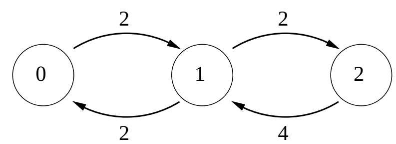
Problem 1 (Larson, 2001)

(a) One is tempted to say yes by setting $\rho = \frac{\lambda}{N\mu} = \frac{2}{2\times 2} = \frac{1}{2}$. But $\lambda = 2$ is not the rate at which customers are accepted into the system because we have a loss system. Thus the answer is no, and we must derive the correct figure. We can use the following aggregate birth-death process (state transition diagram for an M/M/2 queueing system with no waiting space) to compute the workloads:



The balance equations and the normalization equation are

$$\begin{aligned} 2P_0 &= 2P_1 \\ 2P_1 &= 4P_2 \\ P_0 + P_1 + P_2 &= 1 \end{aligned}$$

Solving the equations, we obtain

$$P_0 = \frac{2}{5}, \quad P_1 = \frac{2}{5}, \quad P_2 = \frac{1}{5}.$$

The workloads of server 1 and server 2 are then given by

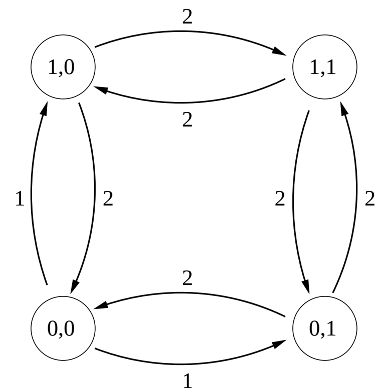
$$\rho_1 = \frac{1}{2}P_1 + P_2 = \frac{2}{5}, \quad \rho_2 = \frac{1}{2}P_1 + P_2 = \frac{2}{5}.$$

(b) The 2-dimensional hypercube state transition diagram is given below. From the steady-state probabilities computed in part (a) and the symmetry of the system, we have

$$P_{00} = P_0 = \frac{2}{5}, \quad P_{11} = P_2 = \frac{1}{5}, \quad P_{10} = P_{01} = \frac{1}{2}P_1 = \frac{1}{5}.$$

The fraction of dispatches that take server 1 to sector 2 is

$$f_{12} = \frac{\lambda_2}{(1 - P_{11})\lambda} P_{10} = \frac{1}{(1-\frac{1}{5})2}\left(\frac{1}{5}\right) = \frac{1}{8}.$$



(c) The mean travel time to a random served customer, $\bar{T}$, is obtained by

$$\bar{T} = f_{11}\,T_1(\text{sector }1) + f_{22}\,T_2(\text{sector }2) + f_{12}\,T_1(\text{sector }2) + f_{21}\,T_2(\text{sector }1)\,.$$

Since the travel speed is constant, let us first compute the mean travel distance to a random customer, $\bar{D}$.

$$\bar{D} = f_{11}\,D_1(\text{sector }1) + f_{22}\,D_2(\text{sector }2) + f_{12}\,D_1(\text{sector }2) + f_{21}\,D_2(\text{sector }1)\,.$$

Using the knowledge of Chapter 3, we have

$$D_1(\text{sector }1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \quad D_2(\text{sector }2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$
$$D_1(\text{sector }2) = 1 + \frac{1}{3} = \frac{4}{3}, \quad D_2(\text{sector }1) = 1 + \frac{1}{3} = \frac{4}{3}.$$

We compute f_{11} as follows:

$$f_{11} = \frac{\lambda_1}{(1-P_{11})\lambda}(P_{00} + P_{10}) = \frac{1}{(1-\frac{1}{5})2}\left(\frac{2}{5} + \frac{1}{5}\right) = \frac{3}{8}\,.$$

Invoking the symmetries, we know

$$f_{21} = f_{12} = \frac{1}{8}, \quad f_{22} = f_{11} = \frac{3}{8}\,.$$

Putting all together,

$$\bar{D} = \frac{3}{8}\cdot\frac{2}{3} + \frac{3}{8}\cdot\frac{2}{3} + \frac{1}{8}\cdot\frac{4}{3} + \frac{1}{8}\cdot\frac{4}{3} = \frac{5}{6}\text{ mile}\,.$$

Hence the mean travel time to a random served customer is $\bar{T} = \frac{\bar{D}}{1000}$ hr $=$ 3.0 sec. This means that changes in total service time due to changes in travel time are insignificant and therefore the Markov models applies. Note that another way to compute $\bar{D}$ is

$$\bar{D} = \frac{P_{00}(\frac{2}{3}) + (P_{01} + P_{10})(\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{4}{3})}{P_{00} + P_{01} + P_{10}} = \frac{\frac{2}{5}(\frac{2}{3}) + \frac{2}{5}(\frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{2}{3})}{\frac{4}{5}} = \frac{5}{6}\,.$$

In fact, we can obtain this form by simplifying the formula for $\bar{D}$ above. However, think about how we can obtain this directly without using the formula for $\bar{D}$ above.

(d) Consider a long time interval T. In the steady state, the average total number of customers served is $\lambda T(1 - P_{11})$. Server 1 is sent to sector 2 in the following cases:

(1) A customer arrives from sector 2, server 2 is busy, and server 1 is idle.

(2) A customer arrives from buffer zone 2, server 2 is idle outside buffer zone 2, and server 1 is idle inside buffer zone 1.

The average number of customers served by the first case is $\lambda_2 T P_{10}$. To compute the average number of customers served by the second case, let us first find the probability that server 2 is idle *outside* buffer zone 2 and server 1 is idle *inside* buffer zone 1. Using geometrical probability and the independence of the two servers, we know that the probability is $(\frac{3}{4})(\frac{1}{4})P_{00}$. Since the arrival rate from buffer zone 2 is $\frac{\lambda_2}{4}$, the average number of customers served by the second case during time interval T is $\frac{\lambda_2}{4}T(\frac{3}{4})(\frac{1}{4})P_{00}$.

Using these quantities, we obtain the fraction of dispatch assignments that send server 1 to sector 2 under the new dispatch policy as follows:

$$f'_{12} = \frac{\lambda_2 T P_{10} + \frac{\lambda_2}{4}T(\frac{3}{4})(\frac{1}{4})P_{00}}{\lambda T(1 - P_{11})} = \frac{1 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{2}{5}}{2(1 - \frac{1}{5})} = \frac{35}{256} = 0.1367\,.$$

f'_{12} is greater than $f_{12} = 0.125$ as expected. Note that the state transition diagram does not change under the new dispatch policy (Why? Invoke symmetries).

(e) Let T_1 be the travel time of server 1 to a random customer and T_2 be the travel time of server 2 to a random customer. Similar to (c), the mean travel time to a random customer under the new dispatch policy is given by

$$\begin{aligned}\bar{T}' = &f'_{11}\, E[T_1 \mid \text{server 1 has been dispatched into sector 1}] + \\ &f'_{22}\, E[T_2 \mid \text{server 2 has been dispatched into sector 2}] + \\ &f'_{12}\, E[T_1 \mid \text{server 1 has been dispatched into sector 2}] + \\ &f'_{21}\, E[T_2 \mid \text{server 2 has been dispatched into sector 1}]\,.\end{aligned}$$

But the existence of buffer zones complicates matters. One way to handle this is as follows:

- Break up f'_{12} (and f'_{21}) into its two constituent parts and compute a conditional mean travel distance for each
- Do the same for f'_{11} and f'_{22}.
- Combine the results for the final answer.

The numerical value is less than that of part (c), because we tend to dispatch the closer available server (not always successful, though).

Although it is not required in the question, let us compute $\bar{T}'$ exactly. We define the following events:

- CB: A customer is in a buffer zone.
- SAB: Server of the adjacent sector is in its buffer zone.
- SHB: Server of home sector is in its buffer zone.

Let us denote by CB^c the complement event of CB, which means that a customer is not in a buffer zone. Other complement events are defined similarly. Then in the state where both servers are available, with probability P_{00}, we have eight mutually exclusive, collective exhaustive events: $(\text{CB}\cap\text{SAB}\cap\text{SHB})$, $(\text{CB}^c\cap\text{SAB}\cap\text{SHB})$, $(\text{CB}\cap\text{SAB}^c\cap\text{SHB})$, $(\text{CB}\cap\text{SAB}\cap\text{SHB}^c)$, $(\text{CB}^c\cap\text{SAB}^c\cap\text{SHB})$, $(\text{CB}^c\cap\text{SAB}\cap\text{SHB}^c)$, $(\text{CB}\cap\text{SAB}^c\cap\text{SHB}^c)$, and $(\text{CB}^c\cap\text{SAB}^c\cap\text{SHB}^c)$.

Let us abbreviate these events in binary, for example, $(\text{CB}\cap\text{SAB}\cap\text{SHB}) = (111)$, $(\text{CB}^c\cap\text{SAB}\cap\text{SHB}^c) = (010)$, etc. Then we can write, using the techniques from Chapter 3 for the conditional mean travel distances,

$$\bar{D}' = \frac{P_{00}A + (P_{01}+P_{10})(\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{4}{3})}{P_{00}+P_{01}+P_{10}},$$

where A is

$$\begin{aligned}A = &\left(\frac{1}{3}+\frac{1}{4}\right)P(110) + \left[\frac{1}{3}+\left(\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}\right)\right]P(100) + \\ &\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}\right)\left(P(101)+P(111)\right) + \left(\frac{1}{3}+\frac{1}{3}\cdot\frac{3}{4}\right)\left(P(010)+P(000)\right) + \\ &\left[\frac{1}{3}+\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\right)\right]\left(P(001)+P(011)\right).\end{aligned}$$

We have

$$P(110) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}, \quad P(100) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}, \quad P(101) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}, \quad P(111) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4},$$

$$P(010) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}, \quad P(000) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}, \quad P(001) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}, \quad P(011) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}.$$

Plugging all numbers, we obtain $\bar{D}' = \frac{1271}{1536} = 0.82747 < \bar{D} = \frac{5}{6} = 0.83333$. The mean travel time to a random customer is $\bar{T}' = \frac{\bar{D}'}{1000} = 2.9789$ sec $< \bar{T} = 3$ sec. So, we do get an expected improvement in mean response distance (time), but not a large one. The fact that we have more inter-sector dispatches *does not* necessarily mean that mean response distance (time) will increase.

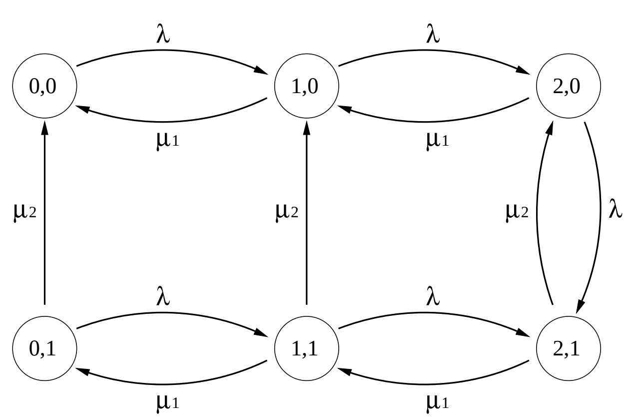
(f) First, do not use Carter, Chaiken, and Ignall formula (Equation (5.18)). It only applies when server locations are fixed. The best option is to compute $\bar{T}(x)$, where x is the location of a boundary line, and use calculus to find an optimal value of x (as we did in the 2-server numerical example in the book and in class). The problem with Equation (5.18) is that $T_1(B)$ and $T_2(B)$ depend on the location of the boundary line separating sectors 1 and 2. This is because each available server patrols uniformly its sector while it is idle and thus its travel time in B depends on sector design.

Problem 2 (Odoni, 2001)

(a) There are several possible ways to define the system's state. All of them lead to essentially the same state-transition diagram. One possible definition of state is (i, j), where

- i is the number of people being serviced by or waiting for Vincent,
- j is the state of Al (either idle or busy).

Then we have the following state-transition diagram:



(b) Let N denote the number of customers served by Vincent per hour. All customers who, on arrival, find the system in state $(0,0)$, $(1,0)$, $(0,1)$, or $(1,1)$ are served by Vincent. Therefore,

$$E[N] = \lambda(P_{00} + P_{10} + P_{01} + P_{11})\,.$$

From the service rate point of view, we can also write $E[N]$ as follows:

$$E[N] = \mu_1(P_{10} + P_{20} + P_{11} + P_{21})\,.$$

This follows from the following balance equations:

$$\lambda P_{00} + \lambda P_{01} = \mu_1 P_{10} + \mu_1 P_{11}$$
$$\lambda P_{10} + \lambda P_{11} = \mu_1 P_{20} + \mu_1 P_{21}$$

(c) Since we have just observed a customer enter the barbershop and sit Al's chair, the system is now in state $(2,1)$. If one of the following two events happens, the next customer who will enter the shop will be served by Vincent:

(1) Vincent finishes his service for the customer in his barber chair before Al does.

(2) Al finishes his service *and then* Vincent finishes his service before the next customer arrives.

The probability of the first event is $\frac{\mu_1}{\mu_1+\mu_2}$ (Think about two competing Poisson processes). The probability of the second event is $(\frac{\mu_2}{\mu_1+\mu_2})(\frac{\mu_1}{\mu_1+\lambda})$. Since the two events are mutually exclusive, the probability we want to know is

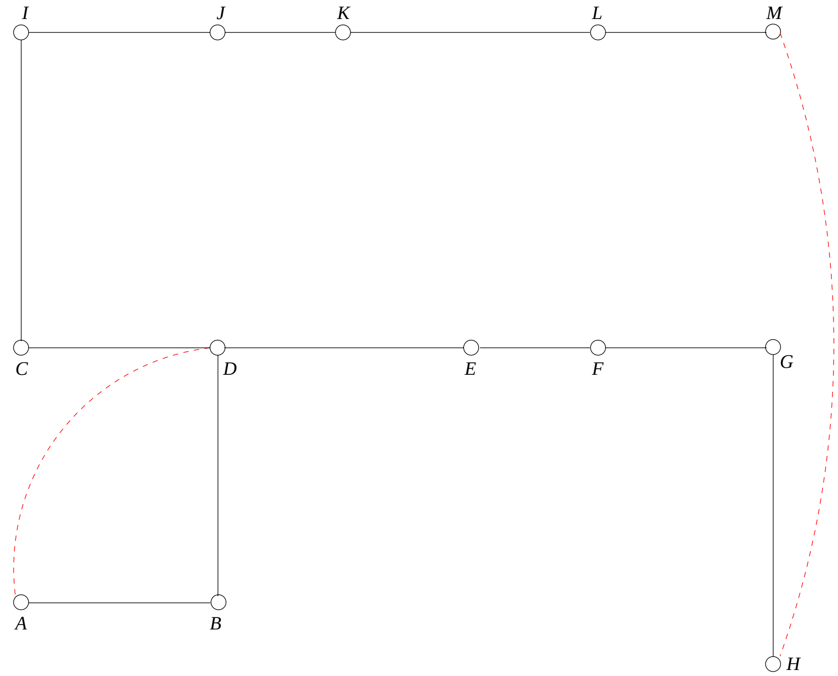
$$P(\cdot) = \frac{\mu_1}{\mu_1+\mu_2} + \left(\frac{\mu_2}{\mu_1+\mu_2}\right)\left(\frac{\mu_1}{\mu_1+\lambda}\right)\,.$$

Problem 3 (Odoni, 2001)

(a) There are many possible minimum spanning trees, some better than others for getting a good TSP solution. The following is one good minimum spanning tree, where the dashed lines indicate a matching of odd-degree nodes, $A-D$ and $H-M$. This results in a tour (skipping nodes already visited, if possible) that looks like

$$\{A,\, B,\, D,\, E,\, F,\, G,\, H,\, G,\, M,\, L,\, K,\, J,\, I,\, C,\, A\}\,.$$

The only edge covered twice in this tour is (G,H).



(b) The above tour is also an optimal tour! The length of the tour is 2,060.