Urban Operations Research **1999 Quiz 1 Solutions**

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Problem 1: M/D/1

The arrival process is Poisson and the service is deterministic. Therefore, This system can be modeled by an M/D/1.

(i)

$$E[W] = E[W_q] + E[W_s]$$

Where, Wq = Average time in the queue
Ws = Average time in service = 1

We know that from M/G/1 note that:

$$E[W_q] = \frac{\lambda \times E[X^2]}{2 \times (1 - \rho)}$$

Therefore,

$$E[W] = \frac{0.5 \times 1}{2 \times (1 - 0.5)} + 1 = 1.5$$

(ii)

The time that Mendel will stay in the system is: T = 5 + 1 + V
Because Mendel has to wait 5 time units for service completion of the customers in the queue and 1 unit time for his service completion with probability 1. V is the time until completion of the customer currently in service when Mendel arrives. This is a case of random incidence. V is uniformly distributed between 0 and 1 and as a consequence T is uniformly distributed between 6 and 7.

(iii)

If a busy period has just begun, the minimum time it could last is 1. The probability that this event happens is the probability that nobody arrives in this unit time interval. Therefore, the probability is: P(A)=$e^{-0.5} \approx 0.607$

(iv)

Let call B the busy period random variable and A, the event: 'the busy period is longer than 1.

For M/G/1, we know that $E[B] = \frac{1}{\mu - \lambda} = 2$

We already know P(A) from (iii) and E[B | A]=1
Therefore,

$$E[B|\bar{A}].P(\bar{A}) = E[B]-E[B|A].P(A)$$

$$E[B|\bar{A}] = \frac{2 - e^{-0.5}}{1-e^{-0.5}} = 3.54$$

Problem 2: Crofton's Method

The objective is to compute $J(a)=E(x_1^2+x_2^2)$ with, x_1 and x_2 U(0,a).

Crofton's Method boils down to solving the differential equation:

$$\frac{dJ(a)}{da} + \frac{2}{a} \times J(a) = \frac{2}{a} \times J_1(a)$$

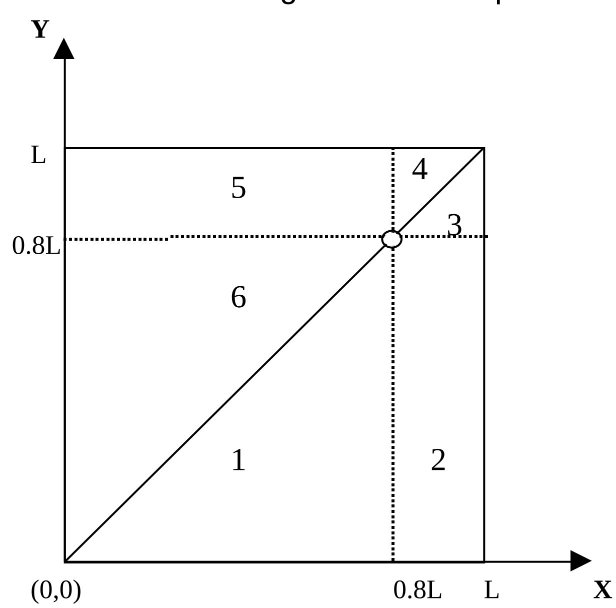
Where, $J_1(a)= E(x_1^2+x_2^2 | x_1=a)= \int_0^a \frac{(x^2+a^2)}{a} dx = \frac{4}{3}a^2$

A particular solution of the form Ka^2 gives: K=2/3

Therefore, $J(a) = \frac{2}{3} \times a^2$

Problem 3: The Diagonal Barrier

One can denote the different regions of the square the following way.



We use the perturbation method:

$$E[D'] = E[D]+E[D_e]$$

First, we know that $E[D] = 2L/3$

Let find $E[D_e]$, the average extra distance traveled due to the barrier.

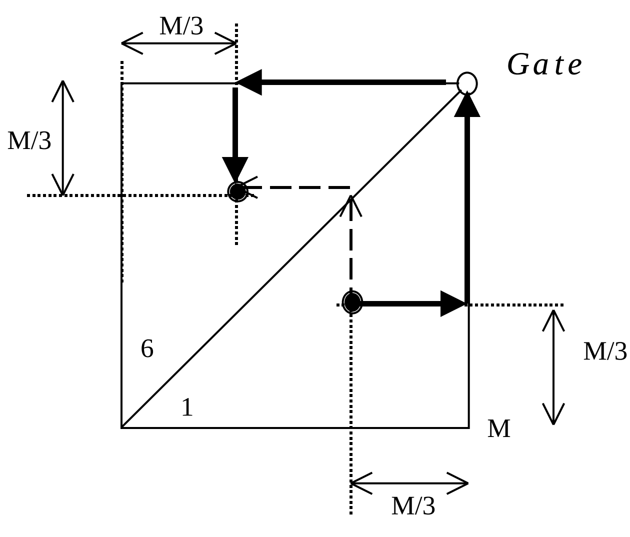
We present in details the work for the event A_1

Let's A_1 be the event that the emergency and the response vehicle are located in (1,6) or (6,1). The probability of this event happening is:

$$P(A_1) = 2 \times \left[\frac{(0.8L)^2}{2}\right]^2 \times \frac{1}{L^4} = \frac{(0.8)^4}{2}$$

Now, Let compute $E[D_e \,|\, A_1]$:

We define $M=0.8L$



Looking at the picture above it requires 4M/3 additional distance to go from the center of mass of 1 to the center of mass of 6.

Hence,

$$E[D_e \,|A_1].P(A_1) = \frac{2}{3} \times (0.8)^5 L$$

Using the same reasoning we find:

Event A_k	$P(A_k)$	$E[D' \mid A_k]$
{1,5}	$0.2(0.8)^3$	2(0.8)L/3
{1,6}	$(0.8)^4/2$	4(0.8)L/3
{2,4}	$0.8(0.2)^3$	2(0.2)L/3
{2,6}	$0.2(0.8)^3$	2(0.8)L/3
{3,4}	$(0.2)^4/2$	4(0.2)L/3
{3,5}	$0.8(0.2)^3$	2(0.2)L/3

Note that {1,5} = {2,6} and {2,4}={3,5}. Moreover, {3,4} is the same problem as {1,6} except that M=0.2L instead of 0.8L.

Finally,

$$E[D_e] = \sum_{k=1}^{6} E[D_e \mid A_k] \times P(A_k)$$

After some algebra one finds,

$$E[D_e] = \frac{2L}{3} \times \left\{2(0.2)(0.8)^4 + 2(0.2)^4(0.8) + (0.8)^5 + (0.2)^5\right\}$$

Therefore, $E[D']=2L/3 + E[D_e]=0.996L$