MASSACHUSETTS INSTITUTE OF TECHNOLOGY
1.203J/6.281J/13.665J/15.073J/16.76J/ESD.216J
Logistical and Transportation P lanning Methods
Fall 2002

Quiz 1 Solutions

1

(a)(i) Without loss of generality we can pin down X_1 at any fixed point. X_2 is still uniformly distributed over the square. Assuming that the police car will always follow a shortest route to the emergency incident, the max possible distance between X_1 and X_2 is 2 km. The travel distance is thus uniformly distributed between 0 and 2 km.

(a)(ii) Following similar logic, the max possible distance is now 4 km. The travel distance is thus uniformly distributed between 0 and 4 km.

(b) Let's number the links as shown in Figure 1. There is a $\frac{1}{12}$ chance that the emergency incident will be on any one of the 12 links. Thus if we can determine the conditional pdf for the travel distance from X_1 (conditioned to be uniformly distributed on link 7) to X_2 for each possible link for X_2, we are done. All we do then is add the resulting conditional pdfs, multiplying each by $\frac{1}{12}$, the probability of occurrence. Careful inspection of the problem reveals that with regard to computing the conditional travel distance pdf between X_1 and X_2 there are three sets of links

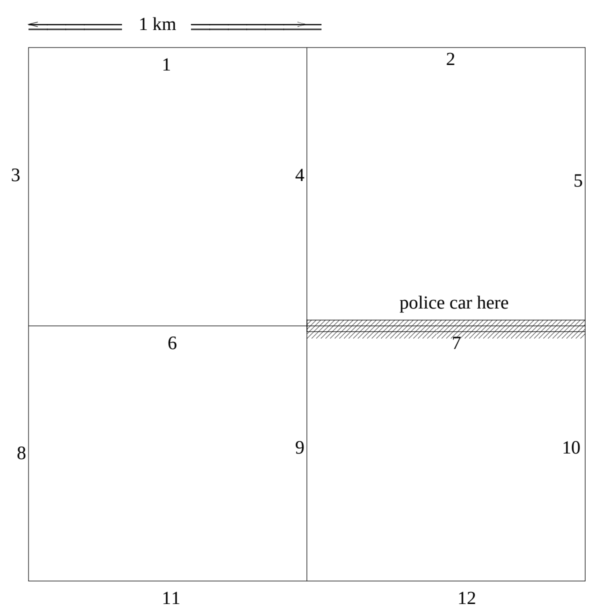

Figure 1: Link Numbering

Set 1: 1, 3, 4, 5, 6, 8, 9, 10, 11

Consider link 1. Define X_2 to be the distance from the right most point of link 1. Then the conditional travel distance, given that X_1 is defined to be the distance on link 7 from its left most point and that X_2 is on link 1, is $(D|1,7) = X_1 + X_2 + 1$. Let's define $V = X_1 + X_2$. Note that $X_1, X_2 \sim \mathrm{U}(0,1)$ i.i.d. Either by convolution, or by the "never fail" sample space method using cumulative distribution functions, or by recalling problem 2(e)(i) of HW1, we find

$$f_V(d) = \begin{cases} d, & d \in [0,1] \\ 2-d, & d \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

Now the conditional pdf we want for link 1 is $f_V(d)$ "shifted to the right" by one unit of distance. Call this conditional pdf $f(D|1,7)(d)$. Then we have for link 1, $f(D|1,7)(d) = f_V(d-1)$. By inspection we also have $f(D|3,7)(d) = f(D|8,7)(d) = f(D|11,7)(d) = f(D|1,7)(d) = f_V(d-1)$. For the remaining links in Set 1, links 4, 5, 6, 9 and 10 "touch" link 7, so there is no shifting of the pdf by one. That is, there is no intermediate link between them that would add 1.0 km to the travel distance. Hence, $f(D|4,7)(d) = f(D|5,7)(d) = f(D|6,7)(d) = f(D|9,7)(d) = f(D|10,7)(d) = f_V(d)$.

Set 2: 2, 12

Consider link 2. (Link 12 is probabilistically the same as link 2.) Say that X_2 is the distance defined from the left most point on link 2 and that, as before, X_1 is the distance from the left most point on link 7. The police car, which can make U-turns, will follow a minimum distance path from its location at X_1 to the emergency incident at X_2. Call this distance $(D|2,7)$. Then we can write

$$\begin{aligned} (D|2,7) &= \min\{X_1 + X_2 + 1, (1-X_1) + (1-X_2) + 1\} \\ &= 1 + \min\{X_1 + X_2, 2 - (X_1 + X_2)\} \end{aligned}$$

Let $W = \min\{X_1 + X_2, 2-(X_1+X_2)\}$. Again, $X_1, X_2 \sim \mathrm{U}(0,1)$ i.i.d. Now use the never-fail cumulative distribution function method.

$$\begin{aligned} F_W(d) &= P(\min\{X_1 + X_2, 2-(X_1+X_2)\} \le d) \\ &= P(X_1 + X_2 \le 2 - (X_1+X_2),\ X_1 + X_2 \le d) \\ &\quad + P(X_1 + X_2 > 2 - (X_1+X_2),\ 2 - (X_1+X_2) \le d) \end{aligned}$$

where the last equality follows since $X_1 + X_2 \le 2 - (X_1+X_2)$ and $X_1 + X_2 > 2 - (X_1+X_2)$ are mutually exclusive and exhaustive events. First note that $\min\{X_1 + X_2, 2-(X_1+X_2)\}$ is always nonnegative and less than or equal to 1. Now consider $d \in [0,1]$. Because the joint distribution of (X_1, X_2) was uniform over the unit square, we can compute the above probabilities by calculating the areas of the smaller triangles shown in Figure 2. So, finally, we obtain that

$$F_W(d) = \begin{cases} 0, & d \le 0 \\ d^2, & d \in [0,1) \\ 1, & d \ge 1 \end{cases}$$

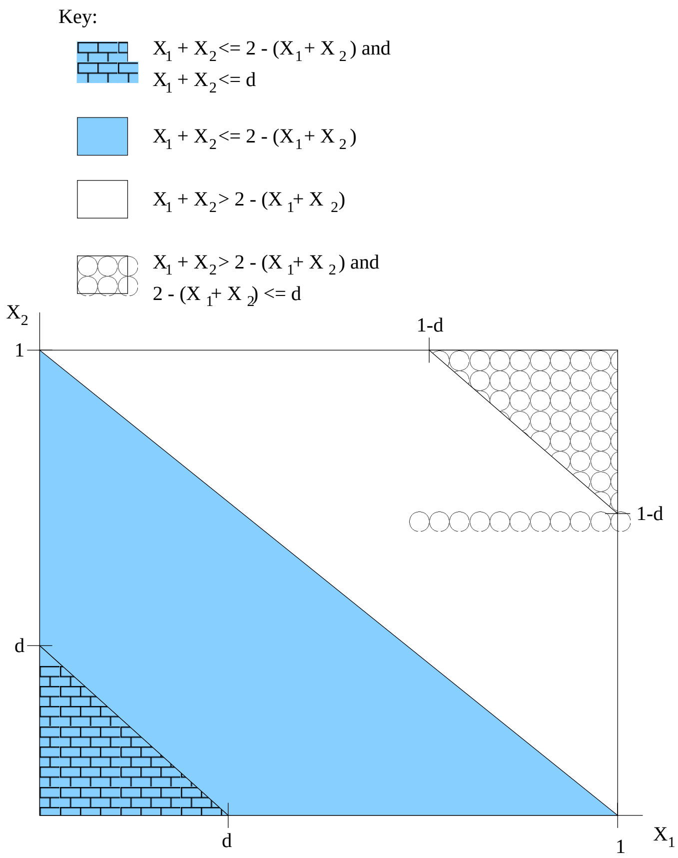

Figure 2: Joint Sample Space of X_1 and X_2

Taking derivatives, we obtain that the pdf of W is given by

$$f_W(d) \quad = \quad \begin{cases} 2d, & d \in [0,1) \\ 0, & \text{otherwise} \end{cases}$$

Now, recalling that we need to add 1.0 km of travel distance between the respective links, the desired answer to this part of the problem is

$$f_{(D|2,7)}(d) \quad = \quad f_{(D|12,7)}(d) \quad = \quad f_W(d-1)$$

Set 3: 7

Here we have the textbook problem of finding the pdf for $(D|7,7) = |X_1 - X_2|$, where X_1 and X_2 are uniformly independently distributed on the interval $[0,1]$. The answer, from p. 82 of the text (Eq.(3.2)) is

$$f_{(D|7,7)}(d) \quad = \quad \begin{cases} 2(1-d), & d \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

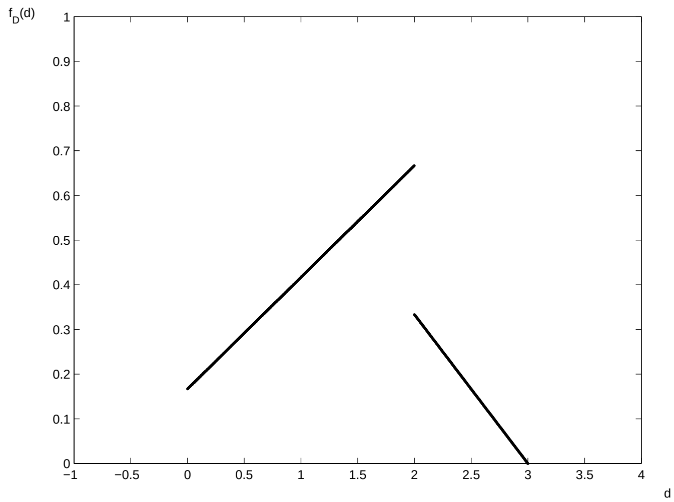

Figure 3: Sketch of $f_D(d)$

Putting this all together, we obtain

	$d \in [0,1]$	$d \in [1,2)$	$d \in [2,3]$
$f_{(D\|1,7)}(d)$	0	$d-1$	$3-d$
$f_{(D\|2,7)}(d)$	0	$2(d-1)$	0
$f_{(D\|3,7)}(d)$	0	$d-1$	$3-d$
$f_{(D\|4,7)}(d)$	d	$2-d$	0
$f_{(D\|5,7)}(d)$	d	$2-d$	0
$f_{(D\|6,7)}(d)$	d	$2-d$	0
$f_{(D\|7,7)}(d)$	$2(1-d)$	0	0
$f_{(D\|8,7)}(d)$	0	$d-1$	$3-d$
$f_{(D\|9,7)}(d)$	d	$2-d$	0
$f_{(D\|10,7)}(d)$	d	$2-d$	0
$f_{(D\|11,7)}(d)$	0	$d-1$	$3-d$
$f_{(D\|12,7)}(d)$	0	$2(d-1)$	0

So,

$$f_D(d) = \begin{cases} \frac{d}{4} + \frac{1}{6}, & d \in [0,2) \\ -\frac{d}{3} + 1, & d \in [2,3] \\ 0, & \text{otherwise} \end{cases}$$

The graph of this pdf is shown in Figure 3.

2

(i) Because the two arrival processes are independent, the total arrival process is also a Poisson process, but with rate $\lambda_1 + \lambda_2$. If the server has just finished an idle period, the time until he is next idle is the length of the busy period that has just commenced. Therefore, the average time until the server is next idle is just the average length of a busy period in an $M/M/1$ queueing system with Poiss$(\lambda_1 + \lambda_2)$ arrivals and exp(μ) services. That is,

$$E[\text{time until next idle}] \;=\; \bar{B} \;=\; \frac{1}{\mu - (\lambda_1 + \lambda_2)}$$

(ii) Each busy period begins with an arrival to an empty system. For a given busy period, let us use the term "trigger customer" to refer to the customer whose arrival begins the busy period. The server will handle only one customer during this busy period iff no additional arrivals occur during the service time of this "trigger customer." Thus, we must compute the probability of no arrivals during the "trigger customer's" service time.

Let us begin by noting that the next event following a service initiation is determined by the following two independent, competing exponentials: the time until the next arrival, and the time until service completion. The probability that a service completion occurs before the next arrival is given by $\frac{\mu}{\lambda_1+\lambda_2+\mu}$. To justify this result, consider a time interval of infinitesimal length ϵ. Then, Pr(next event is svc completion | next event occurs in $[0, \epsilon]$)

$$\begin{aligned} &= \frac{\text{Pr(next event occurs in } [0,\epsilon] \text{ and is svc completion)}}{\text{Pr(next event occurs in } [0,\epsilon])} \\ &= \frac{\mu\epsilon}{\epsilon(\lambda_1 + \lambda_2 + \mu)} \\ &= \frac{\mu}{\lambda_1 + \lambda_2 + \mu} \end{aligned}$$

Therefore, the probability that the server will handle exactly one customer during a busy period is $\frac{\mu}{\lambda_1+\lambda_2+\mu}$.

(iii) The imposition of a non-preemptive priority policy does not affect the arrivals process and does not cause the server to work any more slowly or quickly than in the absence of this priority policy. Therefore, the average fraction of time the server spends busy is the same as under a no-priority policy. That is, it is given by

$$\rho \;=\; \frac{\lambda_1 + \lambda_2}{\mu}$$

(iv) In deriving the preemptive priority queueing results we saw in class, we noted that the average busy period under FCFS is the same as under LCFS. In fact, the average busy period is invariant under all service disciplines, as along as they are work-conserving (the server is never idle when there are customers in the system). To see why, note that, in order to empty the system (and thereby end a busy period), the server must serve all customers currently in the system and all of their "descendents." If we change the service discipline, we do not change the arrivals process. The only thing that changes is the "bookkeeping" of recording new arrivals as "descendents" of one customer rather than another. This change in bookkeeping does not alter the workload in the system and therefore, does not affect the length of the busy period. So, the average busy period under this non-preemptive priority arrangement will be the same as under the no-priority arrangement. That is,

$$\bar{B} = \frac{1}{\mu - (\lambda_1 + \lambda_2)}$$

(v) z gives the total amount of time that both of the following conditions hold

1. Mendel is in the system
2. a type 2 customer is currently being served

To get some intuition for z, consider the following two cases

Case 1) Mendel arrives to find the server either idle or serving a type 1 customer.

Mendel is a type 1 customer, and therefore has non-preemptive priority over any type 2 customers. Accordingly, in this case, no type 2 customer will begin service while Mendel is in the system. Therefore, none of Mendel's time in the system will coincide with a type 2 service. Hence z will be 0.

Case 2) Mendel arrives to find a type 2 customer in service.

Mendel may or may not be the first type 1 customer in the queue. However, this is irrelevant for the purpose of determining z. In either case, since type 1 customers have non-preemptive priority over type 2, once the current type 2 customer in service completes his/her service, no additional type 2 customers will be served until after Mendel leaves the system. Therefore, z is simply the remaining service time of the type 2 customer in service at the time of Mendel's arrival. Because of the memorylessness of the exponential distribution, $z \sim \exp(\mu)$.

- $\Pr(z = 0)$

$$\Pr(z = 0) = \Pr(z = 0 \mid \text{Case 1}) P(\text{Case 1}) + \Pr(z = 0 \mid \text{Case 2}) P(\text{Case 2})$$

As already noted above, $\Pr(z = 0 \mid \text{Case 1}) = 1$, since in Case 1, no type 2 customer will be served while Mendel is in the system. For Case 2, since the remaining service time is a continuous RV, there is a zero probability that its value is exactly 0. That is, $\Pr(z = 0 \mid \text{Case 2}) = 0$. Finally, note that $P(\text{Case 1})$ is equal to the fraction of the time that the server is NOT busy serving type 2 customers. This is given by $1 - \rho_2 = 1 - \frac{\lambda_2}{\mu}$. Putting this all together, we obtain

$$\Pr(z = 0) = 1 - \frac{\lambda_2}{\mu}$$

- $\Pr(z > 2)$

$$\Pr(z > 2) = \Pr(z > 2 \mid \text{Case } 1)P(\text{Case } 1) + \Pr(z > 2 \mid \text{Case } 2)P(\text{Case } 2)$$

Since, in Case 1, no type 2 customer will be served while Mendel is in the system $\Pr(z > 2 \mid \text{Case } 1) = 0$. As already noted, in Case 2, $z \sim \exp(\mu)$. Therefore,

$$\Pr(z > 2) = e^{-2\mu}\frac{\lambda_2}{\mu}$$

- $E[z]$

$$\begin{aligned} E[z] &= E[z \mid \text{Case } 1]P(\text{Case } 1) + E[z \mid \text{Case } 2]P(\text{Case } 2) \\ E[z \mid \text{Case } 1] &= 0 \\ E[z \mid \text{Case } 2] &= E[X], \text{ where } X \sim \exp(\mu) \\ &= \frac{1}{\mu} \\ E[z] &= \frac{\lambda_2}{\mu^2} \end{aligned}$$