Too Close for Comfort: Geometrical Probability in the Sky

Suppose that two aerial routes--one Eastbound and one Northbound--cross at an altitude of 35,000 feet at junction J (Figure 1). In the absence of air-traffic control, the times at which eastbound planes would arrive at the junction would reflect a Poisson process with parameter λ_E (per minute). Likewise, northbound planes would arrive under an independent Poisson process with parameter λ_N. All planes move at a speed of 600 miles per hour along their routes.

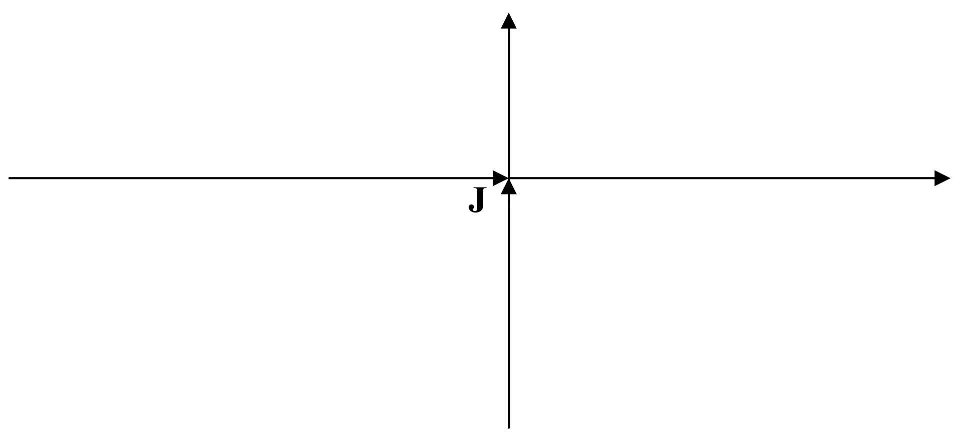

Figure1:Eastbound and Northbound Air Routes Cross at J

The Federal Aviation Administration thinks it dangerous if two planes cruising at the same altitude get within 5 miles of one another (in which case they are said to **conflict**). The idea is that, if a conflict arises, the planes are traveling so fast that they could collide if one of them deviates from its course. With the FAA standard in mind, we calculate the probabilities of three interesting events:

E: the chance that an eastbound plane that has just reached J is in conflict <u>at that moment</u> with a northbound plane

N: the chance that a northbound plane that has just reached J is at that moment in conflict with an eastbound plane

EE: the chance that a given eastbound plane that passes through J is ***at any time*** in conflict with a northbound plane that passes through J.

<u>P(E)</u>

To find P(E), we note that the conflict occurs if, at the time the eastbound plane reaches J, there is a northbound plane within five miles of J. If E* is the complement of E, then E* requires that there be no northbound plane within five miles of the junction. It is easier to find P(E*) than P(E), so we will do so and then invoke the rule P(E) = 1 - P(E*).

We aren't told anything about planes that are not at the junction, so how can we determine whether an aircraft is within five miles of J? We can exploit the clue that planes travel at 600 miles per hour (which works out to ten miles per minute, or one mile every six seconds). Suppose that a plane is north of J and within five miles of it. Then, the plane must have passed through J within the last thirty seconds. Similarly, if a northbound plane is still south of J but less than five miles away, it will reach J within the next thirty seconds.

Thus, if an eastbound plane reaches J at time t, there will be a conflict at t if a northbound plane passes through J between t-0.5 (in minutes) and t + 0.5. And there will be no conflict if no northbound plane reaches t over the interval (t-0.5,t+0.5).

We can therefore write:

P(E*) = P(no northbound arrivals at J over (t-0.5,t+0.5)) = $\exp(-\lambda_N)$

and thus that **P(E) = 1 - $\exp(-\lambda_N)$**

P(N):

The reasoning is the same as for P(E), so we can write:

$$\mathbf{P(N) = 1 - exp(-\lambda_N)}$$

It might seem surprising that P(E) and P(N) differ, given that each conflict we are considering involves one eastbound and one northbound plane. If, however, $\lambda_N > \lambda_E$ (for example), then more northbound planes reach J per hour than do eastbound planes. Thus, if equal numbers of northbound and eastbound planes face conflicts, the percentage of conflicts is lower for northbound planes passing through J than eastbound ones. And P(E) and P(N) reflect these percentages.

P(EE):

The reader might be wondering: what is the difference between P(EE) and P(E)? The definitions of the events differ a bit: P(E) requires that a conflict be in progress when an eastbound plane reaches J; P(EE) requires some east/north conflict, but allows for the possibility that the conflict is already over (or has not yet begun) when the eastbound plane passes through J. Still, does this distinction really matter?

Well, yes. Suppose that, when an eastbound plane arrives at J, there is a northbound plane six miles north of J. The two planes are not then in conflict. But consider the situation twelve seconds earlier, when the northbound plane was four miles north of J and the eastbound plane two miles west of it. The Pythagorean theorem reminds us that the two planes were $\sqrt{20} = 4.5$ miles apart at that time (i.e. that they were in conflict, even though they no longer are).

More generally, suppose that, when an eastbound plane reaches J, the nearest northbound plane to the north of the junction is **x** miles away. What was the situation **s** minutes ago? The northbound plane was x-10s miles north of J, while the eastbound plane was 10s miles west of it. (Recall that all planes travel ten miles per minute.) Thus $r^2(s)$, the squared distance between the planes s minutes earlier follows:

$$r^2(s) = (x-10s)^2 + (10s)^2$$

To determine the minimum distance between the two planes, we find the value of s at which $r^2(s)$ (and thus r(s)) is minimized. We obtain this quantity by setting the derivative of $r^2(s)$ with respect to s equal to zero, a process which reveals that the minimum distance between the northbound and eastbound planes occurred when $s = x/20$, at which time the distance between the planes was $x/\sqrt{2}$ = **.71x**. (Use the second derivative to convince yourself that we have a minimum, rather than a maximum or a point of inflection.)

If we set .71x=5, we find the distance (namely 7.1 miles) at which the minimum distance between the aircraft was exactly 5. For $x > 7.1$, the planes never get within 5 miles of each other; for x between 0 and 7.1, the planes always do. By symmetry, conflict occurs when the nearest northbound plane is between 0 and 7.1 miles *south* of J when the eastbound plane hits the junction. In summary, EE occurs if, at the moment the eastbound plane reaches J, there is a northbound aircraft within $10/\sqrt{2} \approx 7.1$ miles of the junction either way . (If the plane is between 5 and 7.1 miles away, then the conflict has already ended or not yet started.)

Because planes travel ten miles per minute, 7.1 miles corresponds to .71 minutes of flight. Therefore, the eastbound aircraft reaching J at time t will face a conflict if a northbound plane passes through J over the interval (t-.71,t+.71). In consequence,

$$\mathbf{P(EE) = 1 - P(EE^*) = 1 - exp(-1.42\lambda_E)}$$

This exercise assumed the absence of air-traffic control, and random arrivals at J under Poisson processe.. These calculations suggest the frequency at which **potentially** hazardous situations would arise based on activity levels in the air-traffic system. In reality, aircraft arrival times at junctions would never be left to chance alone: air traffic controllers would take steps to postpone a plane's arrival at J to avoid a conflict with another plane on a perpendicular path. The magnificence with which the controllers perform this task is suggested by a statistic: over the 1990's, 5 billion passengers travelled in commercial jet aircraft in the United States. The number killed in midair collisions was zero.