Urban OR: A Barrier Example
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Wall of Shame

Assume a square city of side L, with the borders going North→South and East→West. A malicious mayor decides to create an East→West barrier a distance x north of the city's southern border, which will have only one opening through which vehicles can pass. Assume that his objective is to increase to the greatest extent possible the mean grid distance between two random points in the square. In pursuit of this goal:

(1) Where should he place the opening in the barrier?

(2) What value of x should he choose?

Solution

As is intuitively clear, the opening should be at the end of the barrier, either on the Eastern edge of the city or the Western. That way, every trip that requires crossing the barrier will involve greater East→West distance than it would absent the barrier. The extra East→West distance will average 2L/3 (L/2 + L/2 – L/3). An opening elsewhere would have less than 100% chance of increasing East→West distance when the barrier is crossed, and the mean increase given that one occurred would be less than 2L/3.

If the barrier is x units north of the city's Southern border, then the city is divided into two rectangles, one of area Lx and the other of area L(L-x). (We assume, of course, that x < L.) Thus, if two points are chosen at random over the city, the probability that one of them is north of the barrier and the other south is $2Lx[L(L-x)]/L^4$. (See why?) This quantity is a numerical constant times x(L-x). The value of x that maximizes the expression is found by setting the appropriate derivative equal to 0, which quickly leads to the outcome x = L/2.

In short, the barrier should divide the city into two halves, and travelers between the two sectors should have to travel to the edge of the city to get through. As in Berlin, however, such a barrier cannot be expected to last forever.