Spatially Distributed Queues II

M/G/1

2 Servers

N servers: Hypercube Queueing Model

Approximations

Setup: Hypercube Queueing Model

- Region comprised of geographical atoms or nodes
- Each node *j* is an independent Poisson generator, with rate λ_j
- Travel times: τ_{il} = travel time from node *i* to node *j*
- *N* servers
- Server locations are random: l_{nj}

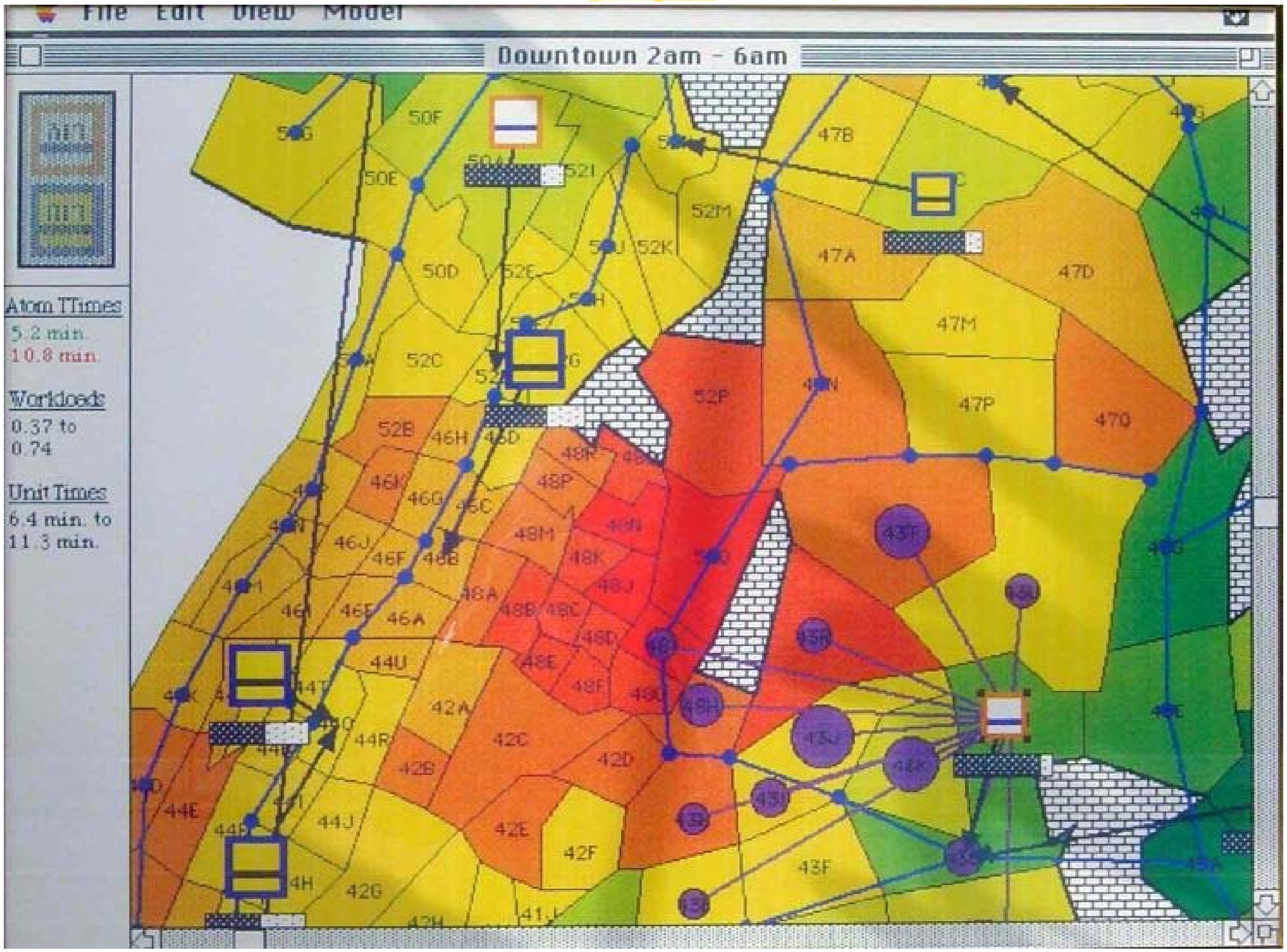
Setup: Hypercube Queueing Model - con't.

- Server assignment: one assigned
- State dependent dispatching
- Service times: mean = $1/\mu_n$; *negative exponential density*
- Service time dependence on travel time
- We allow a queue (FCFS, infinite capacity)

Fixed Preference Dispatch Policies for the Model

- Idea: for each atom, say Atom 12, there exists a vector of length *N* that is the preference-ordered list of servers to assign to a customer from that atom
- Example: {3,1,7,5,6,4,2}, for *N*=7.
- Dispatcher always will assign the most preferred ***available*** server to the customer
- Usually order this list in terms of some travel time criterion.

New York City EMS Hypercube



New York City EMS Hypercube

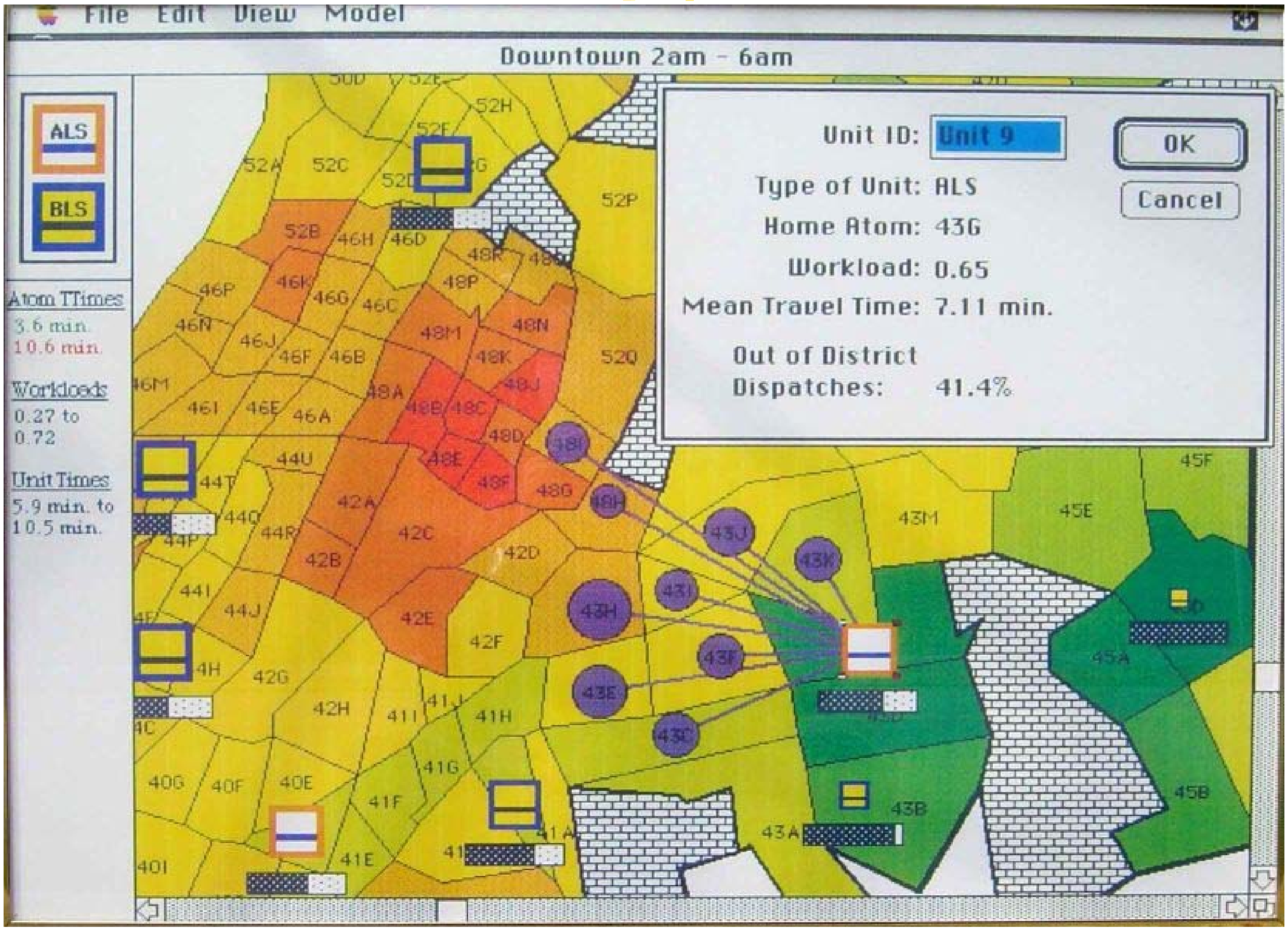


Illustration of Desktop Hypercube with ARCVIEW

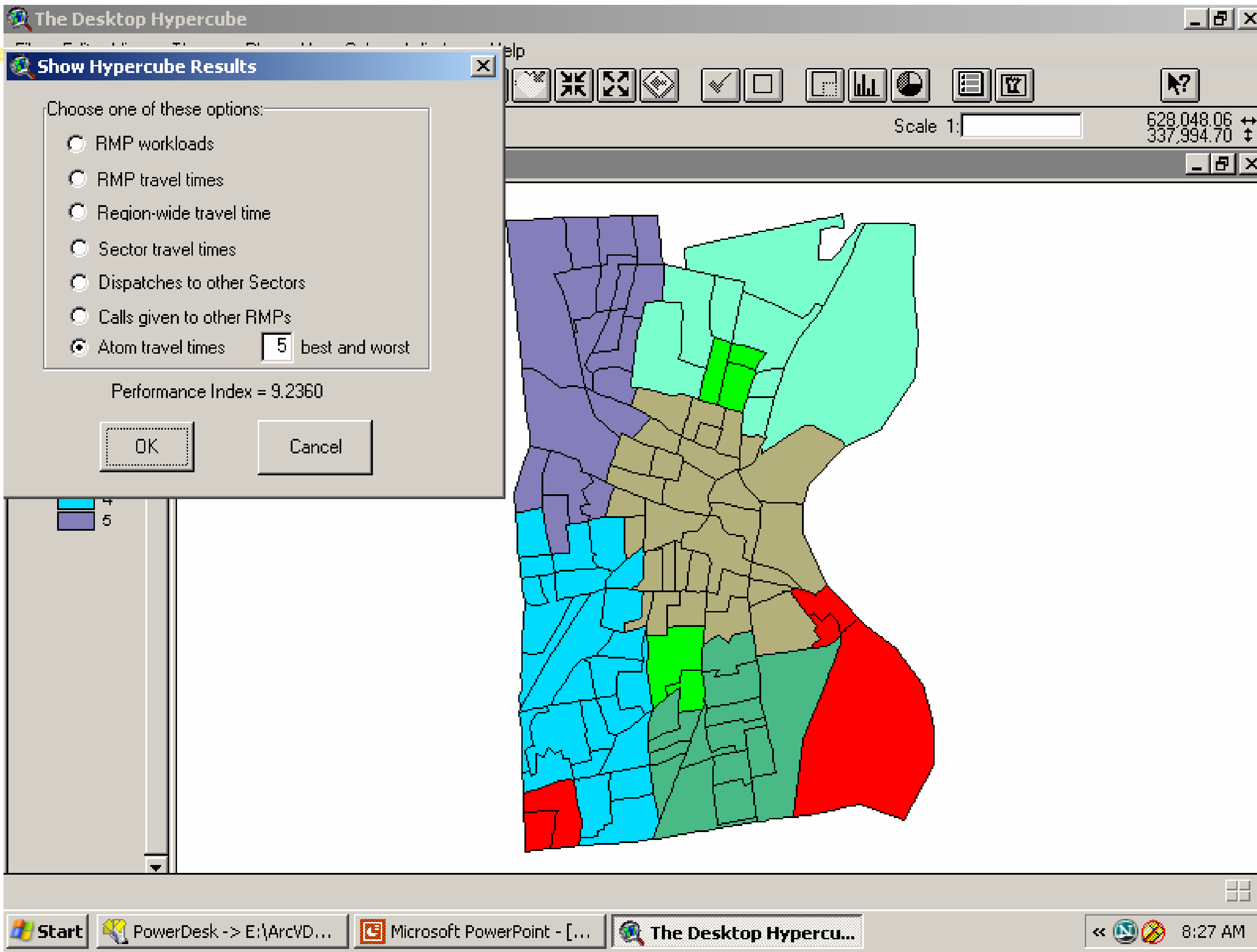
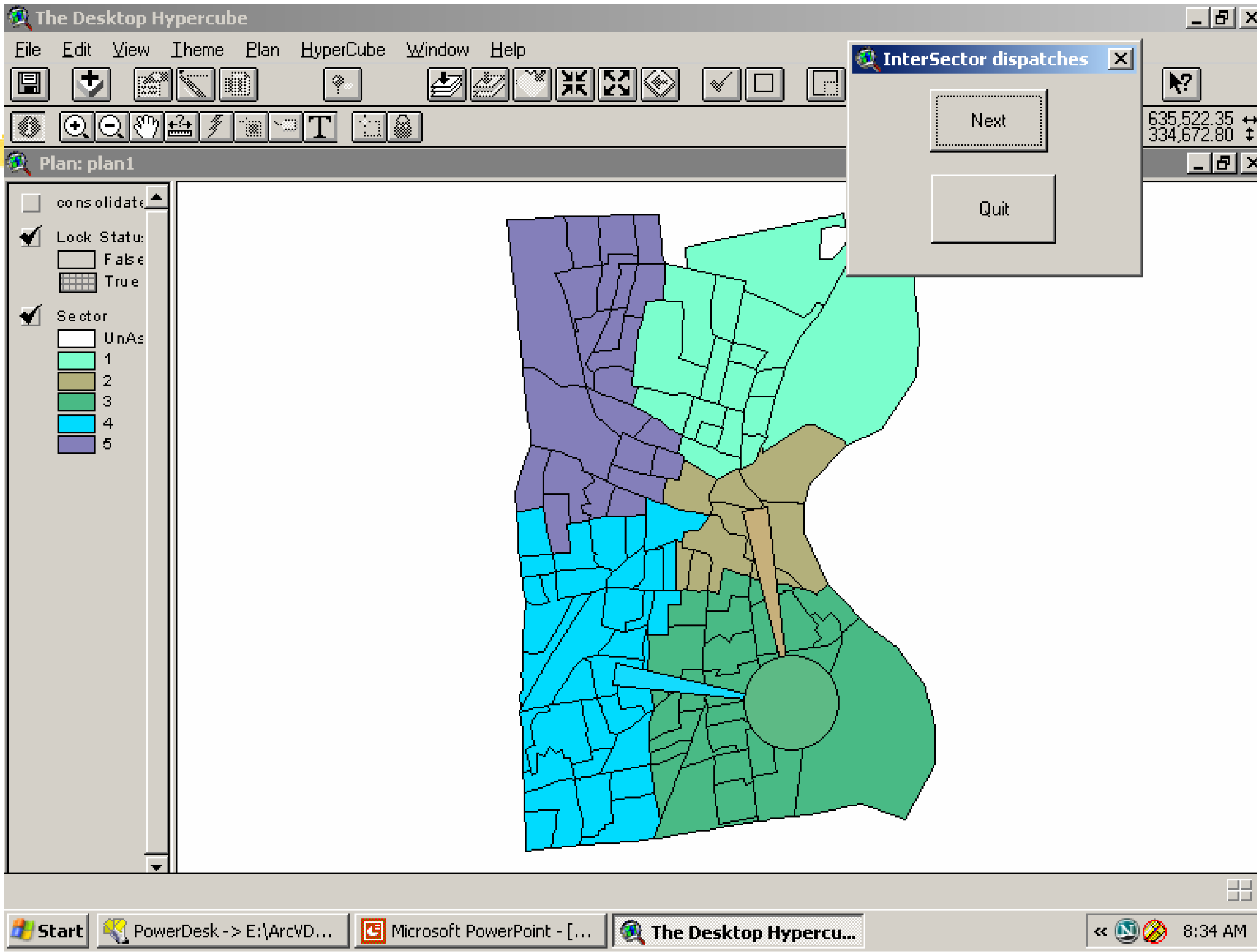


Illustration of Desktop Hypercube with ARCVIEW



Example Dispatch Policies

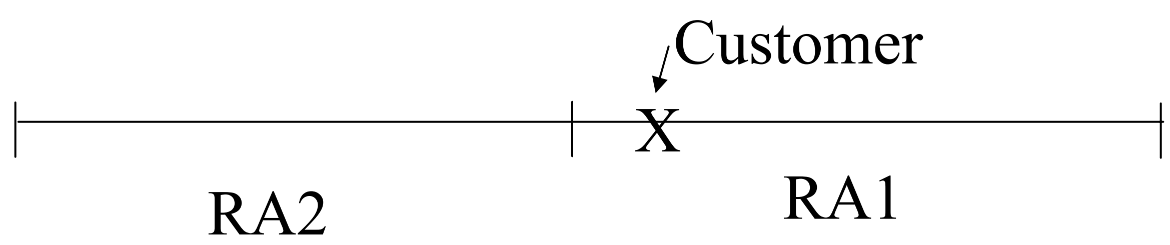
- SCM: Strict Center of Mass
  - Place server at its center of mass
  - Place customer at its center of mass
  - Estimate travel times: center of mass to center of mass
- MCM: Modified Center of Mass
  - Place server at its center of mass
  - Keep customer at centroid of atom
  - Estimate travel times: center of mass to centroid of atom

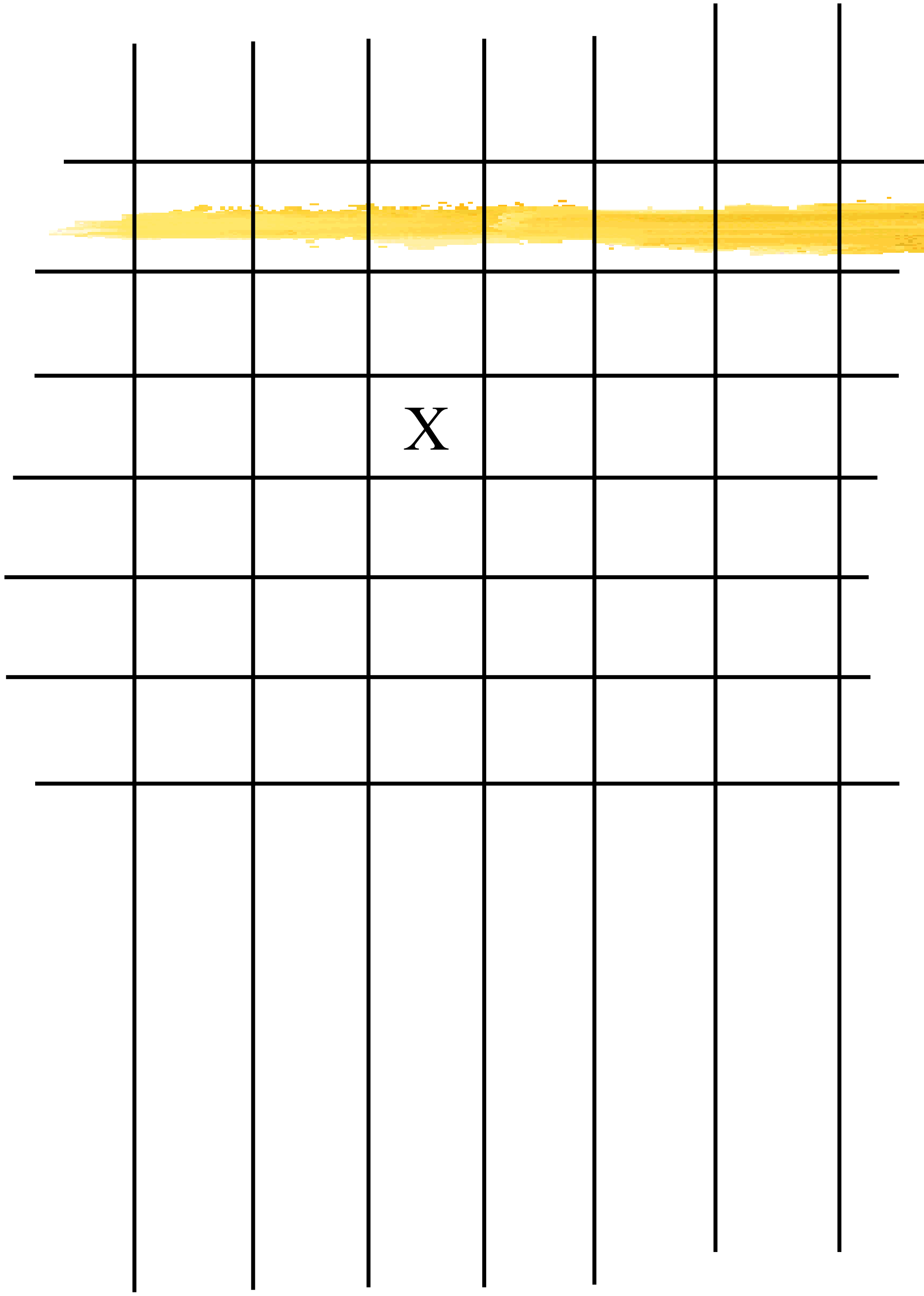
Example Dispatch Policies

- EMCM: Expected Modified Center of Mass
  - Do the conditional expected travel time calculation correctly, conditioned on the centroid of the atom containing the customer

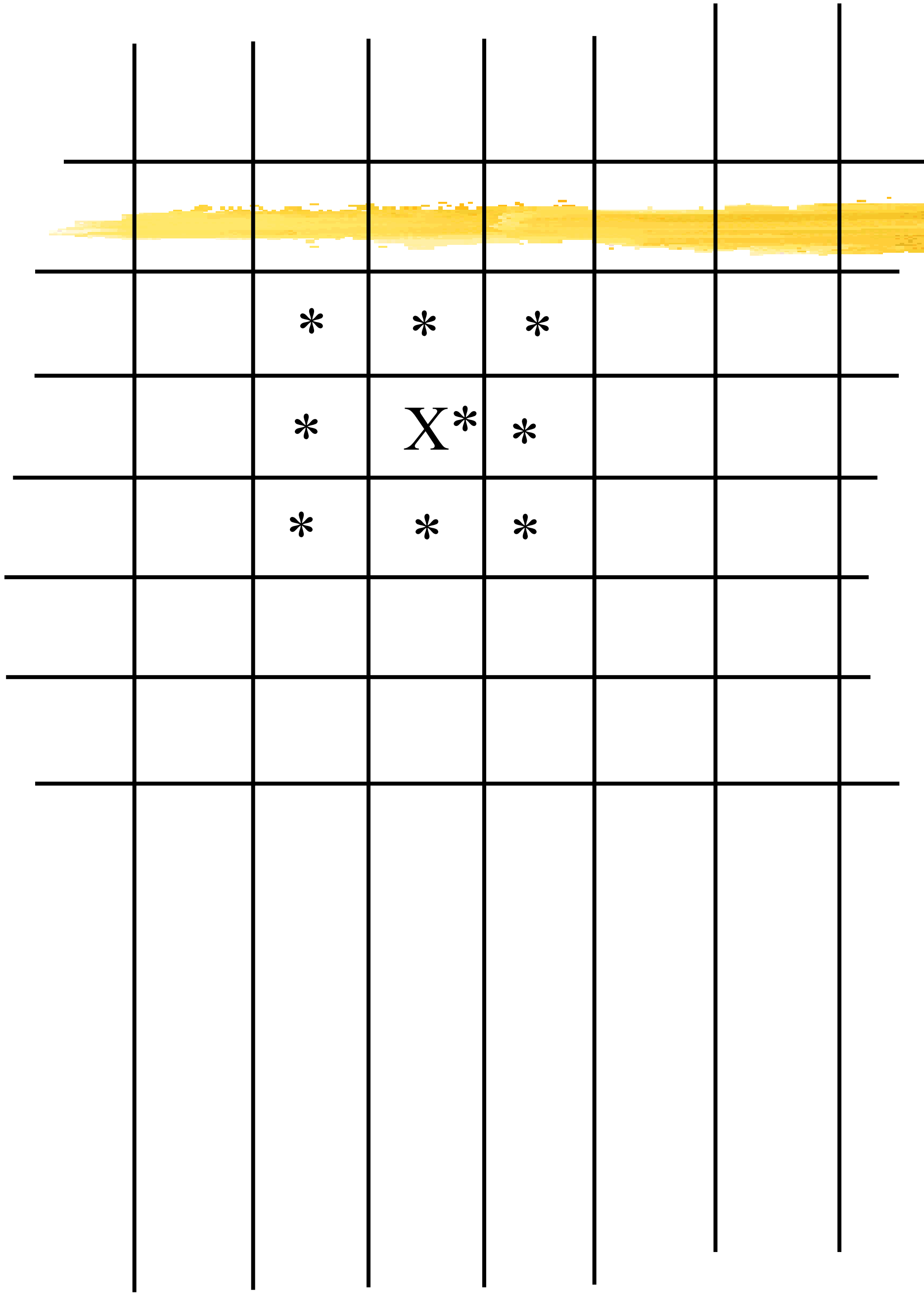
Are fixed preference policies optimal?

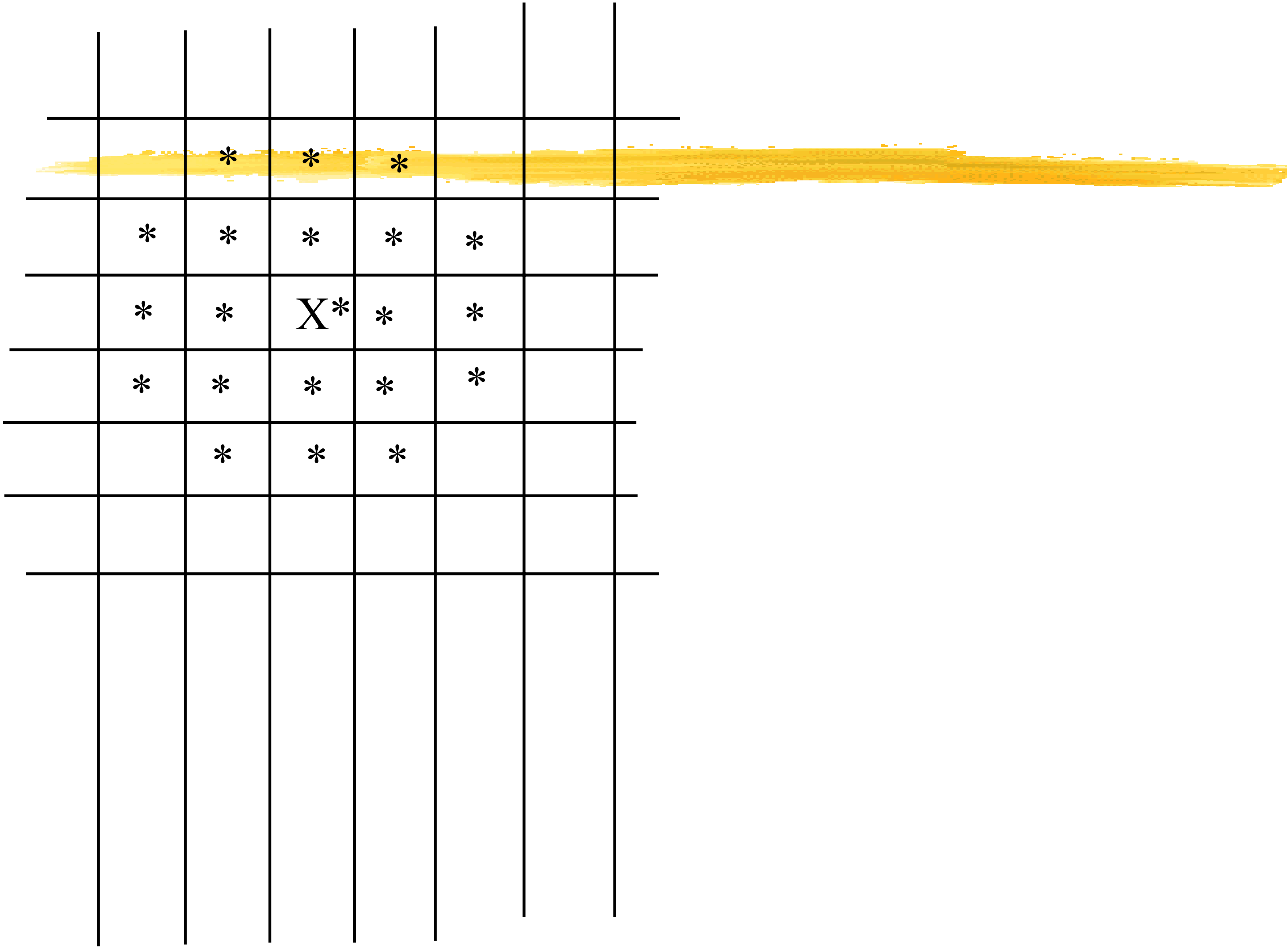
- AVL: Automatic Vehicle Location: dispatch the real time nearest server
  - This can be incorporated into the Hypercube framework, but very carefully!
  - Consider two servers:





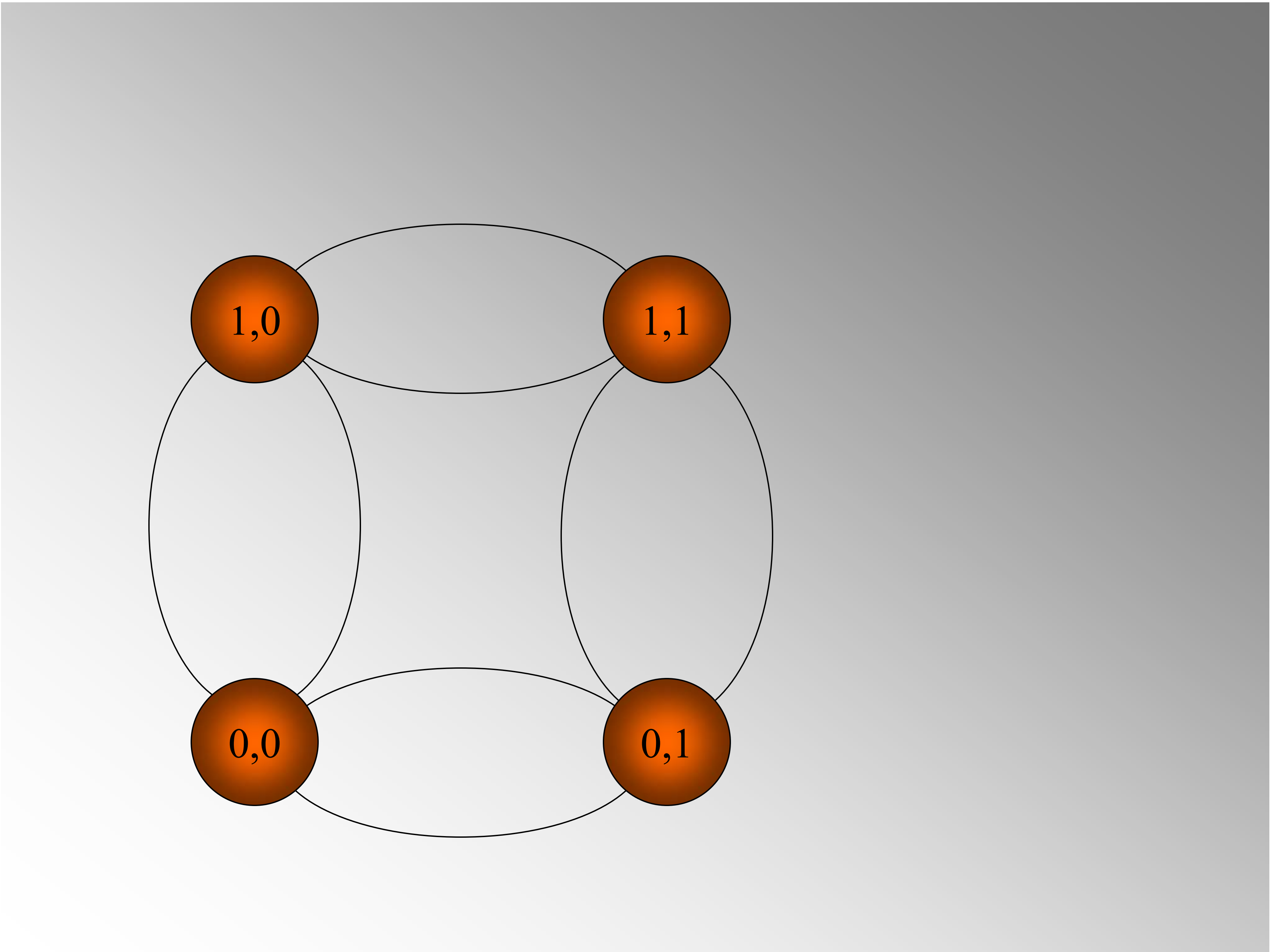
Customer in square marked X. Place an asterisk in each square that could have the closest server.
Assume each server is available and is located 'somewhere' in his/her square "police beat."

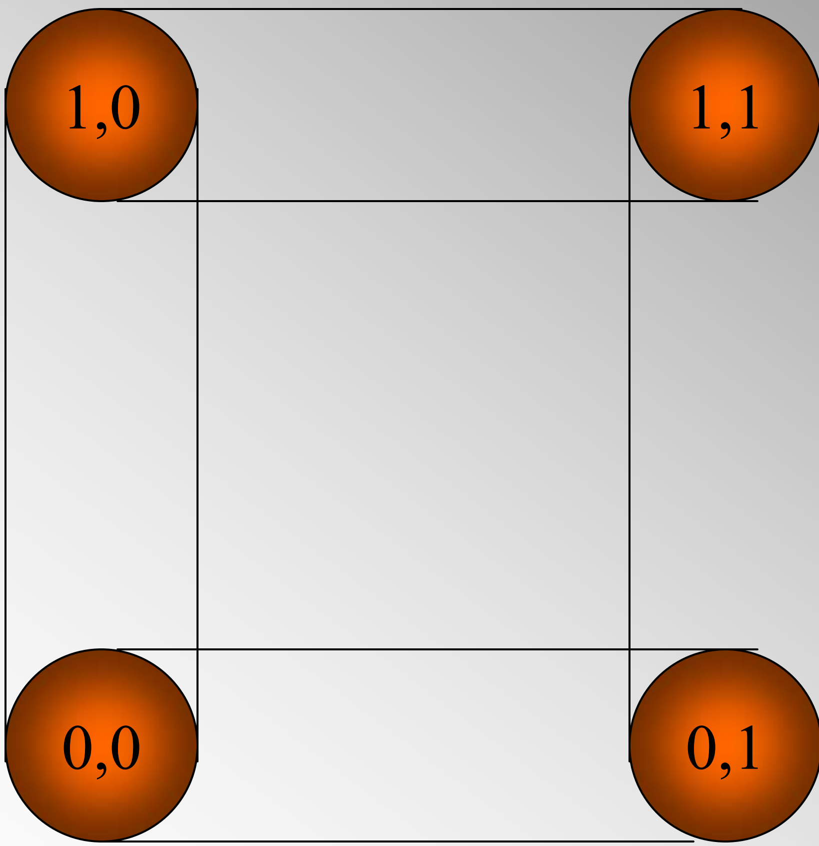

* * *
* X* *
* * *


X*

What to know about the Hypercube Queueing Model

- Know the 2-server setup
- Be able to work with a 3-server model
  - Read in the text the formulas to apply
- Forget the cases for $N>3$ servers.
- Know Hypercube Approximation Procedure (still to come -- fasten your seat belts!)


1,0
1,1
0,0
0,1


1,0
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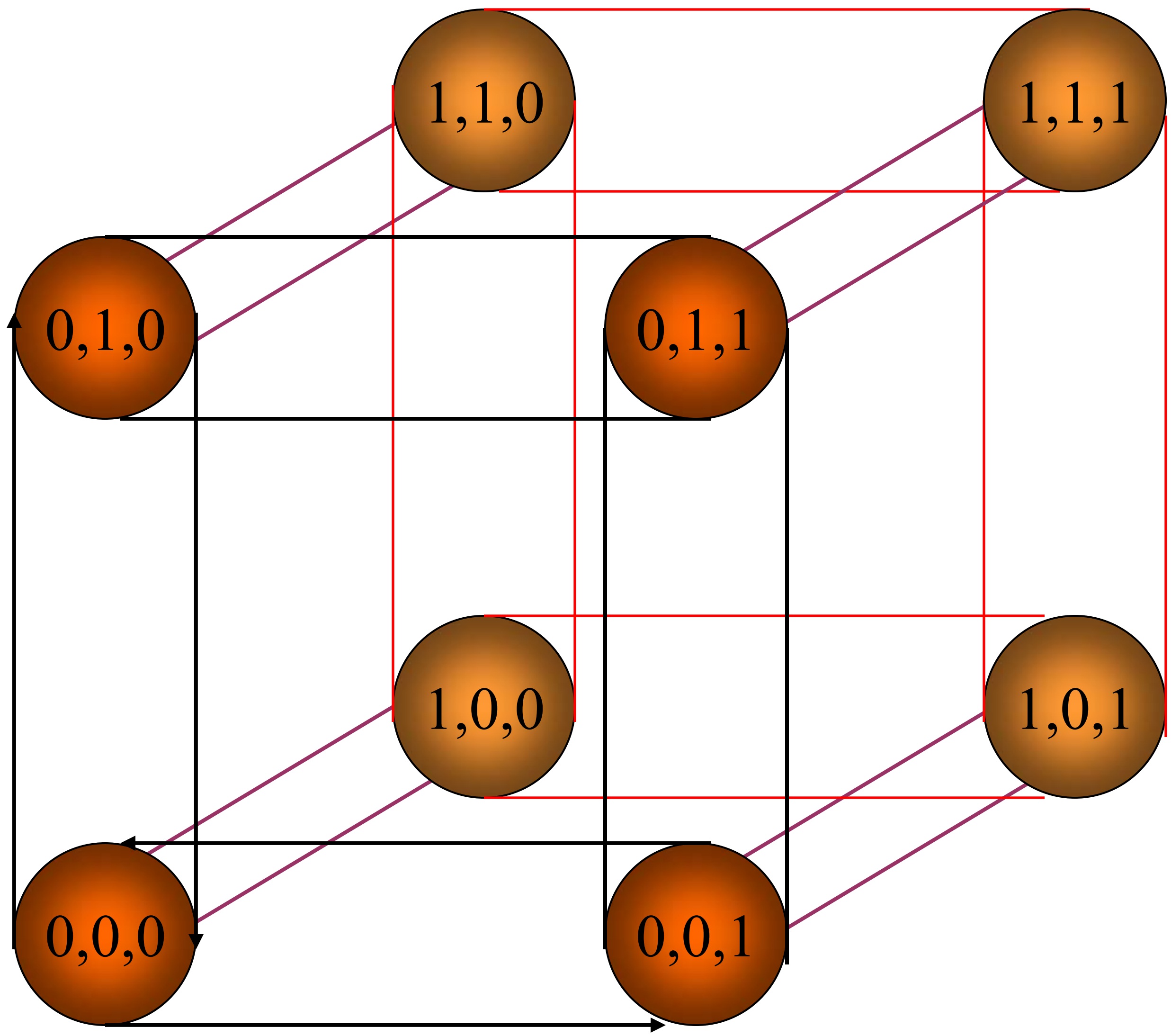

1,1,0
1,1,1
0,1,0
0,1,1
1,0,0
1,0,1
0,0,0
0,0,1

Hypercube Approximation Procedure: A General Technique

- Want to reduce dramatically the number of simultaneous equations to solve
- The procedure reduces the number of equations from 2^N simultaneous linear equations to N simultaneous nonlinear equations.
- We look at only those performance measures we need, not at micro-structure of the binary state space

Hypercube Approximation Procedure
A General Technique

Theory: Sampling Servers Without Replacement from *M/M/N* Queue

From $M/M/N/\infty$ we know the aggregate state probabilities:

$$P\{S_k\} \equiv P_k = N^k \rho^k P_0 / k! \qquad k = 0,1,2,\ldots,N-1$$

$$P\{S_N\} \equiv P_N = N^N \rho^N P_0 /(N![1-\rho])$$

$$P\{S_0\} \equiv P_0 = [\sum_{i=0}^{N-1} N^i \rho^i / i! + N^N \rho^N /(N!\{1-\rho\})]^{-1}$$

The Hypercube Model, when the state space is compressed from its cube in *N* dimensions to a 'line' birth and death process, always reduces to an *M/M/N* queue (assuming service times are not server-specific)

Key expression: $P\{B_1, B_2, \ldots, B_j, F_{j+1}\}$

For our applications, we do not need to know the fine grained binary state probabilities. Rather we need dispatch probabilities and server workloads.

What about 'B-' probability reasoning?

"Flips coins" until first Heads is obtained:

$$P\{B_1, B_2, \ldots, B_j, F_{j+1}\} \approx \begin{cases} \rho^j(1-\rho) & j = 0,1,2,\ldots,N-1 \\ \rho^N & j = N \end{cases}$$

Incompatible with known state probability P_N

Doesn't include biases.

Let's "Divide and conquer":

$$P\{B_1, B_2, \ldots, B_j, F_{j+1}\} = \sum_{k=0}^{k=N} P\{B_1, B_2, \ldots, B_j, F_{j+1} \mid S_k\} P_k \qquad (*)$$

Working carefully and slowly to find the state-conditioned dispatch probabilities:

$$P\{B_1, B_2, \ldots, B_j, F_{j+1} \mid S_k\} = P\{F_{j+1} \mid B_1, B_2, \ldots, B_j, S_k\} \ldots P\{B_2 \mid B_1 S_k\} P\{B_1 \mid S_k\}$$

$$P\{B_1, B_2, \ldots, B_j, F_{j+1} \mid S_k\} = \frac{N-k}{N-j} \ldots \qquad \ldots \frac{k-1}{N-1} \ldots\ldots \frac{k}{N} \quad (**)$$

Can plug (**) back into (*) and obtain an exact expression. Manipulate it to obtain a convenient form as "B-" probability reasoning with an 'A+" correction term:

$$P\{B_1, B_2, \ldots, B_j, F_{j+1}\} = Q(N, \rho, j)\rho^j (1-\rho) \qquad (***)$$

"Correction factor"

Explore properties of Correction Factor

The desired dispatch probabilities can be written as a telescoped expression:

$$P\{B_1, B_2, \ldots, B_j, F_{j+1}\} = P\{F_{j+1} \mid B_1 B_2 \ldots B_j\} P\{B_j \mid B_1 B_2 \ldots B_{j-1}\} \ldots P\{B_1\}$$

Use above in Eq.(***) to obtain:

$$Q(N, r, j) = \left[\frac{P\{F_{j+1} \mid B_1 \ldots B_j\}}{1-\rho}\right]\left[\frac{P\{B_j \mid B_1 \ldots B_{j-1}\}}{\rho}\right] \ldots \left[\frac{P\{B_1\}}{\rho}\right]$$

≤ 1 $\qquad$ ≥ 1 $\qquad$ $= 1$

$G_n^k \equiv$ set of geographical atoms for which unit n is the k^{th} preferred dispatch alternative

$n_{lj} \equiv$ id # of the j^{th} preferred unit for atom l

Set $\mu = 1$

$$\rho_n = \sum_{j \in G_n^1} \lambda_j P\{F_n\} + \sum_{j \in G_n^2} \lambda_j P\{B_{n_{j1}} F_n\} + \sum_{j \in G_n^3} \lambda_j P\{B_{n_{j1}} B_{n_{j2}} F_n\} + \ldots + \lambda P_N / N$$

$$\rho_n = \sum_{j \in G_n^1} \lambda_j (1 - \rho_n) + \sum_{j \in G_n^2} \lambda_j Q(N, \rho, 1) \rho_{n_{j1}} (1 - \rho_n) + \sum_{j \in G_n^3} \lambda_j Q(N, \rho, 2) \rho_{n_{j1}} \rho_{n_{j2}} (1 - \rho_n) + \ldots + \lambda P_N / N$$

The last equation gives N nonlinear simultaneous equations in the unknown workloads, ρ_n, *subject to the constraint that*

$$\sum_{n=1}^{N} \rho_n = \lambda \quad \text{"normalization"}$$

Typically converges in 3 to 5 iterations, within 1 to 2% of 'exact Hypercube' results

Response patterns:

$$f_{n_{kj}k} = \frac{\lambda_k}{\lambda} Q(N, \rho, j-1) \left\{ \prod_{l=1}^{j-1} \rho_{n_{kl}} \right\} (1 - \rho_{n_{kj}})$$

- $f_{n_{kj}k}$: id # of j^{th} preferred unit for atom k
- $\prod_{l=1}^{j-1} \rho_{n_{kl}}$: *j-1* more preferred units
- $(1 - \rho_{n_{kj}})$: j^{th} preferred unit

Square Root Laws (approximations)

In Chapter 3 we found

$$E[D] = C\sqrt{\frac{A}{N_0}}$$

Area of service region (A)

Number of mobile servers (N_0)

C: depended on distance metric and location strategy

Assumes all N_0 servers are available or free (not busy)

Now consider *N* to be a R.V.

Might we expect the following to be true?

$$E[D \mid N = k] = C\sqrt{\frac{A}{k}} \qquad k = 1, 2, \ldots, N_0$$

What if the locations of servers were determined by a homogenous spatial Poisson process, with busy servers selected by "random erasers"?

Getting to Expected Travel Distance

$$E[D] = P_0 D_0 + \sum_{k=1}^{N_0} P_k C \sqrt{\frac{A}{k}}$$

From $M/M/N_0$ queueing model

where P_k = Probability of k servers ***available*** *($M/M/N_0$)*

Moving to E[*D*]

Since $P_0 \cong 0$, we can write

$$E[D] \cong C\sqrt{A}E_{\substack{\text{states of} \\ M/M/N_0}}[1/\sqrt{N}]$$

We now apply "B-" probability reasoning, to get

$$E[D] \cong C\sqrt{\frac{A}{E[N]}}$$

(Jensen's Inequality shows that this Eq. is a lower bound to true E[*D*].)

Finishing

$$E[N] \approx N_0 - N_0\rho = N_0(1-\rho)$$

$$E[D] \approx C\sqrt{\frac{A}{N_0(1-\rho)}}$$

$$E[T] \approx \frac{C}{v}\sqrt{\frac{A}{N_0(1-\rho)}} + \frac{v}{a}$$

↑ Acceleration term

Great results in practice

Jensen's Inequality

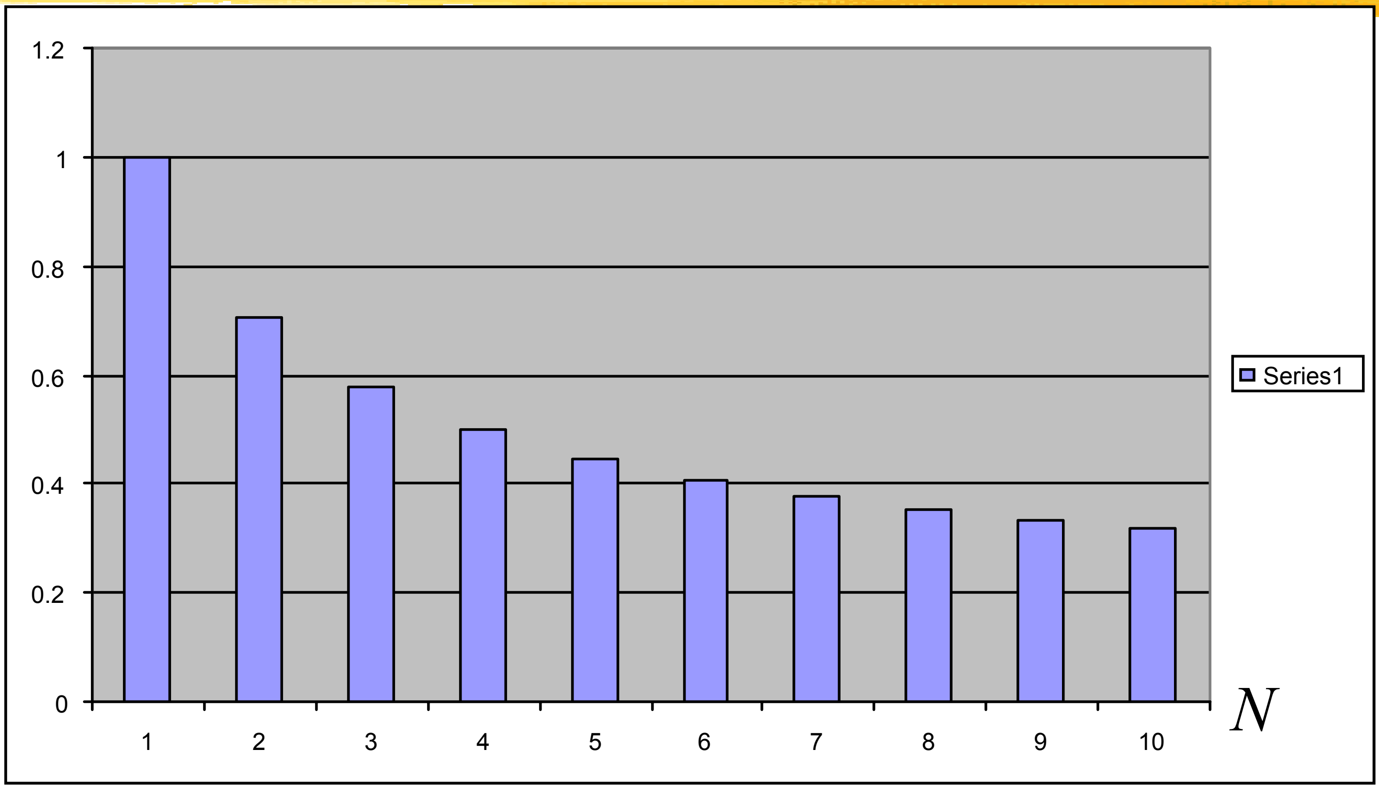
If g(X) is a convex function over the region of non-zero probability, then

$$E[g(X)] \geq g(E[X])$$

(Problem 5.5 explores this further.)

Jensen's Inequality

$1/\sqrt{N}$

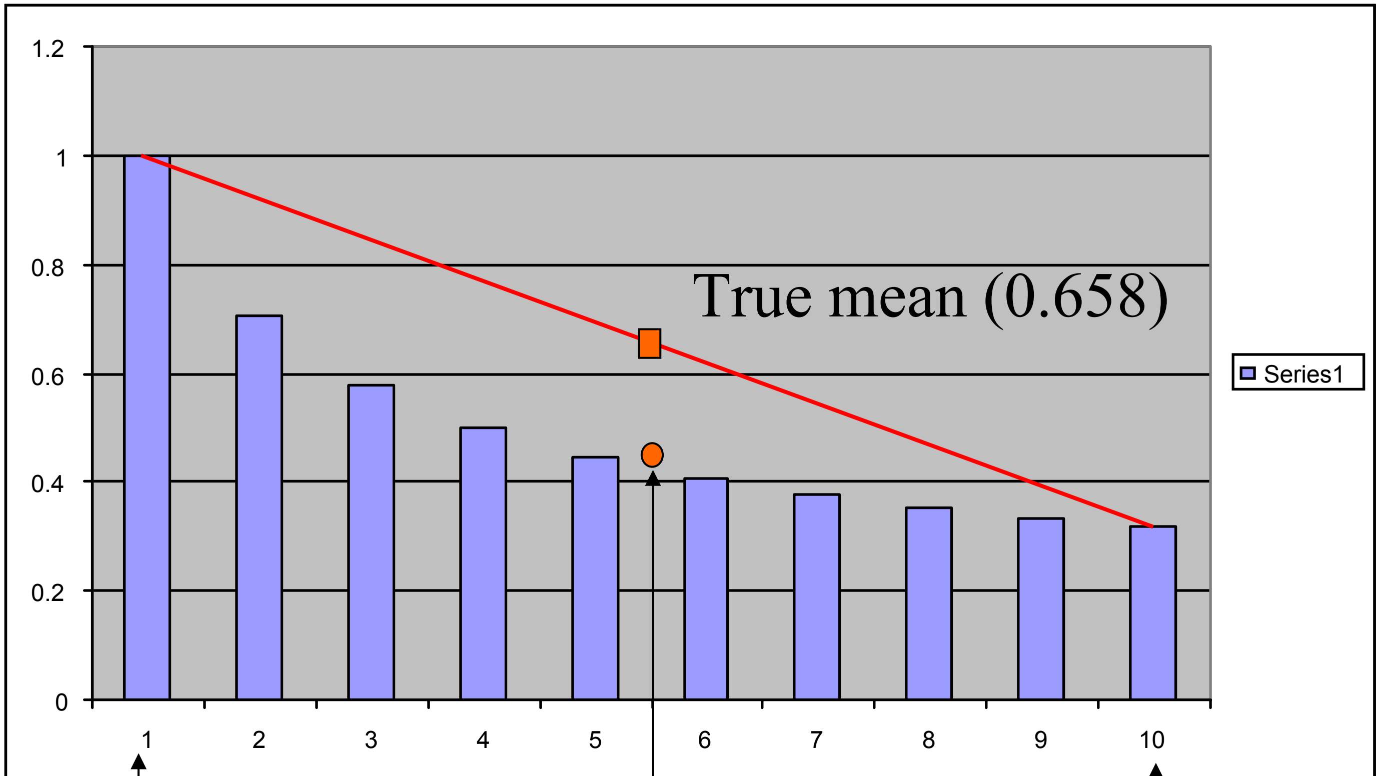


N

$$E[1/\sqrt{N}] = 0.5*(1) + 0.5*(10)^{-0.5} = 0.5*(1 + 0.316) = 0.658$$

$$1/E[N]^{-0.5} = 1/(1*0.5 + 10*0.5)^{-0.5} = 1/(5.5)^{-0.5} = 1/2.345 = 0.426$$

$1/\sqrt{N}$



Suppose 50% of probability mass here

"B-" mean (0.426)

And suppose 50% of probability mass here