Crofton's Method

Let X_1 and X_2 be independent random variables that are uniformly distributed over the interval $[0, a]$. We are interested in computing $E[|X_1 - X_2|]$. For instance, in an urban setting, X_1 and X_2 may denote the location of an accident and the location where an emergency vehicle is currently parked in a road segment of length a, respectively. In this case, we want to know the distance (or the travel time) on average between the two locations, $E[|X_1 - X_2|]$. We may solve this question using a joint distribution of X_1 and X_2, but Crofton's method is quite useful for the question.

Let $G(a) \equiv E[|X_1 - X_2|]$. Now consider the following question: *If the interval were* $[0, a+\varepsilon]$ *where* ε *is small, what would* $G(a+\varepsilon)$ *be?* Table 1 summarizes $G(a+\varepsilon)$ depending on the locations of X_1 and X_2.

Table 1: $G(a+\varepsilon)$

Case	Probability of a case	$G(a+\varepsilon)$ given a case
$0 \le X_1 \le a,\ 0 \le X_2 \le a$	$\frac{a}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = (\frac{a}{a+\varepsilon})^2$	$G(a)$
$a < X_1 \le a+\varepsilon,\ 0 \le X_2 \le a$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$a + \frac{\varepsilon}{2} - \frac{a}{2} = \frac{a+\varepsilon}{2}$
$0 \le X_1 \le a,\ a < X_2 \le a+\varepsilon$	$\frac{a}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$a + \frac{\varepsilon}{2} - \frac{a}{2} = \frac{a+\varepsilon}{2}$
$a < X_1 \le a+\varepsilon,\ a < X_2 \le a+\varepsilon$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = (\frac{\varepsilon}{a+\varepsilon})^2$	We do not care.

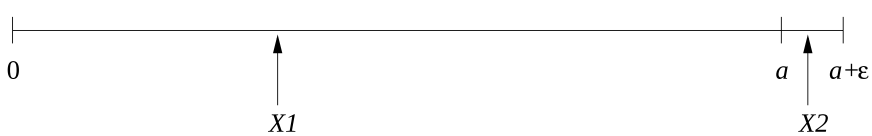

Figure 1: Case where $0 \le X_1 \le a$ and $a < X_2 \le a+\varepsilon$

Note that we did not specify $G(a+\varepsilon)$ for the case where $a < X_1 \le a+\varepsilon$ and $a < X_2 \le a+\varepsilon$, because the probability of that case, $(\frac{\varepsilon}{a+\varepsilon})^2$, is negligible when ε is small ("If ε is small, ε^2 is pathetic.").

To compute $G(a+\varepsilon)$ from Table 1, we invoke the total expectation theorem (or the conditional expectation rule). When a sample space is divided into $A_1, A_2, \ldots, A_n$ that are mutually exclusive and collectively exhaustive events, the expected value of a random variable Z is computed by

$$E(Z) = \sum_{i=1}^{n} E(Z \mid A_i) P(A_i) \ .$$

Using the total expectation theorem, we obtain

$$\begin{aligned} G(a+\varepsilon) &= G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + \frac{a+\varepsilon}{2}\frac{\varepsilon a}{(a+\varepsilon)^2} + \frac{a+\varepsilon}{2}\frac{\varepsilon a}{(a+\varepsilon)^2} + o(\varepsilon^2) \\ &= G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + \frac{\varepsilon a}{a+\varepsilon} + o(\varepsilon^2) \ , \end{aligned}$$

where $o(\varepsilon^2)$ is a collection of terms of order ε^2 or higher. If ε is small, we can ignore $o(\varepsilon^2)$. Hence we have

$$G(a+\varepsilon) \approx G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + \frac{\varepsilon a}{a+\varepsilon} \ .$$

From the formula of the sum of an infinite geometric series, we know

$$\frac{a}{a+\varepsilon} = \frac{1}{1+\frac{\varepsilon}{a}} = 1 - \frac{\varepsilon}{a} + \left(\frac{\varepsilon}{a}\right)^2 - \left(\frac{\varepsilon}{a}\right)^3 + \cdots \ .$$

Ignoring higher order terms of ε, we get

$$\frac{a}{a+\varepsilon} \approx 1 - \frac{\varepsilon}{a} \ .$$

This gives the following approximations:

$$\begin{aligned} \left(\frac{a}{a+\varepsilon}\right)^2 &\approx \left(1-\frac{\varepsilon}{a}\right)^2 = 1 - \frac{2\varepsilon}{a} + \frac{\varepsilon^2}{a^2} \approx 1 - \frac{2\varepsilon}{a} \ , \\ \frac{\varepsilon a}{a+\varepsilon} &\approx \varepsilon\left(1-\frac{\varepsilon}{a}\right) = \varepsilon - \frac{\varepsilon^2}{a} \approx \varepsilon \ . \end{aligned}$$

Therefore, we can rewrite $G(a+\varepsilon)$ as

$$G(a+\varepsilon) \approx G(a)\left(1-\frac{2\varepsilon}{a}\right) + \varepsilon \ .$$

Rearranging terms, we have

$$G(a+\varepsilon) - G(a) = \varepsilon\left(-\frac{2G(a)}{a} + 1\right) \ .$$

$$\Rightarrow \quad \frac{G(a+\varepsilon) - G(a)}{\varepsilon} = -\frac{2G(a)}{a} + 1 \ .$$

If $\varepsilon \to 0$, we have the following differential equation:

$$G'(a) = -\frac{2G(a)}{a} + 1 \ .$$

Now let us "guess" that $G(a) = C + Ba$, where C and B are some constants. Since $G(0) = 0$, we have $C = 0$. From the differential equation above, we obtain

$$B = -\frac{2Ba}{a} + 1 \quad \Rightarrow \quad B = \frac{1}{3} \ .$$

Therefore

$$G(a) = E[|X_1 - X_2|] = \frac{a}{3} \ .$$

Crofton's method can be used to compute $E[\max(X_1, X_2)]$ as well. In this case, there are slight changes in $G(a+\varepsilon)$ as shown in Table 2. Following a procedure similar to one we just used, we can show that

$$G(a) = E[\max(X_1, X_2)] = \frac{2a}{3} \ .$$

Table 2: $G(a+\varepsilon)$

Case	Probability of a case	$G(a+\varepsilon)$ given a case
$0 \le X_1 \le a, \ \ 0 \le X_2 \le a$	$\frac{a}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = (\frac{a}{a+\varepsilon})^2$	$G(a)$
$a < X_1 \le a+\varepsilon, \ \ 0 \le X_2 \le a$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$a + \frac{\varepsilon}{2}$
$0 \le X_1 \le a, \ \ a < X_2 \le a+\varepsilon$	$\frac{a}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$a + \frac{\varepsilon}{2}$
$a < X_1 \le a+\varepsilon, \ \ a < X_2 \le a+\varepsilon$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = (\frac{\varepsilon}{a+\varepsilon})^2$	We do not care.