THE COUPLING APPROXIMATION OF LOCAL DISPERSION RELATIONS*

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The propagation of waves of a given frequency in one-dimensional non-uniform media (coordinate x) with general local dispersion functions, e.g. $D(k, x)$ (such as those from the Vlasov equation), and not expressible as a polynomial in wavevector k has not previously been solved due to the difficulty of treating mode coupling regions where k values come together rapidly, and the eikonal approximation must fail. Providing the modes couple only two at a time, a general method is given for generating appropriate and unambiguous second-order differential equations for each coupling. The differential equation requiring solution is essentially the Schrodinger equation with potential $-2(D(k)/\partial^2 D/\partial k^2)_{k=k_c(x)}$, where $k_c(x)$ satisfies $\partial D(k, x)/\partial k = 0$, x real.

Lower-hybrid heating of plasmas relies on the mode-conversion of incident electromagnetic power to a normal plasma mode which is more readily absorbed via non-collisional damping processes. The usual mode-coupling theory[1] based on the fourth- or sixth-order differential equations corresponding to the warm-plasma approximation of the Vlasov dispersion relation is inapplicable in the presence of kinetic damping. Specifically, the appearance, due to damping, of terms which are not powers of the wavenumber k breaks down the possibility of direct representation of the dispersion relation by a differential equation. Nevertheless, the concept can be recovered, and we present here a method of analysis resulting in a system of coupled-mode differential equations, representing any dispersion relation which is an entire analytic function of k. We are dealing here with "local" dispersion relations, valid for waves in a non-uniform medium (along x) satisfying the eikonal approximation.

Consider a plasma which is weakly nonuniform in one spatial dimension x, supporting waves excited at a fixed frequency ω. The local dispersion relation $D(k, z; \omega) = 0$ representing these waves as they propagate under the influence of nonuniformities, defines a many valued mapping of the complex spatial variable $z = x + iy$ onto the complex wavenumber plane $k = k_r + ik_i$. We have defined $k \equiv k_x$, and we have suppressed the explicit dependence on parameters such as the other two components of the wavevector, etc.; y and z are not to be confused with spatial dimensions. We will assume that the inverse mapping $k \rightarrow z$ is single-valued. It poses no principal difficulty to relax this assumption, and it is useful to do so on occasion, since the case of many valued z is frequently encountered in practice. An arbitrary contour C in the z-plane thus maps onto n branches, $n > 1$, $k = f(z)$, and with such a mapping there are associated saddle and branch points, k_s and z_B, respectively. The branch points z_B are those for which $\frac{df}{dz}$ diverges, and they are, by definition, the only ones which map onto a single point k_s in the k-plane, the latter being the saddle points of the inverse mapping $z = f^{-1}(k)$.

The general procedure of mapping $z \rightarrow k$ through the dispersion relation involves two steps. The first is to determine the saddle points k_s and branch points z_B. The second is to specify the branch lines. The saddle and branch points are inherent to the mapping. Each saddle point defines a distinct coupling event between the branches; we will assume throughout that the branches couple pair-wise, which occurs when

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$$\frac{df^{-1}}{dk} = 0, \qquad \frac{d^2f^{-1}}{dk^2} \neq 0. \tag{1}$$

The crucial step in the analysis of the dispersion relation regards the construction of branch cuts. On the one hand, the branch cuts define the process of mode coupling and the process of analytic continuation of a branch, but on the other, the dispersion relation alone does not provide any ready-made information on how to specify the branch lines. This is, of course, a source of major concern since mode coupling occurs, by definition, every time the wave-propagation contour C (the real axis $x = \mathrm{Re}\, z$) crosses a branch cut. The intuitive reaction in this situation would be to trace the branch cuts in some arbitrary manner away from the real axis, to avoid crossings other than through the branch points themselves. But what if all branch points lie away from the real axis? Does this imply that now no mode coupling takes place? The answer to these questions lies in the proper definition of mode-coupling and of a mode itself. The relevant question to ask is which are the physical conditions for coupling and find their analytical counterparts. Branches couple, by definition, whenever the real x-axis crosses a branch cut (Fig. 1). The usual definition of a mode, in terms of a branch of the dispersion relation, is thus not complete without associating with it a properly cut Riemann sheet. The Riemann sheet then maps onto a simply connected region of the k-plane, whose boundaries (given by the maps of the branch cuts) limit the extent of wavenumbers accessible to the branch under deformations of the x-axis. The required branch cuts C_b and the mode-boundaries k_c are given by the mappings[2]

$$D(k_c, C_b) = 0, \qquad \frac{\partial D(k_c, z)}{\partial k} = 0, \qquad z \in C, \tag{2}$$

where C is required to pass through the branch points and, therefore, might occasionally deviate from the real x-axis (Fig. 2). The only coupling events which materialize under these conditions are those defined by the branch (and saddle) points of the dispersion relation. We emphasize that it is actually through the mapping $\frac{\partial D}{\partial k} = 0$ that we make contact with the physical requirements for mode-conversion. The argument goes as follows. Consider a wave $A(x)\exp[-i\omega t + ik_r(x)x]$ propagating along the x-direction, and characterized by a slowly varying amplitude A (incorporating the effect of $k_i(x)$) and wavenumber $k_r(x)$. The wave has associated with it an averaged energy flux proportional to $v_g AA^*$, where A^* is the conjugate to A, and v_g is the local group velocity

$$v_g = \frac{d\omega}{dk} = -\frac{\frac{\partial D}{\partial k}}{\frac{\partial D}{\partial \omega}}, \tag{3}$$

evaluated implicitly from the dispersion relation $D(k, z; \omega) = 0$. The energy flux thus defined depends on position not only through the amplitude A, but also via v_g. Under these conditions, energy ceases to flow in the form of a particular mode where either $A \to 0$ or $v_g \to 0$. The former case is indicative of dissipative processes and A vanishes gradually. We are concerned with the latter case as that is the one which is indicative of mode-conversion, at least as far as the coupling of energy between different wave-types is concerned. This is because when $v_g \to 0$ without A vanishing simultaneously, the global conservation of energy flux requires that the energy be either transformed into a different wave form, or the amplitude diverges and we have a resonance. The branch points and branch cuts are instrumental in obtaining a representation of the dispersion relation in the form of a system of second-order differential equations. To proceed let us define the function $\mathfrak{D}(k) \equiv D(k, z)$ of the independent variable k, with z as a parameter. To obtain an approximation for the two roots k of the dispersion relation which couple at $k = k_c$, we expand $\mathfrak{D}(k)$ around the corresponding mode-boundary $k_c(z)$ to second-order in k: $\mathfrak{D}(k) = \mathfrak{D}(k_c) + \frac{1}{2}(k - k_c)^2\mathfrak{D}''(k_c)$.

This approximation for $\mathfrak{D}(k)$ is excellent near the coupling points k_s and z_B, by virtue of the definition of $k_c(z)$, providing that $\mathfrak{D}''(k_s) \neq 0$. Moreover, it is a good approximation away from k_s and z_B, as long as $\mathfrak{D}''(k_c)$ does not vanish in the region of interest. The latter condition is particularly important when the branch point is not real, since then the approximation must be extended all the way from z_B to the real axis, which is where the waves actually propagate. It follows that the particular path in the z-plane must avoid passing through a branch point of the mapping $\frac{\partial D}{\partial k} = 0$, since $\mathfrak{D}''(k_c)$ vanishes there and $k_c(z)$ joins onto a different branch. Except for this restriction, the path can be quite arbitrary, and the transition to the real axis can be therefore performed by substituting x for z and making sure that $k_c(x)$ belongs to the correct branch as specified by the appropriate saddle point $k_c(z_B) = k_s$. The dispersion relation $\mathfrak{D}(k) = 0$ on the real axis Re z is then approximated by

$$k^2 - 2k_c(x)k + k_c^2(x) - Q(x) = 0, \tag{4}$$

where

$$Q(x) = -2\,\mathfrak{D}(k)/\mathfrak{D}''(k)\big|_{k_c(x)}. \tag{5}$$

is the scattering potential. We now note that Eq. (4) contains the ambiguous combination $k_c k$ which permits the interpretations $(k \to i\frac{d}{dx})\ ik_c\frac{d\Phi}{dx}$, $i\,d(k_c\Phi)/dx$, or a linear combination thereof, where Φ is the wave-potential.

The criterion for deciding the existence of a unique combination of these terms is that the resulting equation must have regular turning points where $Q = 0$, as this is where, by definition, the mapping represented by the dispersion relation has branch points. Such a combination does indeed exist, and the differential equation is

$$\Phi'' + ik_c\Phi' + i(k_c\Phi)' + (Q - k_c^2)\Phi = 0\,, \tag{6}$$

where the prime now denotes differentiation with respect to x. To prove that Eq. (6) has the desired turning points it suffices to write $\Phi = \varphi \exp\left(-i\int k_c(x)\,dx\right) \equiv \varphi F_c$, giving finally

$$\varphi'' + Q(x)\varphi = 0. \tag{7}$$

The system, composed of the elementary local dispersion relations (4), together with their corresponding differential equations (7) can be appropriately termed as the "coupling approximation" of the dispersion relation. An application of the method pertaining to the mode-conversion of lower-hybrid waves in the presence of ion-cyclotron damping is given in a companion paper of these Proceedings.

References

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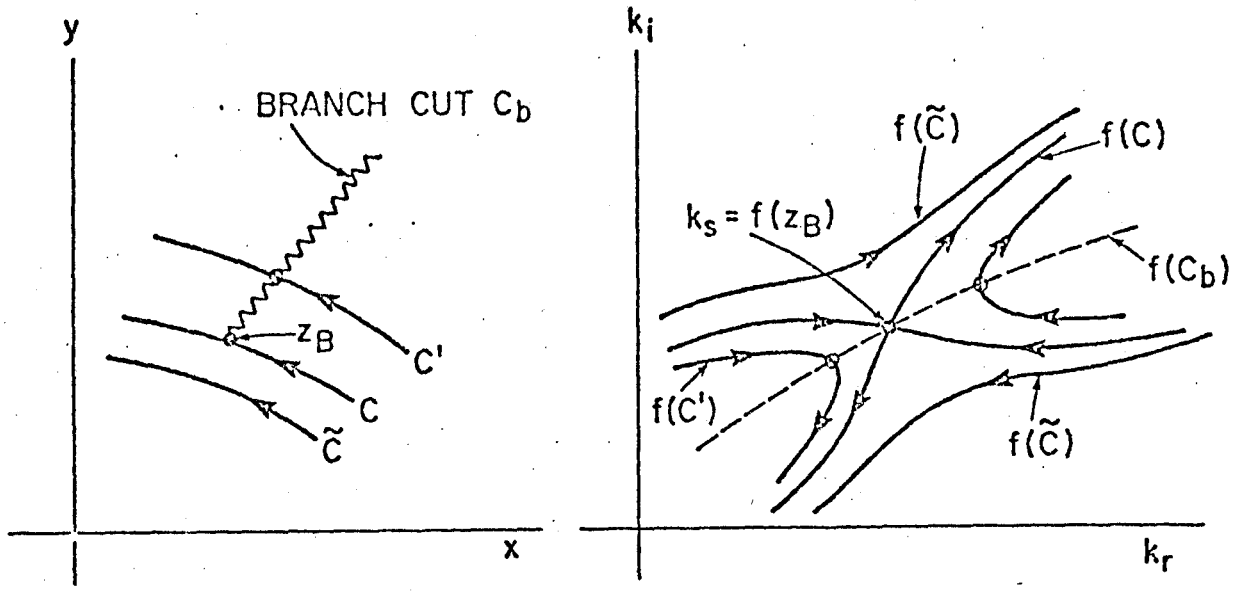

Figure 1. For a contour $\tilde{C}$ not crossing the branch cut, C_b, the branches $f(\tilde{C})$ lie, unconnected, on either side of the map, $f(C_b)$, of the cut. A branch cut itself maps onto the k-plane as extending from the saddle point k_s into two directions, and represents the boundary between the maps of the two Riemann sheets it connects. In the other case, of C' crossing the branch cut, the branches connect through the boundary $f(C_b)$ signifying a mode-coupling event.

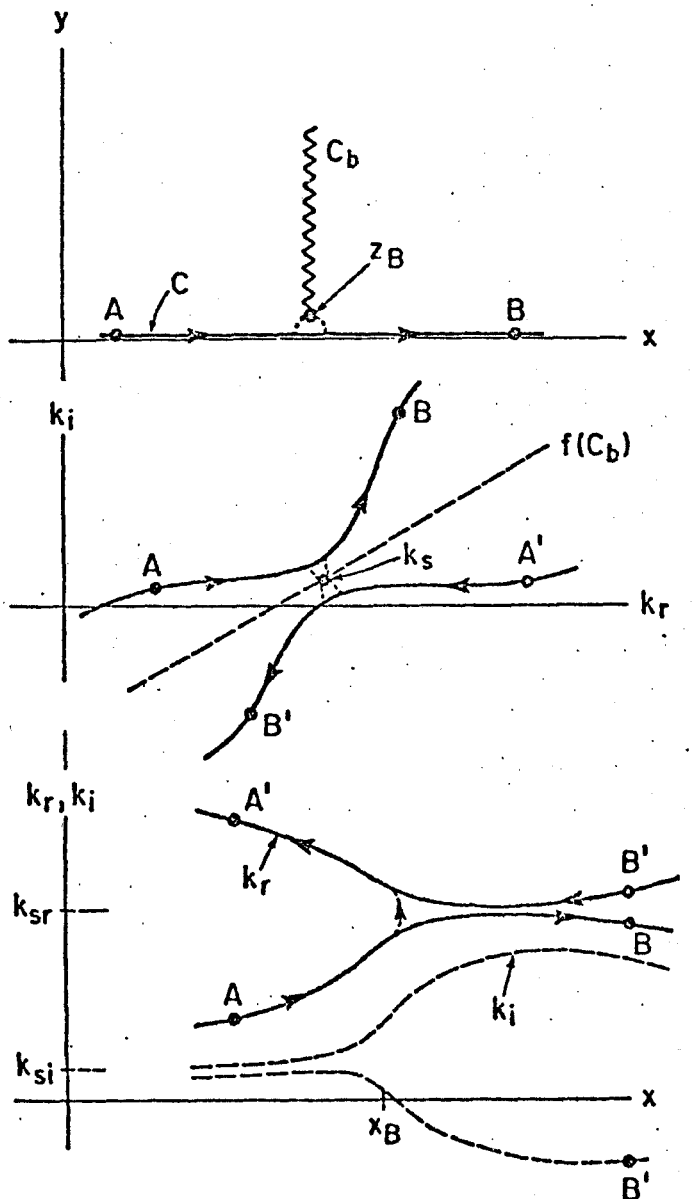

Figure 2. The assumption of pairwise coupling reduces the problem of an n-valued mapping into a series of two-valued maps. If the branch point lies away from the real x-axis, the branches appear uncoupled in both the k_i versus k_r, as well as in the more familiar k_i, k_r versus x, representations.