14.384 Time Series Analysis
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Lecture 15

Breaks and Cointegration

Breaks

Suppose $y_t = \beta_t' x_{t-1} + \epsilon_t$, where $\beta_t = \begin{cases} \beta & t \le t_0 \\ \beta + \gamma & t > t_0 \end{cases}$, x_t is stationary, and $E[\epsilon_t | I_t] = 0$. We want to test $H_0 : \gamma = 0$. There is a nuisance parameter, t_0, which is identified under the alternative, but not under the null hypothesis. If we knew t_0, we could just do an F-test.

$$F_T(\frac{t_0}{T}) = \frac{SSR_{1,T} - (SSR_{1,t_0} + SSR_{t_0+1,T})}{(SSR_{1,t_0} + SSR_{t_0+1,T})/(T-k)} \Rightarrow \chi^2_k \text{ if } t_0 = [\delta T]$$

where $SSR_{t,s}$ is the sum of squared residuals from OLS using the sample from time t to time s, and k is the number of restrictions (the dimension of γ). This test is valid when t_0 is known.

When t_0 is not known, we must use a different test. One test-statistic is the *Quant* statistic

$$Q = \sup_{[\delta T] \le t_0 \le [(1-\delta)T]} F_T(\frac{t_0}{T}) \tag{1}$$

Andrews (1993) derived the distribution of this statistic. Other test statistics are the *mean Wold*:

$$MW = \frac{1}{T - 2\tau} \sum_{t_0=r}^{T-\tau} F_T(\frac{t_0}{T}) \text{ , } \tau = [\delta T] \tag{2}$$

and the Andrews-Ploberger (1994)

$$AP = \ln \left[\frac{1}{T - 2\tau} \sum_{t_0=r}^{T-\tau} \exp\left(\frac{1}{2} F_T(\frac{t_0}{T}) \right) \right] \tag{3}$$

To derive the limiting distribution of these statistics, we must look at the behavior of SSR. Let $\hat{\beta}$ be the OLS estimate from the sample from $t = 1, .., \tau$.

$$\begin{aligned} SSR_{1,\tau} &= \sum_{t=1}^{\tau} (y_t - \hat{\beta}' x_{t-1})^2 \\ &= \sum_{t=1}^{\tau} \epsilon_t^2 - 2(\hat{\beta} - \beta)' \sum x_{t-1}\epsilon_t + (\hat{\beta} - \beta)' \sum x_{t-1} x_{t-1}' (\hat{\beta} - \beta) \end{aligned}$$

Then, since $(\hat{\beta} - \beta)' = (\sum_{t=1}^{\tau} x_{t-1} x_{t-1}')^{-1} (\sum_{t=1}^{\tau} x_{t-1}\epsilon_t)$, we have

$$= \sum_{t=1}^{\tau} \epsilon_t^2 - (\sum_{t=1}^{\tau} x_{t-1}\epsilon_t)' (\sum_{t=1}^{\tau} x_{t-1} x_{t-1}')^{-1} (\sum_{t=1}^{\tau} x_{t-1}\epsilon_t)$$

Applying a functional central limit theorem (for which we need some additional assumptions, e.g. sufficient conditions would be that x and ϵ are independent and x_t are iid with finite fourth moments), we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[\delta T]} x_{t-1}\epsilon_t = \xi_T(\delta) \Rightarrow \sigma\Sigma_{xx}^{1/2}W_k(\cdot)$$

where $W_k(\cdot)$ is k-dimensional Brownian motion, and σ^2 is the variance of ϵ_t. Also, a law of large numbers implies $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x'_{t-1} \xrightarrow{p} \Sigma_{xx}$. If we make additional assumptions, then uniformly in τ, we'll have:

$$\Psi_T(\delta) = \frac{1}{T}\sum_{t=1}^{[T\delta]} x_{t-1}x'_{t-1} \xrightarrow{p} \delta\Sigma_{xx}$$

Combining this, we have, for $[\delta T] = \tau$:

$$\begin{aligned} SSR_{1,\tau} - \sum_{t=1}^{\tau}\epsilon_t^2 =& (\frac{1}{\sqrt{T}}\sum_{t=1}^{\tau} x_{t-1}\epsilon_t)'(\frac{1}{T}\sum_{t=1}^{\tau} x_{t-1}x'_{t-1})^{-1}(\frac{1}{\sqrt{T}}\sum_{t=1}^{\tau} x_{t-1}\epsilon_t) \\ =& \xi_T(\delta)'\Psi_T(\delta)^{-1}\xi_T(\delta) \\ \Rightarrow& \frac{\sigma^2}{\tau}W_k(\delta)'W_k(\delta) \end{aligned}$$

Now, looking at the numerator of F, we have

$$\begin{aligned} \tilde{T}(\tau) =& SSR_{1,T} - (SSR_{1,t_0} + SSR_{t_0+1,T}) \\ \Rightarrow& \sigma^2\left(-W_k(1)'W_k(1) + \frac{W_k(\tau)'W_k(\tau)}{\tau} + \frac{(W_k(1) - W_K(\tau))'(W_k(1) - W_k(\tau))}{1-\tau}\right) \\ \Rightarrow& \sigma^2\frac{(W_k(\tau) - \tau W_k(1))'(W_k(\tau) - \tau W_k(1))}{\tau(1-\tau)} \end{aligned}$$

$B_k(\tau) = W_k(\tau) - W_k(1)$ is called a Brownian bridge. It is a linear transformation of a Brownian motion that is required to be 0 at $t = 0$ and $t = 1$. The denominator of the F-statistic converges to σ^2, thus

$$F_T(\tau) \Rightarrow \frac{B_k(\tau)'B_k(\tau)}{\tau(1-\tau)}$$

Also,

$$Q \Rightarrow \sup_{\delta\le\tau\le 1-\delta}\frac{B_k(\tau)'B_k(\tau)}{\tau(1-\tau)}$$

We can simulate this distribution to find critical values. The test for H_0 : break vs. H_a : no breaks can be performed by calculating the Q statistic in sample and comparing it to the simulated critical values. We will reject large values of Q. An alternative approach is bootstrapping the Q statistic.

Recursive Estimation

Let:

$$\tilde{\beta}(\tau/T) = (\sum_{t=1}^{\tau} x_{t-1}x'_{t-1})^{-1}(\sum_{t=1}^{\tau} x_{t-1}y_t)$$

One can calculate estimates of $\tilde{\beta}$ recursively and look at their stability. If $\tilde{\beta}$ changes a lot, it is a sign of a break. Formally, we need to find the asymptotic distribution. Suppose H_0 : no breaks is true and β is the true coefficient. Then,

$$\sqrt{T}(\tilde{\beta}(\tau) - \beta) = \Psi_T(\tau)^{-1}\xi_T(\tau) \Rightarrow \frac{\sigma\Sigma_{xx}^{-1/2}W_k(\tau)}{\tau}$$

The problem is the nuisance parameters, β and Σ_{xx}. We can eliminate them from the asymptotic distribution by estimating β by $\tilde{\beta}(1)$ and looking at the associated t-statistic:

$$t_t(\tau) = \hat{\sigma}_\epsilon^{-1}(\frac{1}{T}\sum x_{t-1}x'_{t-1})^{1/2}\sqrt{T}(\tilde{\beta}(\tau/T) - \tilde{\beta}(1)) \Rightarrow \frac{W_k(\tau)}{\tau} - W_k(1)$$

and as above, you can use test statistic $\sup_{\delta\leq\tau}|t_T(\tau)|$ and critical values simulated from $\sup_\tau \frac{W_k(\tau)}{\tau} - W_k(1)$. There are many ways to test for breaks, and the way to derive the limiting distribution of test statistics is by using the FCLT.

Unit Root with a Break

Consider a unit root process with possible break in trend. The model is $y_t^* = y_t + d_t$ where y_t is a random walk and $d_t = c + \gamma\mathbf{1}_{t>t_0}$. How do we then test for a unit root? If the trend in misspecified (y_t^* regressed on const, y_{t-1}^*), then you are likely to accept a unit root, where there is not one. The intuition is that both a break and a unit root lead to permanent changes in y_t. Perron (1989) found that if you allow for breaks during the Great Depression (1929) and during the oil shocks (1973), then you reject the null of unit root in most macro series. Some people objected to this by arguing that it is not fair to treat the break dates as known. If you test for unit roots without assuming the break dates are known, then you cannot reject the null of unit root in most series.

Cointegration

Spurious Regression

Let x_t and y_t be two independent random walks,

$$y_t = y_{t-1} + z_t$$
$$x_t = x_{t-1} + u_t$$

where z_t and u_t are iid and independent of each other. Suppose we run OLS of y_t on x_t.

$$y_t = \beta x_t + e_t$$

The true value of β is 0. Then,

$$\hat{\beta} = \frac{\frac{1}{T^2}\sum y_t x_t}{\frac{1}{T^2}\sum x_t^2}$$

let

$$\xi_T(\tau) = \frac{1}{\sqrt{T}}\begin{pmatrix} x_{[\tau T]} \\ y_{[\tau T]} \end{pmatrix} \Rightarrow W(\tau)$$

where $W(\tau)$ is a 2-dimensional Brownian motion. Also,

$$\frac{1}{T}\sum \xi_T(t/T)\xi_T(t/T)' \Rightarrow \int W(\tau)W(\tau)'d\tau$$

Thus,

$$\hat{\beta} = \frac{\frac{1}{T^2}\sum y_t x_t}{\frac{1}{T^2}\sum x_t^2} \Rightarrow \frac{\int W_1 W_2 dt}{\int W_2^2 dt}$$

That is, $\hat{\beta}$ is not consistent. You won't receive zero even in very large samples. Moreover, $R^2 \xrightarrow{p} 1$. The important point here is that with non-stationary regressors, we get non-standard limit distributions.

Usual Cointegration

Let us have two random walks, x_t and y_t, such that a linear combination of them is stationary. This situation is called cointegration.

$$\begin{aligned} y_t =& \beta x_t + e_t \\ x_t =& x_{t-1} + u_t \end{aligned}$$

with $\text{cov}(e_t, u_t) = \phi$. Despite this correlation, β will be consistently estimated. Moreover, $\hat{\beta}$ is super-consistent.

$$\begin{aligned} T(\hat{\beta} - \beta) =& \frac{\frac{1}{T}\sum x_t e_t}{\frac{1}{T}\sum x_t^2} \\ =& \frac{\frac{1}{T}\sum x_{t-1} e_t + \frac{1}{T}\sum u_t e_t}{\frac{1}{T}\sum x_t^2} \\ \Rightarrow& \frac{\int W_2 dW_1 + \phi}{\int W_2^2 dt} \end{aligned}$$

However, since $\phi \neq 0$, the limiting distribution is shifted. As a result, $\hat{\beta}$ has a finite sample bias of order $\frac{1}{T}$, which could be large. In addition, one can show that the limit of the t-statistic would also depend on ϕ, and thus, not be free of nuisance parameters. This makes inference difficult.