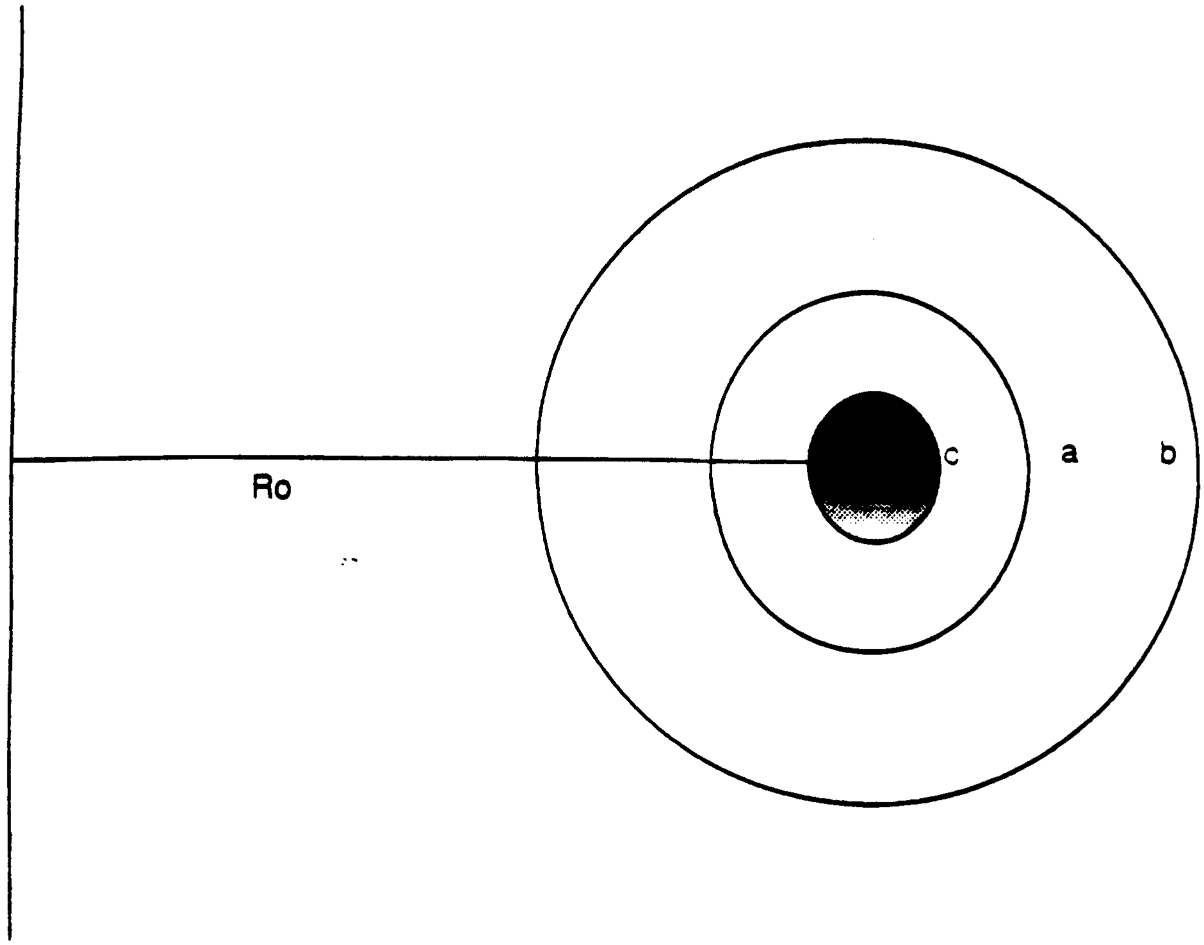


Figure 2.1: Idealized Circular Tokamak

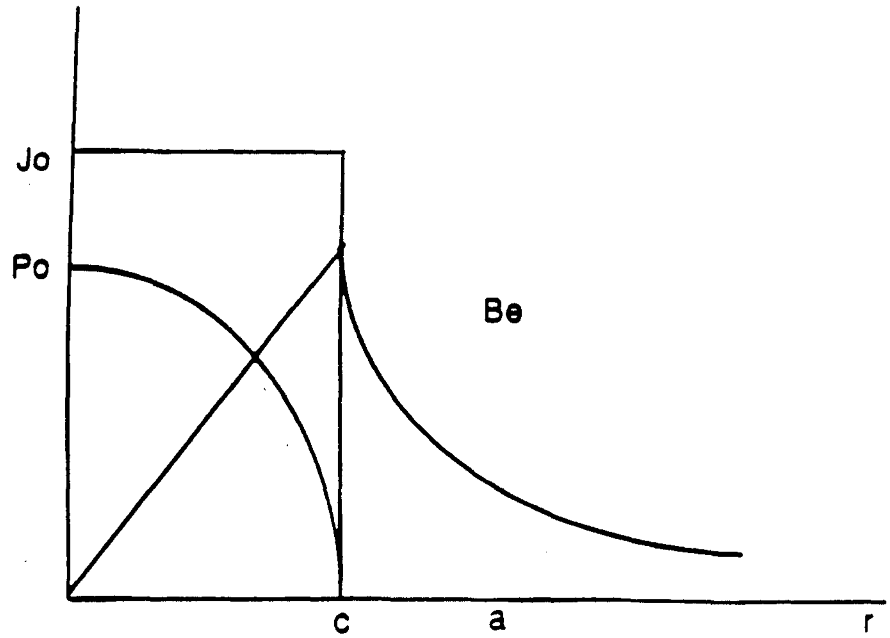


Figure 2.2: Simple Shafranov Profiles

Chapter 3
The Elliptic Limit

3.1 Introduction

In this chapter the Grad-Shafranov equation will be solved to first order in the ohmic tokamak expansion. Then, having explicit formulas for the flux functions $\hat{\psi}_0$ and $\hat{\psi}_1$ outside the plasma, the magnetic fields available to an idealized set of probes are calculated. The dependence of these field amplitudes on β_p and ℓ_i are sought.

3.2 The Zeroth Order Solution

Consider a tokamak of elliptic cross section as illustrated in Fig. 3.1. An elongated plasma limited at a horizontal distance x_b from its center is surrounded by magnetic probes conveniently located on an ellipse characterized by the elliptic coordinate u_m. Before proceeding further, it is useful to review the system of elliptic coordinates that will be used throughout the calculation. The elliptic coordinates are u, v, and ϕ. ϕ is the familiar toroidal angle. Surfaces of constant u are ellipses and v is an angular coordinate varying from 0 to 2π. The transformation from rectangular coordinates to elliptic coordinates is given below.

$$x = c \sinh u \cos v \tag{3.1}$$

$$y = c \cosh u \sin v \tag{3.2}$$

c is a length factor that for the remainder of the problem will be considered **determined** by the actual dimensions and ellipticity of the measurement surface. Solving the two transcendental equations that appear below, knowing the height y_m and width x_m of the measurement surface, uniquely determines c and u_m.

$$x_m = c \sinh u_m \tag{3.3}$$

$$y_m = c \cosh u_m \tag{3.4}$$

For use later in the calculation the two operators ∇ and ∇^2 are given below.

$$\nabla\psi = \frac{1}{c(\frac{\cosh 2u+\cos 2v}{2})^{1/2}}\left(\hat{u}\frac{\partial\psi}{\partial u} + \hat{v}\frac{\partial\psi}{\partial v}\right) + \frac{1}{R}\hat{e}_\phi\frac{\partial\psi}{\partial\phi} \tag{3.5}$$

$$\nabla^2\psi = \frac{1}{c^2(\frac{\cosh 2u+\cos 2v}{2})}\left[\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right] + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{c^2(\frac{\cosh 2u+\cos 2v}{2})R}\left[\frac{\partial R}{\partial u}\frac{\partial\psi}{\partial u} + \frac{\partial R}{\partial v}\frac{\partial\psi}{\partial v}\right] \tag{3.6}$$

As in Chapter 2, the following five plasma parameters are sought.

- I_p total plasma current.
- Δ_a Shafranov shift.
- β_p the poloidal β.
- ℓ_i the normalized internal inductance.
- κ the plasma elongation.

The magnetic probes on the measurement surface u_m sample the tangential and normal magnetic fields during the flat top portion of tokamak operation. It is the aim of this part of the calculation to obtain analytic expressions for the field amplitudes sampled in terms of the plasma parameters sought, thereby trying to uniquely determine β_p and ℓ_i from the field measurements.

Again, the Grad-Shafranov equation describes the plasma equilibrium inside the tokamak.

$$\Delta^*\psi = -\mu_0 R^2\frac{dp}{d\psi} - F\frac{dF}{d\psi} \tag{3.7}$$

$p = p(\psi)$ and $F = F(\psi)$ are free functions that describe the pressure and toroidal field profiles respectively. As in the circular case, assume ohmic tokamak operation and use the appropriate scalings (2.5–2.12) when expanding (3.7) order by order. A perturbed solution for $\psi(u,v)$ is sought.

$$\psi(u,v) = \psi_0(u,v) + \psi_1(u,v) + \dots \tag{3.8}$$

$$\frac{\psi_1}{\psi_0} \sim \epsilon \tag{3.9}$$

This time, however, ϵ, the aspect ratio, is given by the expression below

$$\epsilon = \frac{c}{R_0} \ll 1 \tag{3.10}$$

Analogously to the circular case, $p(\psi)$ and $F^2(\psi)$ are expanded about their ψ_0 values and are linear in ψ_1.

$$p(\psi) = p(\psi_0) + \frac{dp}{d\psi_0}\psi_1 \dots \tag{3.11}$$

$$F^2(\psi) = [B_0^2 + 2B_0B_2(\psi_0) + 2B_0\frac{dB_2}{d\psi_0}\psi_1 \dots] \tag{3.12}$$

$$p(\psi_0) = p_0[1 - \frac{\psi_0(u,v)}{\psi_a}] \tag{3.13}$$

$$B_2(\psi_0) = B_0\alpha[1 - \frac{\psi_0(u,v)}{\psi_a}] \tag{3.14}$$

$$J = J_0 \qquad u \leq u_0 \tag{3.15}$$

$$J = 0 \qquad u > u_0 \tag{3.16}$$

p_0 is the plasma kinetic pressure on axis. B_0 is the toroidal field applied at the edge of the plasma. α represents the paramagnetic rise of toroidal field inside the plasma that characterize ohmic discharges. $\alpha \sim O(\epsilon^2)$. As before, the plasma is modeled as having a hot, current-carrying core and a more diffuse outer region, the area of the former to the latter being some measure of the peakedness of actual, smooth profiles encountered in experiments. ψ_a = const defines the edge of the current carrying core. The core of the plasma is modeled as an ellipse $u = u_0$, of area $\pi\kappa_c a^2$. The x, y coordinates of the core follow immediately.

$$x_c = a = c\sinh u_0 \tag{3.17}$$

$$y_c = \kappa_c a = c\cosh u_0 \tag{3.18}$$

At this point in the calculation the dimensions of the core and hence κ_c and u_0 are unknown.

ψ_0 is not as obvious here as in Chapter 2. In fact, the behavior of ψ_0 is markedly different from the circular case if the ellipticity is zeroth order. Examine (3.7). Expand the right hand side to zeroth order.

$$\Delta^*\psi_0 = -\mu_0 R_0^2(-\frac{p_0}{\psi_a}) + \frac{\alpha B_0^2 R_0^2}{\psi_a}$$

$$\Delta^*\psi_0 = \mu_0 p_0 \frac{R_0^2}{\psi_a} + \frac{\alpha B_0^2 R_0^2}{\psi_a}$$

$$\text{Let} \quad Q \equiv \mu_0 p_0 \frac{R_0^2}{\psi_a} + \frac{\alpha B_0^2 R_0^2}{\psi_a} = const$$

$$\Delta^*\psi_0 = Q \tag{3.19}$$

Now expand the Δ^* operator in the left hand side of (3.19).

$$\nabla^2\psi_0 - \frac{2}{R}\nabla R \cdot \nabla\psi_0 = Q \tag{3.20}$$

The second term on the left hand side of (3.20) is first order and hence should be neglected.

$$\nabla^2\psi_0 = Q \tag{3.21}$$

Curiously, (3.20) is most conveniently solved for the elliptic problem in **rectangular coordinates**. First boundary conditions must be given on the boundary u_0 and at the origin. The boundary of the plasma core u_0, is to be modeled as a flux surface up to and including first order. This specifies the following two conditions on ψ_0 and $\hat{\psi}_0$.

$$\psi_0(u_0, v) = \hat{\psi}_0(u_0, v) = \psi_a = const \tag{3.22}$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0,v} = \left.\frac{\partial\hat{\psi}_0}{\partial u}\right|_{u_0,v} \tag{3.23}$$

ψ_0 must also be regular at the origin.

The equation of the ellipse after which the core is modeled appears below.

$$\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2} = 1 \tag{3.24}$$

Immediately the solution of (3.21) becomes obvious.

$$\psi_0 = \bar{c}\left(\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2}\right) \tag{3.25}$$

Equation (3.25) is regular at the origin and constant on the u_o ellipse.

$$\psi_0(u_0, v) = \bar{c} \tag{3.26}$$

The constant $\bar{c}$ can be determined quite simply from (3.21).

$$\nabla^2\psi_0 = \bar{c}\left(\frac{2}{a^2} + \frac{2}{\kappa_c^2 a^2}\right) = Q$$

$$\bar{c} = \frac{Qa^2}{2}\left(\frac{\kappa_c^2}{\kappa_c^2+1}\right) \tag{3.27}$$

The Grad-Shafranov equation can then be expressed as

$$\Delta^*\psi = -\mu_0 R_0 J_0 = Q \tag{3.28}$$

Observing the convention of defining a positive valued flux function (3.25) can be rewritten in the following form.

$$\psi_0 = \mu_0 R_0 \frac{I_p}{2\pi}\left(\frac{\kappa_c}{\kappa_c^2+1}\right)\left[\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2}\right] \tag{3.29}$$

The zeroth order flux function inside the plasma is fully determined.

The zeroth order flux function in the outer region $u > u_0$, is most conveniently expressed in terms of the coordinates u and v.

Note that in an axisymmetric torus, $\frac{\partial\psi}{\partial\phi} = 0$, and in the outside region $\frac{dp}{d\psi} = \frac{dF}{d\psi} = 0$. Equation (3.7) can be written in an extremely simple form to zeroth order.

$$\frac{\partial^2\hat{\psi}_0}{\partial u^2} + \frac{\partial^2\hat{\psi}_0}{\partial v^2} = 0 \tag{3.30}$$

Equation(3.30) is satisfied by an infinite set of orthogonal complete functions natural to elliptic coordinates.

$$\hat{\psi}_0(u,v) = \sum_n (A_n \sinh nu + B_n \cosh nu)(C_n \sin nv + D_n \cos nv) + Eu + Fv \tag{3.31}$$

Since the problem is up-down symmetric, $F = 0$ and all $C_n = 0$. Keeping in mind the criteria that $\hat{\psi}_0(u_0,v) = \psi_0(u_0,v)$ and that their derivatives must also be matched on the boundary, choose the form of the solution listed below.

$$\hat{\psi}_0 = \bar{c}_2(u-u_0) + \bar{c}_3 \sinh 2[u-u_0]\cos 2v + \bar{c}_4 \tag{3.32}$$

Apply (3.22) to (3.32)

$$\bar{c}_4 = \mu_0 R_0 \frac{I_p}{2\pi}\left(\frac{\kappa_c}{\kappa_c^2+1}\right) \tag{3.33}$$

Now apply (3.23).

$$\left.\frac{\partial\hat{\psi}_0}{\partial u}\right|_{u_0} = \bar{c}_2 + 2\bar{c}_3\cos 2v$$

$$\psi_0 = \bar{c}_4\left[\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2}\right]$$

$$\psi_0 = \bar{c}_4\left[\frac{c^2}{a^2}\sinh^2 u\cos^2 v + \frac{c^2}{\kappa_c^2 a^2}\cosh^2 u\sin^2 v\right]$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0} = \bar{c}_4\left[\frac{c^2}{a^2}2\sinh u_0\cosh u_0\cos^2 v + \frac{c^2}{\kappa_c^2 a^2}2\cosh u_0\sinh u_0\sin^2 v\right]$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0} = \bar{c}_4\left[\frac{c^2 2\sinh u_0\cosh u_0\cos^2 v}{c^2\sinh^2 u_0} + \frac{c^2 2\cosh u_0\sinh u_0\sin^2 v}{c^2\cosh^2 u_0}\right]$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0} = \bar{c}_4\left[2\frac{\cosh u_0}{\sinh u_0}\cos^2 v + 2\frac{\sinh u_0}{\cosh u_0}\sin^2 v\right]$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0} = 2\bar{c}_4\left[\kappa_c\cos^2 v + \frac{1}{\kappa_c}\sin^2 v\right]$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0} = \bar{c}_4\left[\kappa_c + \kappa_c\cos 2v + \frac{1}{\kappa_c} - \frac{1}{\kappa_c}\cos 2v\right]$$

$$\left.\frac{\partial\psi_0}{\partial u}\right|_{u_0} = \bar{c}_4\left[\frac{\kappa_c^2+1}{\kappa_c} + \frac{\kappa_c^2-1}{\kappa_c}\cos 2v\right]$$

$$\bar{c}_2 = \bar{c}_4\left[\frac{\kappa_c^2+1}{\kappa_c}\right] \tag{3.34}$$

$$\bar{c}_3 = \bar{c}_4\left[\frac{1}{2}\frac{\kappa_c^2-1}{\kappa_c}\right] \tag{3.35}$$

$\hat{\psi}_0(u,v)$ is now uniquely determined.

$$\hat{\psi}_0(u,v) = \mu_0 R_0\frac{I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\left[\frac{\kappa_c^2+1}{\kappa_c}[u-u_0] + \frac{1}{2}\frac{\kappa_c^2-1}{\kappa_c}\sinh 2[u-u_0]\cos 2v + 1\right] \tag{3.36}$$

Note the $\cos 2v$ behavior of $\hat{\psi}_0$ and hence of $\hat{B}_{0u}$ and $\hat{B}_{0v}$ disappears as $\kappa_c \rightarrow 1$, corresponding to the circular limit.

3.3 Beta Poloidal and the Internal Inductance

Before calculating the zeroth order fields, develop expressions for ℓ_i and β_p for the profiles given. First consider ℓ_i.

The internal inductance (un-normalized) of the plasma, L_i, is determined from a poloidal magnetic field energy balance inside the plasma.

$$\frac{1}{2}L_i I_p^2 = \frac{1}{2\mu_0}\int \mathbf{B}_{p0}\cdot\mathbf{B}_{p0} d^3V_{plasma} \tag{3.37}$$

Since the exact shape of the plasma boundary is not a simple ellipse if one models the core as such, Eq. (3.37) can be tricky to evaluate. Making use of the vector potential $\mathbf{A}$ simplifies matters considerably.

$$\mathbf{B}_{p0} = \frac{1}{R_0}\nabla\psi_0 \times \hat{e}_\phi \tag{3.38}$$

$$\mathbf{B}_{p0} = \nabla \times A\hat{e}_\phi \tag{3.39}$$

$$\begin{aligned}\frac{1}{2}L_i I_p^2 &= \frac{1}{2\mu_0}\int \nabla\times\mathbf{A}\cdot\nabla\times\mathbf{A} d^3V_p \\ &= \frac{1}{2\mu_0}\int [\nabla\cdot(\mathbf{A}\times\nabla\times\mathbf{A}) + \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{A}] d^3V_p\end{aligned} \tag{3.40}$$

Let

$$T_1 \equiv \frac{1}{2\mu_0}\int \nabla\cdot(\mathbf{A}\times\nabla\times\mathbf{A}) d^3V_p$$

$$T_2 \equiv \frac{1}{2\mu_0}\int \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{A} d^3V_p$$

Examine T_1. Apply Gauss' theorem.

$$T_1 = \frac{1}{2\mu_0}\int \hat{n}\cdot\mathbf{A}\times\nabla\times\mathbf{A} dS_p$$

$$T_1 = -\frac{1}{2\mu_0}\oint \nabla\times\mathbf{A}\cdot d\ell_p 2\pi R_0 A\Big|_{plasma\ boundary}$$

$$= -\frac{1}{2\mu_0}\oint \mathbf{B}_p\cdot d\ell_p 2\pi R_0 A\Big|_{plasma\ boundary}$$

Now employ Ampere's law $\oint \mathbf{B}_p \cdot d\ell = \mu_0 I_p$ and $\hat{\psi}_0 = -AR_0 =$ const.

$$T_1 = \frac{1}{2\mu_0}\mu_0 I_p 2\pi\hat{\psi}_0\Big|_{plasma\ boundary}$$

$$T_1 = \pi I_p \hat{\psi}_0\Big|_{plasma\ boundary} \tag{3.41}$$

Examine T_2.

$$T_2 = \frac{1}{2\mu_0}\int \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{A}d^3V_p$$

Again the differential form of Ampere's law gives

$$\nabla\times\nabla\times\mathbf{A}_\phi = \mu_0 J\hat{e}_\phi \tag{3.42}$$

$$T_2 = \frac{1}{2\mu_0}\int A\hat{e}_\phi\cdot\mu_0 J\hat{e}_\phi d^3V_p$$

Recall, however that $J = J_0$ inside the core and is zero everywhere else.

$$T_2 = \frac{1}{2\mu_0}\int A\mu_0 J_0 2\pi R_0 dS_{core}$$

$$= \pi J_0 \int AR_0 dS_{core}$$

$$T_2 = -\pi J_0 \int \psi_0 dS_{core} \tag{3.43}$$

Equation (3.37) simplifies tremendously.

$$\frac{1}{2}L_i I_p^2 = \pi I_p\hat{\psi}_0\Big|_{plasma\ boundary} - \pi J_0\int \psi_0(u,v)dS_{core} \tag{3.44}$$

Equation (3.44) is very easy to evaluate since $\hat{\psi}_0$ and ψ_0 are uniquely determined and the only integral to be evaluated spans the core and not the entire plasma. Examine the integral on the right hand side of (3.44).

$$I = \int \psi_0(u,v)dS_{core} \tag{3.45}$$

$$I = \int_0^{u_0}\int_0^{2\pi} \mu_0 R_0 J_0 \frac{a^2}{2}\left(\frac{\kappa_c^2}{\kappa_c^2+1}\right)\left[\frac{x^2}{a^2}+\frac{y^2}{\kappa_c^2 a^2}\right]\frac{c^2}{2}(\cosh 2u + \cos 2v)dv\ du$$

$$I = \int_0^{u_0}\int_0^{2\pi} \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2(\kappa_c^2-1)}{\kappa_c^2+1}\left[\sinh^2 u\cos^2 v + \frac{1}{\kappa_c^2}\cosh^2 u \sin^2 v\right][\cosh 2u + \cos 2v]\, dv\; du$$

$$I = \int_0^{u_0}\int_0^{2\pi} \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2(\kappa_c^2-1)}{\kappa_c^2+1}\left[\sinh^2 u\cosh 2u\cos^2 v + \sinh^2 u\cos 2v\cos^2 v\right.$$

$$\left. + \frac{1}{\kappa_c^2}\cosh^2 u\cosh 2u\sin^2 v + \frac{1}{\kappa_c^2}\cosh^2 u\cos 2v\sin^2 v\right] dv\; du$$

$$I = \int_0^{\mu_0} \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2(\kappa_c^2-1)}{\kappa_c^2+1}\pi\left[\sinh^2 u\cosh 2u + \frac{1}{2}\sinh^2 u + \frac{1}{\kappa_c^2}\cosh^2 u\cosh 2u - \frac{1}{2\kappa_c^2}\cosh^2 u\right] du$$

$$I = \int_0^{\mu_0} \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2(\kappa_c^2-1)}{\kappa_c^2+1}\pi\left[\frac{\kappa_c^2+1}{\kappa_c^2}\cosh^2 u\cosh 2u + \frac{-3\kappa_c^2-1}{4\kappa^2}\cosh 2u - \frac{\kappa_c^2+1}{4\kappa_c^2}\right] du$$

$$I = \int_0^{\mu_0} \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2(\kappa_c^2-1)}{\kappa_c^2+1}\pi\left[\frac{\kappa_c^2+1}{2\kappa_c^2}\cosh^2 2u + \frac{1-\kappa_c^2}{4\kappa_c^2}\cosh 2u - \frac{\kappa_c^2+1}{4\kappa_c^2}\right] du$$

$$I = \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2}{\kappa_c^2+1}(\kappa_c^2-1)\pi\left[\frac{\kappa_c^2+1}{2\kappa_c^2}\frac{1}{2}(u + \frac{1}{4}\sinh 4u) + \frac{1-\kappa_c^2}{8\kappa_c^2}\sinh 2u - \frac{\kappa_c^2+1}{4\kappa_c^2}u\right]_0^{u_0}$$

$$I = \frac{c^2}{2}\mu_0 R_0 J_0 \frac{a^2}{2}\frac{\kappa_c^2}{\kappa_c^2+1}(\kappa_c^2-1)\pi\left[\frac{\kappa_c^2+1}{16\kappa_c^2}\sinh 4u_0 + \frac{1-\kappa_c^2}{8\kappa_c^2}\sinh 2u_0\right]$$

$$I = \mu_0 R_0 J_0 \frac{a^4}{4}\frac{\kappa_c^3}{\kappa_c^2+1}(\kappa_c^2-1)\pi\left[\frac{\kappa_c^2+1}{16\kappa_c^2}4(1+2\sinh^2 u_0) + \frac{1-\kappa_c^2}{4\kappa_c^2}\right]$$

$$I = \mu_0 R_0 J_0 \frac{a^4}{4}\frac{\kappa_c^3}{\kappa_c^2+1}(\kappa_c^2-1)\pi\left[\frac{\kappa_c^2+1}{4\kappa_c^2} + \frac{1-\kappa_c^2}{4\kappa_c^2} + \frac{\kappa_c^2+1}{2\kappa_c^2}\frac{1}{(\kappa_c^2-1)}\right]$$

$$I = \mu_0 R_0 \frac{I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\frac{\pi\kappa_c a^2}{2}\left[\frac{\kappa_c^2-1}{4\kappa_c^2} + \frac{\kappa_c^2+1}{2\kappa_c^2}\right] \tag{3.46}$$

Now it is possible to obtain an expression for ℓ_i. Evaluate (3.44). Modeling the plasma core as an ellipse does not guarantee that the actual boundary of the plasma will also be an ellipse. In fact, casual examination of (3.36) reveals that the actual plasma boundary (also a flux surface) will not be an ellipse, but some other elongated shape approximately elliptical. Therefore in evaluating (3.44), $\hat{\psi}_0|_{plasma\ boundary}$ will be evaluated at $x = x_b, y = 0$ to which correspond the elliptic coordinates $u = u_b$ and $x = 0$ even though the flux surface upon which x_b lies is not **itself** an ellipse.

$$\frac{1}{2}L_i I_p^2 = \pi I_p \mu_0 R_0 \frac{I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\left[\frac{\kappa_c^2+1}{\kappa_c}[u-u_0] + \frac{1}{2}\frac{\kappa_c^2-1}{\kappa_c}\sinh 2[u-u_0] + 1\right]$$

$$- \pi J_0 \mu_0 \frac{R_0 I_p}{2\pi} \frac{\kappa_c}{\kappa_c^2+1} \frac{\pi \kappa_c a^2}{2} \left[\frac{\kappa_c^2-1}{4\kappa_c^2} + \frac{\kappa_c^2+1}{2\kappa_c^2} \right]$$

$$\frac{1}{2} L_i I_p^2 = \mu_0 R_0 I_p^2 \left\{ \frac{1}{2} \frac{\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2+1}{\kappa_c} [u-u_0] + \frac{1}{2} \frac{\kappa_c^2-1}{\kappa_c} \sinh 2[u-u_0] + 1 \right] \right.$$

$$\left. - \frac{1}{4} \frac{\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2-1}{4\kappa_c^2} + \frac{\kappa_c^2+1}{\kappa_c^2} \right] \right\}$$

$$L_i = \mu_0 R_0 \left\{ \frac{\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2+1}{\kappa_c} [u-u_0] + \frac{1}{2} \frac{\kappa_c^2-1}{\kappa_c} \sinh 2[u-u_0] + 1 \right] \right.$$

$$\left. - \frac{1}{2} \frac{\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2-1}{4\kappa_c^2} + \frac{\kappa_c^2+1}{2\kappa_c^2} \right] \right\}$$

$$\ell_i = \frac{2L_i}{\mu_0 R_0} \tag{3.47}$$

$$\ell_i = \frac{2\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2+1}{\kappa_c} [u-u_0] + \frac{1}{2} \frac{\kappa_c^2-1}{\kappa_c} \sinh 2[u-u_0] + 1 - \frac{\kappa_c^2+1}{4\kappa_c^2} - \frac{\kappa_c^2-1}{8\kappa_c^2} \right]$$

$$\ell_i = \frac{2\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2+1}{\kappa_c} [u-u_0] + \frac{1}{2} \frac{\kappa_c^2-1}{\kappa_c} \sinh 2[u-u_0] + 1 - \frac{3\kappa_c^2+1}{8\kappa_c^2} \right] \tag{3.48}$$

This expression for ℓ_i is exact. Since the limiter position x_b is known, the corresponding point in toroidal coordinates $u = u_b$ and $v = 0$ can be determined from the equation below.

$$x_b = c \sinh u_b \tag{3.49}$$

$$\ell_i = \frac{2\kappa_c}{\kappa_c^2+1} \left[\frac{\kappa_c^2+1}{\kappa_c} [u_b-u_0] + \frac{1}{2} \frac{\kappa_c^2-1}{\kappa_c} \sinh 2[u_b-u_0] + 1 - \frac{3\kappa_c^2+1}{8\kappa_c^2} \right] \tag{3.50}$$

Examine (3.50) as $\kappa_c \to 1$.

$$\ell_i = 2[u_b - u_0] + \frac{1}{2} \tag{3.51}$$

Consider the plasma boundary to be at $r = b$ and the core boundary to be at $r = a$. Then as the circular limit is approached u_b and u_0 approach the following values

$$u_0 \approx \ln \frac{2a}{c} \tag{3.52}$$

$$u_b \approx \ln \frac{2b}{c} \tag{3.53}$$

$$\ell_i = 2 \ln \frac{b}{a} + \frac{1}{2} \tag{3.54}$$

$$\kappa_c \to 1$$

Equation (3.54) is the old circular result found in Chapter 2. Next obtain an expression for β_p.

$$\beta_p \equiv \frac{8\pi}{\mu_0 I_p^2} \int p dS_p \tag{3.55}$$

$$\beta_p = \frac{8\pi}{\mu_0 I_p^2} \int p_0 \left(1 - \frac{\psi_0}{\psi_a}\right) d\ S_p$$

$$\beta_p = \frac{8\pi}{\mu_0 I_p^2} p_0 \pi \kappa_c a^2 \left[\frac{5\kappa_c^2 - 1}{8\kappa_c^2}\right] \tag{3.56}$$

3.4 The Zeroth Order Fields

Now it is necessary to evaluate the information contained in the zeroth order field measurements. The poloidal field at the measurement surface can be expressed as

$$\hat{\mathbf{B}}_p(u_m, v) = \frac{1}{R} \nabla \hat{\psi} \times \hat{e}_\phi \Big|_{u_m, v} \tag{3.57}$$

$\hat{\mathbf{B}}_p$ has both $\hat{v}$ and $\hat{u}$ components in the elliptic limit.

$$|\hat{\mathbf{B}}_{0v}| = \frac{1}{R_0} \nabla_{\hat{u}} \hat{\psi}_0 \tag{3.58}$$

$$|\hat{\mathbf{B}}_{0u}| = \frac{1}{R_0} \nabla_{\hat{v}} \hat{\psi}_0 \tag{3.59}$$

Evaluate (3.58) and (3.59) on the measurement surface u_m.

$$|\hat{\mathbf{B}}_{0v}(u_m, v)| = \frac{1}{c(\frac{\cosh 2u_m + \cos 2v}{2})^{1/2}} \frac{\mu_0 I_p}{2\pi} \left[1 + \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \cosh 2[u_m - u_0] \cos 2v\right] \tag{3.60}$$

$$|\hat{\mathbf{B}}_{0u}(u_m, v)| = \frac{1}{c(\frac{\cosh 2u_m + \cos 2v}{2})^{1/2}} \frac{\mu_0 I_p}{2\pi} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \sinh 2[u_m - u_0] \sin 2v \tag{3.61}$$

Suppose that the data from the magnetic probes located on the measurement surface u_m is Fourier analyzed. To measure the zeroth harmonic or "dc" component of the field, theoretically only one probe is required. To measure higher harmonics, proportionally more probes are required. To compensate for measurement errors and random fluxuations in the data, this minimum number of probes must be supplemented. Therefore, it is advantageous to measure the lowest harmonics accessible to the diagnostics with as much redundancy as is practical. This consideration motivates the remainder of the calculation.

$$|\hat{\mathbf{B}}_p(u_m, v)| = \sum_{n=0}^{\infty} \frac{1}{c(\frac{\cosh 2u_m + \cos 2v}{2})^{1/2}} (B_n \sin nv + C_n \cos nv) \tag{3.62}$$

If the zeroth order fields are so decomposed the following three amplitudes are measured.

$$\hat{B}v_{DC} \equiv C_0 = \frac{\mu_0 I_p}{2\pi} \quad (T - m) \tag{3.63}$$

$$\hat{B}v_{02} \equiv C_2 = \frac{\mu_0 I_p}{2\pi} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \cosh 2[u_m - u_0] \quad (T - m) \tag{3.64}$$

$$\hat{B}u_{02} \equiv B_2 = \frac{\mu_0 I_p}{2\pi} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \sinh 2[u_m - u_0] \quad (T - m) \tag{3.65}$$

Expectedly, as $\kappa_c \rightarrow 1$, the second harmonic behavior of the zeroth order fields (i.e. $\hat{B}v_{02} \rightarrow \hat{B}u_{02} \rightarrow 0$) disappears. As one approaches a circular cross section, $\kappa_c \rightarrow 1$, the zeroth order fields lose their angular dependence. Information is lost.

It is useful to take the difference of the squares of (3.64) and (3.65) applying the identity $\cosh^2 x - \sinh^2 x = 1$.

$$\hat{B}^2 v_{02} - \hat{B}^2 u_{02} = \left(\frac{\mu_0 I_p}{2\pi}\right)^2 \left(\frac{\kappa_c^2 - 1}{\kappa_c^2 + 1}\right)^2 \quad (T^2 - m^2) \tag{3.66}$$

Hence, $\hat{B}v_{02}$ and $\hat{B}u_{02}$ are not independent quantities. In fact, casual examination of (3.64) and (3.65) reveals that in the limit of large u_m, that is, when the measurement surface is far away from the plasma, $\hat{B}v_{02} \rightarrow \hat{B}u_{02}$ and the combination $\hat{B}v_{02}^2 - \hat{B}u_{02}^2$ cannot be used to find κ_c. Let

$$\gamma \equiv \left(\hat{B}v_{02}^2 - \hat{B}u_{02}^2\right)^{1/2} \frac{2\pi}{\mu_0 I_p} \tag{3.67}$$

I_p can be determined immediately from the Bv_{DC} measurement and Eq. (3.63).

$$I_p = \frac{2\pi \hat{B}v_{DC}}{\mu_0} \tag{3.68}$$

Assuming that the measurement surface is not extremely far from the plasma edge, and having determined I_p, measuring $\hat{B}v_{02}$ and $\hat{B}u_{02}$ uniquely determines γ.

$$\frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} = \gamma$$

$$\kappa_c = \left(\frac{1+\gamma}{1-\gamma}\right)^{1/2} \tag{3.69}$$

The ellipticity of the hot, current-carrying core is determined. Having κ_c, the u_0 coordinate that describes the boundary of the core can be found by solving the transcendental equation below.

$$\tanh u_0 = \frac{1}{\kappa_c} \tag{3.70}$$

The actual dimensionality of the core follows.

$$a = c \sinh u_0 \tag{3.71}$$

At this point in the calculation, the dimensionality, area, elongation, and current of the core have been uniquely specified by **zeroth order measurements.**

Returning to the actual edge of the plasma specified by the limit position x_b given, the elliptic coordinate of that point, u_b, can be determined by solving the transcendental equation given below

$$x_b = c \sinh u_b \tag{3.72}$$

u_0, κ_c, u_b are known quantities. ℓ_i follows immediately from Eq. (3.48).

$$\ell_i = \frac{2\kappa_c}{\kappa_c^2+1}\left[\frac{\kappa_c^2+1}{\kappa_c}[u_b - u_0] + \frac{1}{2}\frac{\kappa_c^2-1}{\kappa_c}\sinh 2[u_b - u_0] + 1 - \frac{3\kappa_c^2+1}{8\kappa_c^2}\right] \tag{3.73}$$

Because of the additional information available in the zeroth order field measurements, ℓ_i can be determined independent of the first order fields and β_p for a finite ellipticity. It can be demonstrated from the formulae above that the ability to determine ℓ_i independently from zeroth order measurements disappears as one approaches the circular limit.

Before moving on to determine β_p from the first order field measurements, it is necessary to extract yet another plasma parameter from the zeroth order flux function $\hat{\psi}_0$. The plasma boundary is also a flux surface. This implies the following.

$$\hat{\psi}_0(\hat{u}_b, \frac{\pi}{2}) = \hat{\psi}_0(u_b, 0) \tag{3.74}$$

$$\frac{\kappa_c^2+1}{\kappa_c}\hat{u}_b - \frac{1}{2}\frac{\kappa_c^2-1}{\kappa_c}\sinh 2[\hat{u}_b - u_0] = \frac{\kappa_c^2+1}{\kappa_c}u_b + \frac{1}{2}\frac{\kappa_c^2-1}{\kappa_c}\sinh 2[u_b - u_0] \tag{3.75}$$

Solve (3.75) numerically for $\hat{u}_b$. The elongation of the plasma κ, as distinct from the elongation of the core κ_c, is then uniquely determined

$$\kappa = \frac{\cosh \hat{u}_b}{\sinh u_b} \tag{3.76}$$

3.5 The First Order Solution

So far, before appealing to first order measurements, ℓ_i, κ, and I_p have been determined from zeroth order measurements. Now return to the Grad-Shafranov equation and solve it to first order obtaining ψ_1 and $\hat{\psi}_1$. β_p lies buried in the first order field measurements. Inside the core, ψ_1 is most easily obtained using rectangular coordinates. In the R, Z plane

$$\Delta^*\psi = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} \tag{3.77}$$

$$\Delta^*\psi = -\frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial R^2} + \frac{\partial^2\psi}{\partial Z^2} \tag{3.78}$$

In the R, Z plane make the following transformation

$$Z = y \tag{3.79}$$

$$R = R_0 + x$$

Keeping the large aspect ratio limit in mind, $x/R_0 \ll 1$, the first order Grad-Shafranov equation inside the plasma can be cast in the following form.

$$\nabla^2\psi_1 - \frac{1}{R_0}\frac{\partial\psi_0}{\partial x} = -\frac{2x}{R_0}\mu_0 R_0^2\frac{dp}{d\psi_0} \tag{3.80}$$

∇^2 is now the familiar $\partial^2/\partial x^2 + \partial/\partial y^2$.

$$p = p_0(1 - \frac{\psi_0}{\psi_a})$$

$$\frac{dp}{d\psi_0} = -\frac{p_0}{\psi_a} \tag{3.81}$$

$$\psi_0 = \frac{\mu_0 R_0 I_p}{2\pi}\left(\frac{\kappa_c}{\kappa_c^2+1}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2}\right)$$

$$\frac{\partial\psi_0}{\partial x} = \frac{\mu_0 R_0 I_p}{2\pi}\left(\frac{2\kappa_c}{\kappa_c^2+1}\right)\frac{x}{a^2} \tag{3.82}$$

Combine (3.80-3.82) to generate the ψ_1 equation.

$$\nabla^2\psi_1 = \frac{2}{R_0}\left(\frac{\mu_0 R_0 I_p}{4\pi a^2}\frac{2\kappa_c}{\kappa_c^2+1} - \mu_0 R_0^2\left[-\frac{p_0}{\mu_0 R_0 I_p}\frac{2\pi(\kappa_c^2+1)}{\kappa_c}\right]\right)x \tag{3.83}$$

$$\nabla^2\psi_1 = \left(\frac{\mu_0 I_p}{\pi a^2}\frac{\kappa_c}{\kappa_c^2+1} + 4\pi\frac{p_0}{I_p}\frac{\kappa_c^2+1}{\kappa_c}\right)x \tag{3.84}$$

Using (3.56), eliminate p_0 in favor of β_p.

$$\beta_p = \frac{8\pi}{\mu_0 I_p^2}p_0\pi\kappa_c a^2\left[\frac{5\kappa_c^2-1}{8\kappa_c^2}\right] \tag{3.85}$$

$$\frac{p_0}{I_p} = \frac{\mu_0 I_p}{8\pi^2\kappa_c a^2}\left[\frac{8\kappa_c^2}{5\kappa_c^2-1}\right]\beta_p \tag{3.86}$$

$$\nabla^2\psi_1 = \left(\frac{\mu_0 I_p}{\pi a^2}\frac{\kappa_c}{\kappa_c^2+1} + 4\pi\frac{\mu_0 I_p}{8\pi}\frac{1}{\pi\kappa_c a^2}\left[\frac{8\kappa_c^2}{5\kappa_c^2-1}\right]\beta_p\frac{\kappa_c^2+1}{\kappa_c}\right)x$$

$$\nabla^2\psi_1 = \frac{\mu_0 I_p}{\pi a^2}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right)x \tag{3.87}$$

$$\nabla^2\psi_1 = Zx \tag{3.88}$$

$$Z \equiv \frac{\mu_0 I_p}{\pi a^2}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right) \tag{3.89}$$

In order to solve (3.88), boundary conditions are needed. The first is that ψ_1 must be regular, i.e. does not diverge at the origin. Also, the edge of the core, the u_0 ellipse, is modeled as a flux surface even to first order. Thus ψ_1 is constant on that surface. Choose a convenient value

$$\psi_1(u_0, v) = 0 \tag{3.90}$$

A function that satisfies both conditions appears below.

$$\psi_1 = Qx\left[\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2} - 1\right] \tag{3.91}$$

Plugging the Ansatz for ψ_1 back into (3.88), it is a trivial matter to fix the value of Q.

$$Q = \frac{a^2}{2}\left(\frac{\kappa_c^2}{3\kappa_c^2+1}\right)Z \tag{3.92}$$

$$Q = \frac{\mu_0 I_p}{2\pi}\frac{\kappa_c^2}{3\kappa_c^2+1}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right) \tag{3.93}$$

Finally ψ_1 is uniquely determined.

$$\psi_1 = \frac{\mu_0 I_p}{2\pi}\frac{\kappa_c^2}{3\kappa_c^2+1}\left(\frac{\kappa_c}{\kappa_c^2+1}+\frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right)x\left[\frac{x^2}{a^2}+\frac{y^2}{\kappa_c^2 a^2}-1\right] \quad u \le u_0 \tag{3.94}$$

It is useful to check these results against those obtained in Chapter 2 by taking the circular limit $\kappa_c \to 1$.

$$\psi_1 = \frac{\mu_0 I_p}{2\pi}\frac{1}{4}\left(\frac{1}{2}+2\beta_p\right)x\left[\frac{x^2+y^2}{a^2}-1\right] \tag{3.95}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2+y^2 = r^2$$

$$\psi_1(r,\theta) = \frac{\mu_0 I_p}{8\pi}(1+4\beta_p)\left(\frac{r^3}{a^2}-r\right)\cos\theta \tag{3.96}$$

Equation (3.96) agrees nicely with (2.39). The next step involves solving the first order Grad-Shafranov equation outside the core for $\hat{\psi}_1$. Again, u, v coordinates serve best in this region.

$$\frac{\partial^2\hat{\psi}_1}{\partial u^2}+\frac{\partial^2\hat{\psi}_1}{\partial v^2} = \frac{c}{R_0}(\cosh u\cos v\frac{\partial\hat{\psi}_0}{\partial u} - \sinh u\sin v\frac{\partial\hat{\psi}_0}{\partial v}) \tag{3.97}$$

$$\epsilon \equiv \frac{c}{R_0} \tag{3.98}$$

$$T_1 = \epsilon\cosh u\cos v\frac{\partial\hat{\psi}_0}{\partial u} \tag{3.99}$$

$$T_2 = \epsilon\sinh u\sin v\frac{\partial\hat{\psi}_0}{\partial v} \tag{3.100}$$

Evaluate T_1.

$$T_1 = \epsilon\cosh u\cos v C_s\left(\frac{\kappa_c^2+1}{\kappa_c}+\frac{\kappa_c^2-1}{\kappa_c}\cosh 2[u-u_0]\cos 2v\right) \tag{3.101}$$

$$C_s \equiv \frac{\mu_0 R_0 I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1} \tag{3.102}$$

$$T_1 = C_s\epsilon\left\{\frac{\kappa_c^2+1}{\kappa_c}\cosh u\cos v + \frac{\kappa_c^2-1}{\kappa_c}\cosh u\cosh 2[u-u_0]\cos v\cos 2v\right\}$$

$$T_1 = C_s\epsilon\left\{\frac{\kappa_c^2+1}{\kappa_c}\cosh u\cos v\right.$$

$$\left.+\frac{\kappa_c^2-1}{\kappa_c}\cosh u\,[\cosh 2u_0\cosh 2u - \sinh 2u_0\sinh 2u]\cos v\cos 2v\right\}$$

$$T_1 = C_s\epsilon\left\{\frac{\kappa_c^2+1}{\kappa_c}\cosh u\cos v\right.$$

$$\left.+\frac{\kappa_c^2-1}{2\kappa_c}[\cosh 2u_0\cosh 3u + \cosh 2u_0\cosh u - \sinh 2u_0\sinh 3u - \sin 2u_0\sinh u]\cos v\cos 2v\right\}$$

$$T_1 = C_s\epsilon\left\{\frac{\kappa_c^2+1}{\kappa_c}\cosh u\cos v\right.$$

$$+\frac{\kappa_c^2-1}{4\kappa_c}\Big[\cosh 2u_0\cosh 3u\cos v + \cosh 2u_0\cosh 3u\cos 3v + \cosh 2u_0\cosh u\cos v$$

$$+\cosh 2u_0\cosh u\cos 3v - \sinh 2u_0\sinh 3u\cos v - \sinh 2u_0\sinh 3u\cos 3v$$

$$\left.-\sinh 2u_0\sinh u\cos v - \sinh 2u_0\sinh u\cos 3v\Big]\right\} \tag{3.103}$$

Evaluate T_2.

$$T_2 = \epsilon C_s\sinh u\sin v\left\{-\frac{(\kappa_c^2-1)}{\kappa_c}\sinh 2[u-u_0]\sin 2v\right\} \tag{3.104}$$

$$T_2 = -\epsilon C_s\frac{\kappa_c^2-1}{\kappa_c}\{[\cosh 2u_0\sinh u\sinh 2u - \sinh 2u_0\sinh u\cosh 2u]\sin v\sin 2v\}$$

$$T_2 = -\epsilon C_s\frac{\kappa_c^2-1}{2\kappa_c}\Big\{[\cosh 2u_0\cosh 3u - \cosh 2u_0\cosh u$$

$$-\sinh 2u_0\sinh 3u + \sinh 2u_0\sinh u]\sin v\sin 2v\Big\}$$

$$T_2 = -\epsilon C_s\frac{\kappa_c^2-1}{4\kappa_c}\Big\{\cosh 2u_0\cosh 3u\cos v - \cosh 2u_0\cosh 3u\cos 3v - \cosh 2u_0\cosh u\cos v$$

$$+\cosh 2u_0\cosh u\cos 3v - \sinh 2u_0\sinh 3u\cos v + \sinh 2u_0\sinh 3u\cos 3v$$

$$+\sinh 2u_0\sinh u\cos v - \sinh 2u_0\sinh u\cos 3v\Big\} \tag{3.105}$$

Calculate the combination $T_1 - T_2$.

$$T_1 - T_2 = \epsilon C_s \left[\frac{\kappa_c^2 + 1}{\kappa_c} \cosh u \cos v \right.$$

$$+ \frac{\kappa_c^2 - 1}{4\kappa_c} \left\{ 2\cosh 2u_0 \cosh 3u \cos v + 2\cosh 2u_0 \cosh u \cos 3v - 2\sinh 2u_0 \sinh 3u \cos v \right.$$

$$\left.\left. - 2\sinh 2u_0 \sinh u \cos 3v \right\} \right]$$

$$T_1 - T_2 = \epsilon C_s \left[\frac{\kappa_c^2 + 1}{\kappa_c} \cosh u \cos v + \frac{\kappa_c^2 - 1}{2\kappa_c} \left\{ [\cosh 2u_0 \cosh 3u - \sinh 2u_0 \sinh 3u] \cos v \right.\right.$$

$$\left.\left. + [\cosh 2u_0 \cosh u - \sinh 2u_0 \sinh u] \cos 3v \right\} \right]$$

$$T_1 - T_2 = \epsilon C_s \left[\frac{\kappa_c^2 + 1}{\kappa_c} \cosh u \cos v \right.$$

$$\left. + \frac{\kappa_c^2 - 1}{2\kappa_c} \cosh[3u - 2u_0] \cos v + \frac{\kappa_c^2 - 1}{2\kappa_c} \cosh[u - 2u_0] \cos 3v \right] \tag{3.106}$$

Equation (3.97) simplifies tremendously.

$$\frac{\partial^2 \hat{\psi}_1}{\partial u^2} + \frac{\partial^2 \hat{\psi}_1}{\partial v^2} = \epsilon C_s \left\{ \frac{\kappa_c^2 + 1}{\kappa_c} \cosh u \cos v + \frac{\kappa_c^2 - 1}{2\kappa_c} \cosh[3u - 2u_0] \cos v + \frac{\kappa_c^2 - 1}{2\kappa_c} \cosh[u - 2u_0] \cos 3v \right\} \tag{3.107}$$

It can be shown that for equations of the form

$$\frac{\partial^2 \hat{\psi}_1}{\partial u^2} + \frac{\partial^2 \hat{\psi}_1}{\partial v^2} = C \cosh mu \cos nv \qquad m \neq n \tag{3.108}$$

$$\hat{\psi}_{1p} = \frac{C}{m^2 - n^2} \cosh mu \cos nv \tag{3.109}$$

The particular solutions for the non-resonant terms in (3.107) can be written down immediately.

The first term on the right hand side of (3.107) is somewhat troublesome. It is resonant. Equations with resonant forcing terms of the form

$$\frac{\partial^2 \hat{\psi}_1}{\partial u^2} + \frac{\partial^2 \hat{\psi}_1}{\partial v^2} = C \cosh u \cos v \tag{3.110}$$

have particular solutions of the form

$$\hat{\psi}_{1p} = \frac{1}{4}C(u \sinh u \cos v + v \sin v \cosh u) \tag{3.111}$$

However, since $\hat{\psi}_{1p}$ must be single valued, solutions linear in v are not allowed.

$$\hat{\psi}_{1p} = \frac{1}{2}Cu \sinh u \cos v \tag{3.112}$$

At this point, the particular solution of (3.107) is fully determined.

$$\hat{\psi}_{1p} = \epsilon C_s \left\{ \frac{\kappa_c^2+1}{2\kappa_c} u \sinh u \cos v + \frac{\kappa_c^2-1}{16\kappa_c} \cosh[3u - 2u_0] \cos v - \frac{\kappa_c^2-1}{16\kappa_c} \cosh[u - 2u_0] \cos 3v \right\} \tag{3.113}$$

The homogeneous solutions must be chosen carefully to match ψ_1 and $\hat{\psi}_1$ at the core boundary. The particular solution (3.113) represents a toroidal **correction** to the essentially straight elliptic plasma column solution $\hat{\psi}_0$. Notice it only depends on the plasma current and core dimensions, not on β_p.

$$\hat{\psi}_1 = \hat{\psi}_{1p} + \hat{\psi}_{1h} \tag{3.114}$$

Choose a convenient form for the homogeneous solution $\hat{\psi}_{1h}$.

$$\hat{\psi}_{1h} = A \cosh[u-u_0] \cos v + B \cosh 3[u-u_0] \cos 3v + C \sinh[u-u_0] \cos v + D \sinh 3[u-u_0] \cos 3v \tag{3.115}$$

$$\hat{\psi}_{1h}\Big|_{u_0} = A \cos v + B \cos 3v \tag{3.116}$$

$$\frac{\partial \hat{\psi}_{1h}}{\partial u}\Bigg|_{u_0} = C \cos v + 3D \cos 3v \tag{3.117}$$

The jump conditions across the core boundary u_0 will fix $A, B, C,$ and D. Since in this problem the core boundary is modeled as a flux surface to first order and there are not surface currents, ψ_1 and its derivatives are continuous across u_0.

$$\left[\hat{\psi}_1 - \psi_1\right]\Bigg|_{u_0, v=0} = 0 \tag{3.118}$$

$$\left[\frac{\partial \hat{\psi}_1}{\partial u} - \frac{\partial \psi_1}{\partial u}\right]\Bigg|_{u_0, v=0} = 0 \tag{3.119}$$

Recall ψ_1 was so chosen so that $\psi_1(u_0, v) = 0$.

$$\hat{\psi}_1(u_o, v) = 0 \tag{3.120}$$

Equation (3.120) in combination with (3.116) and (3.113) specifies A and B.

$$A = -\epsilon C_s \left\{ \frac{\kappa_c^2 + 1}{2\kappa_c} u_0 \sinh u_0 + \frac{\kappa_c^2 - 1}{16\kappa_c} \cosh u_0 \right\}$$

$$A = -\frac{c}{R_0} \frac{\mu_0 R_0 I_p}{2\pi} \frac{\kappa_c}{\kappa_c^2 + 1} \left\{ \frac{\kappa_c^2 + 1}{2\kappa_c} u_0 \sinh u_0 + \frac{\kappa_c^2 - 1}{16\kappa_c} \cosh u_0 \right\}$$

$$c \sinh u_0 = a$$

$$c \cosh u_0 = \kappa_c a$$

$$A = -\frac{\mu_0 a I_p}{2\pi} \left\{ \frac{u_0}{2} + \frac{\kappa_c}{16} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \right\} \tag{3.121}$$

$$B = \epsilon C_s \frac{\kappa_c^2 - 1}{16\kappa_c} \cosh u_0$$

$$B = \frac{c}{R_0} \frac{\mu_0 R_0 I_p}{2\pi} \frac{\kappa_c}{\kappa_c^2 + 1} \frac{\kappa_c^2 - 1}{16\kappa_c} \cosh u_0$$

$$B = \frac{\mu_0 a I_p}{2\pi} \frac{\kappa_c}{16} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \tag{3.122}$$

Next, match the derivatives. Calculate $\partial\psi_1/\partial u\Big|_{u_0, v}$

$$\psi_1 = Qx \left[\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2} - 1 \right]$$

Use the chain rule.

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} \tag{3.123}$$

$$\frac{\partial \psi_1}{\partial u} = \frac{\partial x}{\partial u} \left(Qx \frac{2x}{a^2} + Q \left[\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^1} - 1 \right] \right) + \frac{\partial y}{\partial u} Q \frac{2xy}{\kappa_c^2 a^2}$$

$$\left[\frac{x^2}{a^2} + \frac{y^2}{\kappa_c^2 a^2} - 1 \right] = 0 \qquad u = u_0$$

$$\left.\frac{\partial \psi_1}{\partial u}\right|_{u_0,v} = 2Q\left(c\cosh u_0 \cos v \frac{c^2\sinh^2 u_0}{a^2}\cos^2 v + c\sinh u_0 \frac{c^2 \sinh u_0}{\kappa_c^2 a^2}\cosh u_0 \cos v \sin^2 v\right)$$

$$\left.\frac{\partial \psi_1}{\partial u}\right|_{u_0,v} = 2Q\left(c\cosh u_0 \cos v \cos^2 v + \frac{c}{\kappa_c^2}\cosh u_0 \cos v \sin^2 v\right)$$

$$\left.\frac{\partial \psi_1}{\partial u}\right|_{u_0,v} = Qa\kappa_c \cos v\left(1 + \cos 2v + \frac{1}{\kappa_c^2} - \frac{1}{\kappa_c^2}\cos 2v\right)$$

$$\left.\frac{\partial \psi_1}{\partial u}\right|_{u_0,v} = Qa\kappa_c \cos v\left(\frac{\kappa_c^2+1}{\kappa_c^2} + \frac{\kappa_c^2-1}{\kappa_c^2}\cos 2v\right)$$

$$\left.\frac{\partial \psi_1}{\partial u}\right|_{u_0,v} = Qa\kappa_c\left(\left[\frac{\kappa_c^2+1}{\kappa_c^2} + \frac{\kappa_c^2-1}{2\kappa_c^2}\right]\cos v + \frac{\kappa_c^2-1}{2\kappa_c^2}\cos 3v\right)$$

$$\left.\frac{\partial \psi_1}{\partial u}\right|_{u_0,v} = Qa\left(\frac{3\kappa_c^2+1}{2\kappa_c}\cos v + \frac{\kappa_c^2-1}{2\kappa_c}\cos 3v\right) \tag{3.124}$$

$$\left.\frac{\partial \hat{\psi}_1}{\partial u}\right|_{u_0,v} = \epsilon C_s\left\{\frac{\kappa_c^2+1}{2\kappa_c}(u_0\cosh u_0 + \sinh u_0)\cos v + \frac{3}{16}\frac{\kappa_c^2-1}{\kappa_c}\sinh u_0 \cos v + \frac{\kappa_c^2-1}{16\kappa_c}\sinh u_0 \cos 3v\right\}$$
$$+ C\cos v + 3D\cos 3v \tag{3.125}$$

Together, (3.124) and (3.125) specify C and D. Match the $\cos v$ terms.

$$\epsilon C_s\left\{\frac{\kappa_c^2+1}{2\kappa_c}(u_0\cosh u_0 + \sinh u_0) + \frac{3}{16}\frac{\kappa_c^2-1}{\kappa_c}\sinh u_0\right\} + C = Qa\frac{3\kappa_c^2+1}{2\kappa_c}$$

$$C = Qa\frac{3\kappa_c^2+1}{2\kappa_c} - \epsilon C_s\left\{\frac{\kappa_c^2+1}{2\kappa_c}u_0\cosh u_0 + \frac{\kappa_c^2+1}{2\kappa_c}\sinh u_0 + \frac{3}{16}\frac{\kappa_c^2-1}{\kappa_c}\sinh u_0\right\}$$

$$C = \frac{\mu_0 a I_p}{2\pi}\frac{\kappa_c^2}{3\kappa_c^2+1}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right)\frac{3\kappa_c^2+1}{2\kappa_c}$$
$$- \frac{c}{R_0}\frac{\mu_0 R_0 I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\left\{\frac{\kappa_c^2+1}{2\kappa_c}u_0\cosh u_0 + \frac{\kappa_c^2+1}{2\kappa_c}\sinh u_0 + \frac{3}{16}\frac{\kappa_c^2-1}{\kappa_c}\sinh u_0\right\}$$

$$C = \frac{\mu_0 a I_p}{2\pi}\frac{\kappa_c}{2}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right)$$
$$- \frac{\mu_0 a I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\left\{\frac{\kappa_c^2+1}{2}u_0 + \frac{\kappa_c^2+1}{2\kappa_c} + \frac{3}{16}\frac{\kappa_c^2-1}{\kappa_c}\right\}$$

$$C = \frac{\mu_0 a I_p}{2\pi}\left[-\frac{\kappa_c u_0}{2} + \frac{\kappa_c^2}{2(\kappa_c^2+1)} - \frac{1}{2} + \frac{2\kappa_c(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p - \frac{3}{16}\frac{\kappa_c^2-1}{\kappa_c^2+1}\right] \tag{3.126}$$

Match the $\cos 3v$ terms.

$$\epsilon C_s \frac{\kappa_c^2-1}{16\kappa_c}\sinh u_0 + 3D = Qa\frac{\kappa_c^2-1}{2\kappa_c}$$

$$D = Qa\frac{\kappa_c^2-1}{6\kappa_c} - \epsilon C_s \frac{\kappa_c^2-1}{48\kappa_c}\sinh u_0$$

$$D = \frac{\kappa_c^2-1}{6\kappa_c}\left[Qa - \frac{1}{8}\epsilon C_s \sinh u_0\right]$$

$$D = \frac{\kappa_c^2-1}{6\kappa_c}\left[\frac{\mu_0 a I}{2\pi}\frac{\kappa_c^2}{3\kappa_c^2+1}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right) - \frac{1}{8}\frac{c}{R_0}\frac{\mu_0 R_0 I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\sinh u_0\right]$$

$$D = \frac{\mu_0 a I_p}{2\pi}\frac{\kappa_c^2-1}{6\kappa_c}\left[\frac{\kappa_c^2}{3\kappa_c^2+1}\left(\frac{\kappa_c}{\kappa_c^2+1} + \frac{4(\kappa_c^2+1)}{5\kappa_c^2-1}\beta_p\right) - \frac{1}{8}\frac{\kappa_c}{\kappa_c^2+1}\right] \tag{3.127}$$

$\hat{\psi}_1(u,v)$ is now fully determined.

$$\hat{\psi}_1(u,v) = \frac{\mu_0 c I_p}{2\pi}\frac{\kappa_c}{\kappa_c^2+1}\left\{\frac{\kappa_c^2+1}{2\kappa_c}u\sinh u\cos v \right.$$

$$\left. + \frac{\kappa_c^2-1}{16\kappa_c}\cosh[3u-2u_0]\cos v - \frac{\kappa_c^2-1}{16\kappa_c}\cosh[u-2u_0]\cos 3v\right\}$$

$$+ A\cosh[u-u_0]\cos v + B\cosh 3[u-u_0]\cos 3v$$

$$+ C\sinh[u-u_0]\cos v + D\sinh 3[u-u_0]\cos 3v \tag{3.128}$$

The coefficients A, B, C and D are given by (3.121), (3.122), (3.126) and (3.127), respectively.

In order to have confidence in the calculation of $\hat{\psi}_1,(u,v)$, it is necessary to examine its behavior in the circular limit. Let $\kappa_c \to 1$.

$$A \to -\frac{\mu_0 a I_p}{2\pi}\frac{u_0}{2} \tag{3.129}$$

$$B \to 0 \tag{3.130}$$

$$C \to \frac{\mu_0 a I_p}{2\pi}\left[-\frac{u_0}{2} - \frac{1}{4} + \beta_p\right] \tag{3.131}$$

$$D \to 0 \tag{3.132}$$

As $\kappa_c \to 1$, all third harmonic ($\cos 3v$) behavior disappears. The $\cos v$ behavior persists. This is in perfect agreement with the circular limit.

Evaluate (3.128) in the limit $\kappa_c \to 1$ on the plasma boundary $(u_b, 0)$.

$$\hat{\psi}_{1}\underset{\kappa_c \to 1}{} = \frac{\mu_0 c I_p}{2\pi}\left[\frac{u_b}{2}\sinh u_b\right] - \frac{\mu_0 a I_p}{2\pi}\frac{u_0}{2}\cosh[u_b - u_0] + \frac{\mu_0 a I_p}{2\pi}\left[-\frac{u_0}{2} - \frac{1}{4} + \beta_p\right]\sinh[u_b - u_0] \tag{3.133}$$

In this limit, a and b approach the following.

$$a \approx \frac{ce^{u_0}}{2} \tag{3.134}$$

$$b \approx \frac{ce^{u_b}}{2} \tag{3.135}$$

$$\sinh[u_b - u_0] \approx \cosh[u_b - u_0] \approx \frac{e^{u_b - u_0}}{2} \tag{3.136}$$

$$\hat{\psi}_{1}\underset{\kappa_c \to 1}{} = \frac{\mu_0 b I_p}{2\pi}\frac{u_b}{2} - \frac{\mu_0 a I_p}{2\pi}\frac{u_0}{2}\frac{e^{[u_b - u_0]}}{2} + \frac{\mu_0 a I_p}{2\pi}\left[-\frac{u_0}{2} - \frac{1}{4} + \beta_p\right]\frac{e^{[u_b - u_0]}}{2}$$

$$\hat{\psi}_{1}\underset{\kappa_c \to 1}{} = \frac{\mu_0 b I_p}{4\pi}\left[u_b - u_0 - \frac{1}{4} + \beta_p\right]$$

$$\hat{\psi}_{1}\underset{\kappa_c \to 1}{} = \frac{\mu_0 b I_p}{4\pi}\left[\ln\frac{b}{a} - \frac{1}{4} + \beta_p\right] \text{ Eliminate } \ln b/a \text{ in favor of } \ell_i \text{ using (3.54).}$$

$$\hat{\psi}_{1}\underset{\kappa_c \to 1}{} = \frac{\mu_0 b I_p}{4\pi}\left[\frac{\ell_i}{2} - \frac{1}{4} - \frac{1}{4} + \beta_p\right]$$

$$\hat{\psi}_{1}\underset{\kappa_c \to 1}{} = \frac{bB_{\theta b}}{2}\left[\beta_p + \frac{\ell_i}{2} - \frac{1}{2}\right]b \tag{3.137}$$

Recalling that in Chapter 2 the boundary of the plasma was at $r = a$ and that in taking the limits (3.134–3.136), the decaying exponentials $\alpha 1/r$ were ignored, Eq. (3.137) is in perfect agreement with Eq. (2.52). $\hat{\psi}_1(u, v)$ checks out.

3.6 The First Order Fields

Fewer probes are needed to accurately sample the first harmonic than to sample the third. Analyzing the signals as before, the poloidal field at the measurement surface u_m can be expressed in the form Eq. (3.62).

$$|\hat{B}_p(u_m, v)| = \sum_{n=0}^{\infty} \frac{1}{c\left(\frac{\cosh 2u_m + \cos 2v}{2}\right)^{1/2}} \left(B_n \sin nv + C_n \cos nv\right)$$

Measure only the first harmonic. Two amplitudes corresponding to the tangential and normal fields are found.

$$\hat{B}_{v_{11}} \equiv C_1 = \frac{\mu_0 c I_p}{2\pi} \frac{1}{R_0} \left\{ \frac{1}{2} u_m \cosh u_m + \frac{1}{2} \sinh u_m + \frac{3}{16} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \sinh[3u_m - 2u_0] \right\}$$
$$+ \frac{1}{R_0} A \sinh[u_m - u_0] + \frac{1}{R_0} C \cosh[u_m - u_0] \qquad (3.138)$$

$$\hat{B}_{u_{11}} \equiv B_1 = \frac{\mu_0 c I_p}{2\pi} \frac{1}{R_0} \left\{ \frac{1}{2} u_m \sinh u_m + \frac{3}{16} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \cosh[3u_m - 2u_0] \right\}$$
$$+ \frac{1}{R_0} A \cosh[u_m - u_0] + \frac{1}{R_0} C \sinh[u_m - u_0] \qquad (3.139)$$

The constant C is linear in β_p. Equations (3.138) and (3.139) give the same information. Therefore, only $\hat{B}_{v_{11}}$, the first order tangential field amplitude, need be considered. Once $\hat{B}_{v_{11}}$ is measured, β_p can be calculation directly from (3.138). As in Chapter 2, the Shafranov shift Δ_a was set to zero for simplicity.

β_p is determined from first order measurements for a plasma with finite elongation. Notice, however, that if the magnetic probes are far away from the plasma edge, the large u_m limit, the third term on the right hand side of Eq. (3.138) dominates and the β_p information is lost.

3.7 Qualitative Behavior of the Model

A qualitative picture of how the model derived above behaves in a tokamak with C-Mod-like parameters is shown in Figs. 3.2-3.4. The parameters used in these calculations are listed below.

$$a = .25m \tag{3.140}$$

$$R_0 = .75m \tag{3.141}$$

$$\epsilon = 1/3 \tag{3.142}$$

$$I_p = 4MA \tag{3.143}$$

The elongation of the plasma κ was varied from $\kappa = 1$ to $\kappa = 2$.

Figure 3.2 illustrates the dependence of the second harmonic field on the elongation of the plasma. Notice that the second harmonic field $\hat{B}_{v02}$ quickly disappears as the plasma cross section approaches a circle. Remember, the information contained in the second harmonic led directly to the evaluation of ℓ_i.

Figure 3.3 demonstrates how ℓ_i could possibly be measured from $\hat{B}_{v02}$. It is only meant to show that these particular values of field could be used to infer ℓ_i for conditions (3.140-3.143) using this simple model.

In principle, having ℓ_i, β_p can be determined from the first harmonic. Figure 3.4 illustrates how this might be accomplished. Notice the linear dependence of $\hat{B}_{v_{11}}$ on β_p and that even at very low $\beta_p, \hat{B}_{v11}$ persists as $\kappa \rightarrow 1$. This agrees with the results obtained in Chapter 2.

In conclusion, it has been shown analytically for this model problem that for finite ellipticity, magnetic measurements can be used to measure β_p and ℓ_i separately. This ability is lost as the plasma cross section approaches a circle. Then only the combination $\beta_p + \ell_i/2$ is available to the diagnostics.

3.8 Summary

The equations of interest that describe the fields available to the diagnostics in our simple model problem are summarized below.

$$\hat{B}_{vDC} = \frac{\mu_0 I_p}{2\pi} \quad (T - m) \tag{3.63}$$

$$\hat{B}_{v02} = \frac{\mu_0 I_p}{2\pi} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \cosh 2[u_m - u_0] \quad (T - m) \tag{3.64}$$

$$\hat{B}_{u02} = \frac{\mu_0 I_p}{2\pi} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \sinh 2[u_m - u_0] \quad (T - m) \tag{3.65}$$

$$\ell_i = \frac{2\kappa_c}{\kappa_c^2 + 1} \left[\frac{\kappa_c^2 + 1}{\kappa_c} [u_b - u_0] + \frac{1}{2} \frac{\kappa_c^2 - 1}{\kappa_c} \sinh 2[u_b - u_0] + 1 - \frac{3\kappa_c^2 + 1}{8\kappa_c^2} \right] \tag{3.73}$$

$$\hat{B}_{v_{11}} = \frac{\mu_0 c I_p}{2\pi R_0} \left\{ \frac{1}{2} u_m \cosh u_m + \frac{1}{2} \sinh u_m + \frac{3}{16} \frac{\kappa_c^2 - 1}{\kappa_c^2 + 1} \sinh[3u_m - 2u_0] \right\}$$

$$+ \frac{1}{R_0} A \sinh[u_m - u_0] + \frac{1}{R_0} C \cosh[u_m - u_0] \quad (T - m) \tag{3.138}$$

It has been shown that in the circular limit, when $\kappa_c \to 1$, $\hat{\psi}_1$ and hence $\hat{B}_{v_{11}}$ only depend on the combination $\beta_p + \ell_i/2$.

In the elliptic limit, when the measurement surface is far away from the plasma boundary, $u_m \gg 1$, the combination $\hat{B}_{v02}^2 - \hat{B}_{u02}^2$ can no longer be used to accurately determine κ_c and hence ℓ_i. From far away, the plasma looks circular. Also in the large u_m limit examination of (3.138) reveals that β_p information is lost as non-β_p dependent terms that make up $\hat{B}_{v_{11}}$ dominate.

Finally, for both finite κ and u_m, ℓ_i and I_p can be determined from $\hat{B}_{v02}$ and $\hat{B}_{vDC}$ respectively and the β_p information resides in the $\hat{B}_{v_{11}}$ measurement.

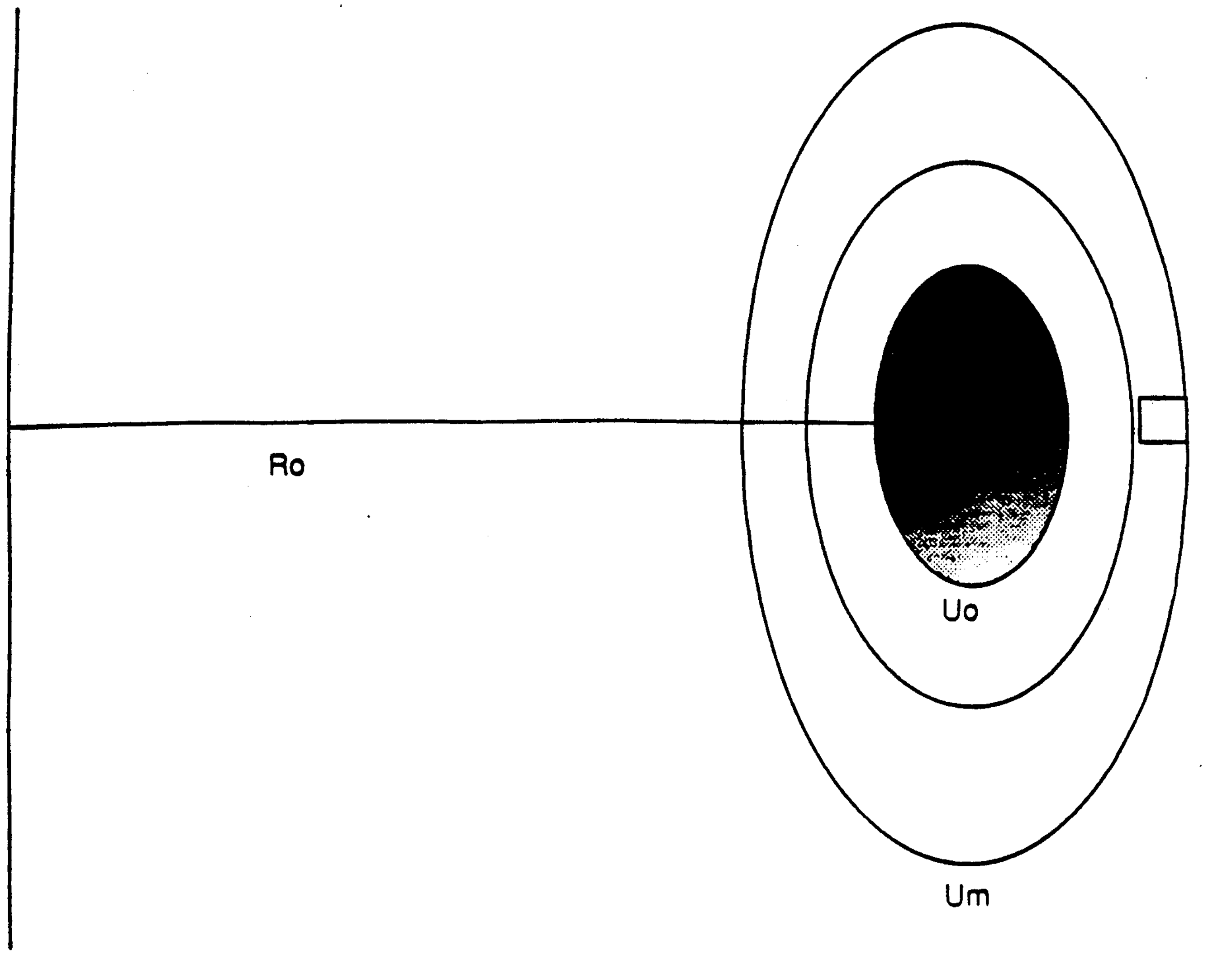


Figure 3.1: Idealized Elliptical Tokamak

Figure 3.2: $\hat{B}$v02 versus Plasma Elongation

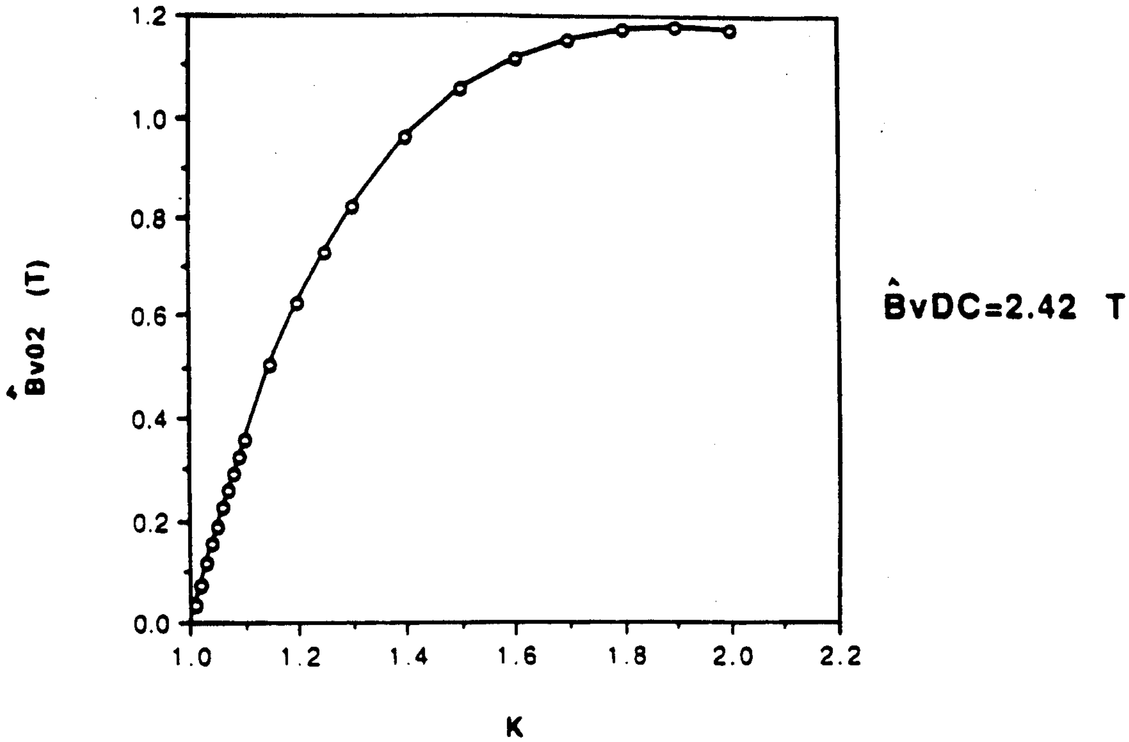


Figure 3.3: $\hat{B}v02$ vs. Internal Inductance

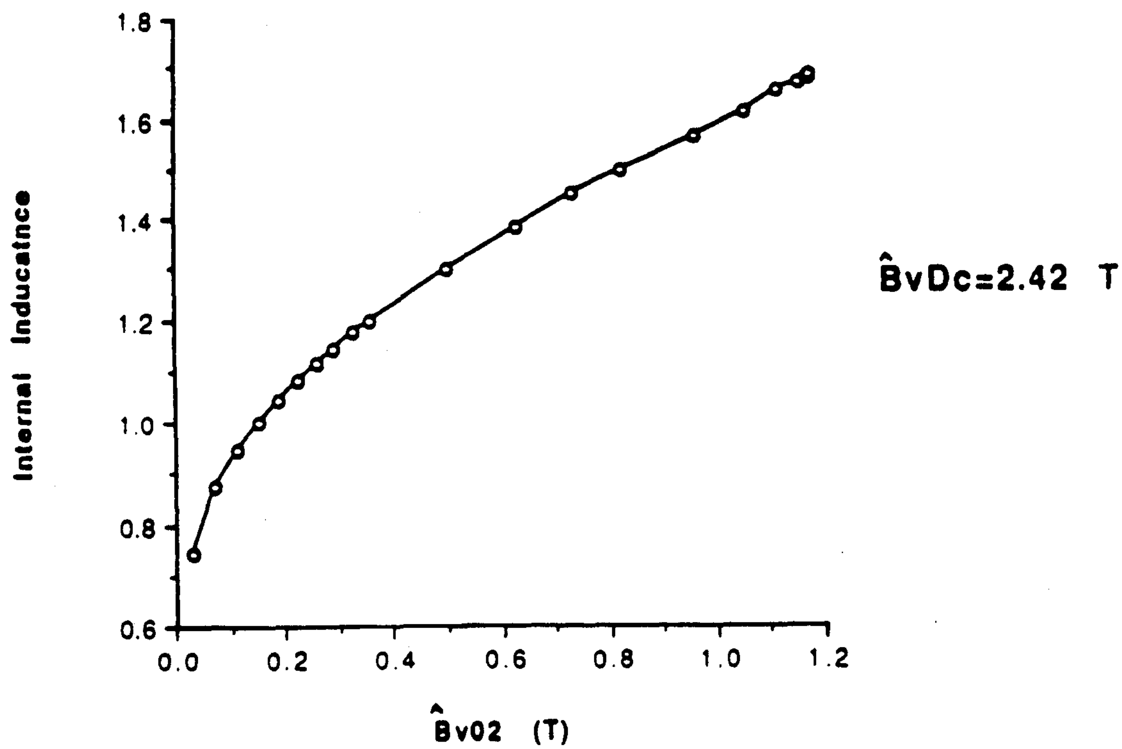
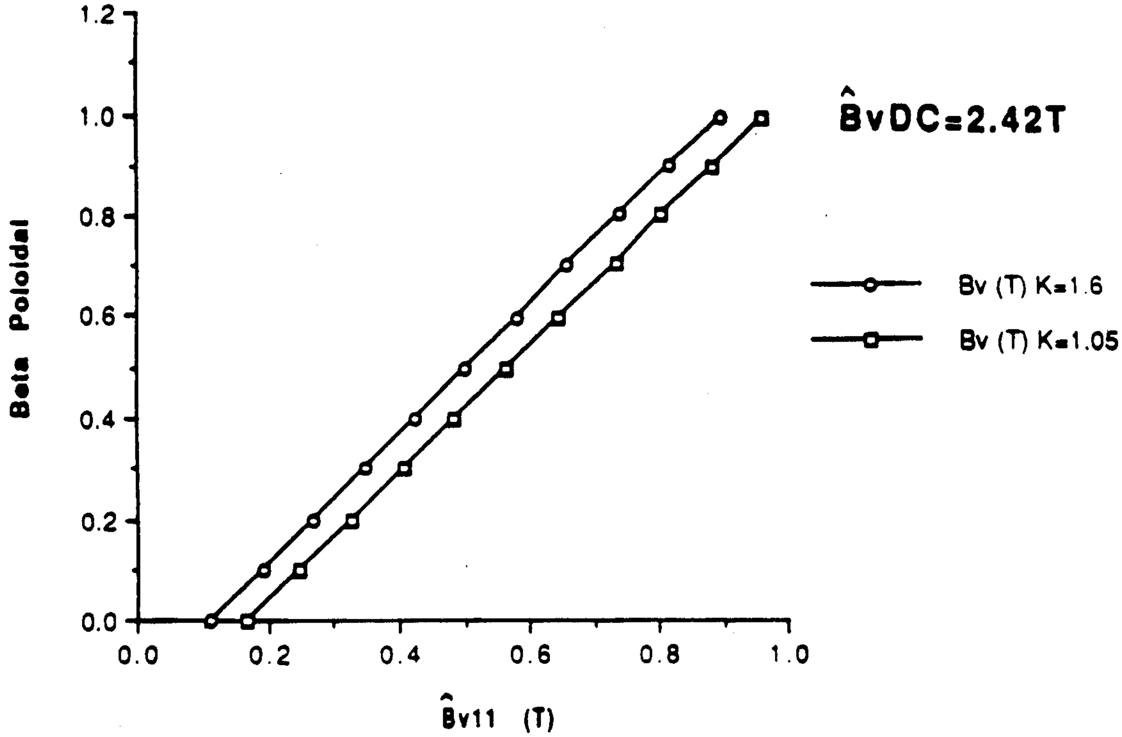


Figure 3.4: $\hat{B}v11$ vs. Beta Poliodal



Chapter 4

Conclusions and Suggestions for Future Work

For tokamaks with circular cross sections, only the combination $\beta_p + \ell_i/2$ is obtainable from first order measurements. Second order field measurements are found to specify β_p and ℓ_i separately, but are too sensitive to be used with any confidence.

For a certain class of idealized tokamaks with elliptical cross sections, it is shown that finite ellipticity introduces robust second harmonics into the zeroth order magnetic fields. From these second harmonics it is possible to deduce ℓ_i. β_p can then be separately determined from the measurement of the first harmonic component of the magnetic field that appears in first order. The second harmonics that determine ℓ_i disappear as the elliptical cross section approaches a circle. Concurrently, the combination $\beta_p + \ell_i/2$ reappears in the first order flux function and hence the first harmonic. The circular degeneracy is recovered.

Future work along these same lines might take the form of solving the Grad-Shafranov equation in an elliptical tokamak under less restrictive, less idealized conditions. Generalization of the calculation to include an arbitrary Shafranov shift and a circular measurement surface would be highly desirable.

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