Order Preserving Properties of Vehicle Dynamics with Respect to the Driver's Input

Mojtaba Forghani and Domitilla Del Vecchio

Massachusetts Institute of Technology

September 19, 2014

1 Introduction

In this report, we consider longitudinal dynamics of a vehicle and prove that its longitudinal position and velocity are order preserving with respect to the driver's (or automatic controller's) input. We also prove that the position is strictly order preserving with respect to the constant term of the acceleration replaced with a linear function of the position and the velocity. This estimation of the acceleration is used for the cases in which no direct measurement of the acceleration is available.

2 System Model

Before introducing the model that we are considering, we define the strict and non-strict order preserving properties.

Definition 1. (Order preserving) For all $w, z \in \mathbb{R}^n$ we have that $w \leq z$ ($w < z$) if and only if $w_i \leq z_i$ ($w_i < z_i$) for all $i \in \{1, 2, ..., n\}$, in which w_i denotes the ith component of w. We denote the piecewise continuous signal on U by $S(U) : \mathbb{R}_+ \to U$. For $U \subset \mathbb{R}^m$ we define the partial order (strict partial order) by component-wise ordering for all times, that is, for all $\mathbf{w}, \mathbf{z} \in S(U)$ we have that $\mathbf{w} \leq \mathbf{z}$ ($\mathbf{w} < \mathbf{z}$) provided $\mathbf{w}(t) \leq \mathbf{z}(t)$ ($\mathbf{w}(t) < \mathbf{z}(t)$) for all $t \in \mathbb{R}_+$. The map $f : P \to Q$ is order preserving (strict order preserving) provided if for $x, y \in P$ we have $x \leq y$ ($x < y$), then $f(x) \leq f(y)$ ($f(x) < f(y)$).

Definition 2. (Continuous system) A continuous system is a tuple $\Sigma = (X, U, \Delta, O, f, h)$, with state $x \in X \subset \mathbb{R}^n$, control input $u \in U \subset \mathbb{R}^m$, disturbance input $d \in \Delta \subset \mathbb{R}^q$, output $y \in O \subset X$, vector field in the form of $f : X \times U \times \Delta \to X$, and output map $h : X \to O$.

We denote the flow of a system Σ at time $t \in \mathbb{R}_+$ by $\phi(t, x, \mathbf{u}, \mathbf{d})$, with initial condition $x \in X$, control input signal $\mathbf{u} \in S(U)$ and disturbance input signal $\mathbf{d} \in S(\Delta)$. We also denote the ith component of the flow by $\phi_i(t, x, \mathbf{u}, \mathbf{d})$.

Definition 3. (Control input/output order preserving) A continuous system $\Sigma = (X, U, \Delta, O, f, h)$ is called input/output order preserving (strict input/output order preserving) with respect to the control input, if the map $h(\phi(t, x, \cdot, \mathbf{d})) : U \to O$, for any fixed t, x and $\mathbf{d}$, is order preserving (strict order preserving).

Definition 4. (Disturbance input/output order preserving) A continuous system $\Sigma = (X, U, \Delta, O, f, h)$ is called input/output order preserving (strict input/output order preserving) with respect to the disturbance input, if the map $h(\phi(t, x, \mathbf{u}, \cdot)) : \Delta \to O$, for any fixed t, x and $\mathbf{u}$, is order preserving (strict order preserving).

We denote the position and the velocity of vehicle by p and v, respectively. We consider two systems Σ^1 and Σ^2. For Σ^1 we assume that the input is the control input u and there is no disturbance. Therefore it is reasonable to represent the flow of Σ^1 by $\phi^1(t, x^1, \mathbf{u})$, where x^1 is the initial state of the system. The deceleration due to the road load (rolling resistance) and the slope of the road are represented by a_r and a_s, respectively, and the drag coefficient is denoted by D. We also impose a condition that the speed of the vehicle must be non-negative. We use the superscript T to denote the transpose of a vector or matrix, e.g., A^T represents the transpose of matrix A. The dynamics of Σ^1 is as the following:

$$x^1 \in \mathcal{X} \subset \mathbb{R}^2, \text{where } x^1 = (p, v)^T, \tag{1}$$

$$u \in \mathcal{U} \subset \mathbb{R}, \text{with } \mathcal{U} = \{u \mid u \in [u_m, u_M]\}, \tag{2}$$

where u_m is the minimal control input, and u_M is the maximal control input,

$$f^1 : \mathcal{X} \times \mathcal{U} \to \mathcal{X}, \text{with } \dot{x}^1 = f^1(x^1, u), \tag{3}$$

where

$$f^1(x^1, u) = \begin{cases} \bar{f}^1(x^1, u) & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases}. \tag{4}$$

The function $\bar{f}^1(x^1, u)$ is also in the following form:

$$\bar{f}^1(x^1, u) = \begin{bmatrix} v \\ u - Dv^2 - a_r - a_s \end{bmatrix}. \tag{5}$$

Since we cannot measure the acceleration for system Σ^2, we assume the total acceleration of the vehicle to be in the form of a linear function of the position and the velocity. Moreover we assume the input term (the constant term of the linear function), be a disturbance input. Therefore it is reasonable to represent the flow of Σ^2 by $\phi^2(t, x^2, \mathbf{d})$, where x^2 is the initial state of the system. The dynamics of system Σ^2 is in the following form:

$$x^2 \in \mathcal{X} \subset \mathbb{R}^2, \text{where } x^2 = (p, v)^T, \tag{6}$$

$$d \in \mathbb{R}, \tag{7}$$

$$f^2 : \mathcal{X} \times \Delta \to \mathcal{X}, \text{with } \dot{x}^2 = f^2(x^2, d), \tag{8}$$

where

$$f^2(x^2, d) = \begin{cases} \bar{f}^2(x^2, d) & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases}. \tag{9}$$

The function $\bar{f}^2(x^2, d)$ is also in the following form:

$$\bar{f}^2(x^2, d) = \begin{bmatrix} v \\ ap + bv + d \end{bmatrix}. \tag{10}$$

The term $ap + bv + d$ is the total acceleration of the vehicle in Σ^2, where a and b are known constants. Since in Σ^2, unlike Σ^1, we cannot measure the acceleration directly, this model can be a good approximation to the acceleration of the vehicle.

3 Order Preserving Properties

We prove that the flows of the position and the velocity in Σ^1, $\phi_1^1(t, x^1, \mathbf{u})$ and $\phi_2^1(t, x^1, \mathbf{u})$, respectively, have order preserving property with respect to the control input signal $\mathbf{u}$.

Proposition 1. The flows $\phi_1^1(t, x^1, \mathbf{u})$ and $\phi_2^1(t, x^1, \mathbf{u})$ are order preserving with respect to the control input signal $\mathbf{u}$.

Proof. If we consider two different input signals $\mathbf{u}_1$ and $\mathbf{u}_2$, such that $\mathbf{u}_1 > \mathbf{u}_2$, then for the velocity of Σ^1 at time t corresponding to these two input signals, with the same initial conditions $p_1(0) = p_2(0) = p(0)$ and $v_1(0) = v_2(0) = v(0)$, we have $\dot{v}_1(t) = \mathbf{u}_1(t) - Dv_1(t)^2 - a_r - a_s$ and $\dot{v}_2(t) = \mathbf{u}_2(t) - Dv_2^2(t) - a_r - a_s$, if both $v_1(t) > 0$ and $v_2(t) > 0$. Let the function $g(t) := v_1(t) - v_2(t)$. At an arbitrary time t we have

$$\dot{g}(t) = \dot{v}_1(t) - \dot{v}_2(t) = (\mathbf{u}_1(t) - \mathbf{u}_2(t)) - D\left(v_1^2(t) - v_2^2(t)\right). \tag{11}$$

Note that since we have chosen the same initial conditions, we have $g(0) = v_1(0) - v_2(0) = 0$. Because of the continuity of flow of the system with respect to time, if order in state v is not preserved, we must have a time $t' \in \mathbb{R}_+$ such that $g(t') = 0$, since otherwise for all $t \in \mathbb{R}_+$, either $g(t) < 0$ or $g(t) > 0$. Therefore we can define $t^* := \min\{t \in \mathbb{R}_+ \mid g(t) = 0\}$. Since $\dot{g}(0^+) = \mathbf{u}_1(0^+) - \mathbf{u}_2(0^+) > 0$, $\dot{g}(t^{*-}) = \mathbf{u}_1(t^{*-}) - \mathbf{u}_2(t^{*-}) > 0$, and $g(0) = g(t^*) = 0$, for the interval $t \in (0, t^*)$ we have

$$\begin{aligned} &\dot{g}(0^+) = \lim_{h \to 0^+} \frac{g(h) - g(0)}{h - 0} = \lim_{h \to 0^+} \frac{g(h)}{h} > 0 \Rightarrow \\ &\text{since } h > 0 : \exists\ h = h_1 \in (0, t^*)\ s.t.\ g(h_1) > 0, \end{aligned} \tag{12}$$

and similarly

$$\begin{aligned} &\dot{g}(t^{*-}) = \lim_{h \to 0^-} \frac{g(t^*) - g(t^* + h)}{t^* - (t^* + h)} = \lim_{h \to 0^-} \frac{g(t^* + h)}{h} > 0 \Rightarrow \\ &\text{since } h < 0 : \exists\ h = h_2 \in (0, t^*)\ s.t.\ g(h_2) < 0, \end{aligned} \tag{13}$$

and because of the continuity of the flow with respect to time, there is a $t \in [h_1, h_2]$ such that $g(t) = 0$, which is in contradiction with the initial assumption that $t^* := \min\{t \in \mathbb{R}_+ \mid g(t) = 0\}$. Therefore there is no such t^*, and for all $t \in \mathbb{R}_+$ such that $v_1(t) > 0$ and $v_2(t) > 0$ we have either $g(t) = v_1(t) - v_2(t) > 0$ or $g(t) = v_1(t) - v_2(t) < 0$. From (12) we conclude that the former is true.

We had assumed initially that $v_1(t) > 0$ and $v_2(t) > 0$. For a case that for some $t' \in \mathbb{R}_+$ we have $v_1(t') = 0$ and $v_2(t') = 0$, we let $\bar{t}_1 := \min\{t \in \mathbb{R}_+ \mid v_1(t) = 0\}$ and $\bar{t}_2 = \min\{t \in \mathbb{R}_+ \mid v_2(t) = 0\}$. Because of the non-negativity of $v_1(t)$, $v_2(t)$ and $v_1(t) - v_2(t)$, we must have $\bar{t}_2 \leq \bar{t}_1$. If an arbitrary time $t \in (0, \bar{t}_2)$, then $g(t) = v_1(t) - v_2(t) > 0$; if $t \in [\bar{t}_2, \bar{t}_1)$, then $v_1(t) - v_2(t) = v_1(t) > 0$; and if $t \in [\bar{t}_1, \infty)$, then $g(t) = v_1(t) - v_2(t) = 0$. Therefore, in any case the order of the flow of the velocity is preserved with respect to the control input signal. Since $p_1(0) = p_2(0) = p(0)$, then based on equation (5), $p_1(t) - p_2(t) = \int_0^t g(s)ds \geq 0$, which implies that the order preserving property of the flow of p is also satisfied with respect to the control input signal. □

In Proposition 2 we prove that the position in Σ^2, $\phi_1^2(t, x^2, \mathbf{d})$, is strictly order preserving with respect to $\mathbf{d}$.

Proposition 2. The flow $\phi_1^2(t, x^2, \mathbf{d})$ is strictly order preserving with respect to the disturbance input $\mathbf{d}$.

Proof. Let $x_0^2 := (p_0, v_0)^T$ be the initial condition, where $v_0 > 0$. Since $\mathbf{d}$ is not a function of time, then by differentiating $\dot{v}$ from equations (8) and (10), we have that the velocity in Σ^2, for $v(t) > 0$, satisfies the following differential equation:

$$\ddot{v} - b\dot{v} - av = 0 \text{ where } v(0) = v_0 \text{ and } \dot{v}(0) = ap_0 + bv_0 + d. \tag{14}$$

The above differential equation has the solution in the form

$$v(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \text{ where,}$$
$$\lambda_1 = 0.5(b + \sqrt{b^2 + 4a}) \text{ and } \lambda_2 = 0.5(b - \sqrt{b^2 + 4a}). \tag{15}$$

Since complex and real values of λ_1 and λ_2 reveal different behaviors for $v(t)$, we consider different possible cases and analyze the behaviors of $v(t)$ and $p(t)$ with respect to d for each of them. We divide the problem into three different cases; (1): $b^2 + 4a > 0$, (2): $b^2 + 4a < 0$ and (3): $b^2 + 4a = 0$. For each case we consider two input signals $\mathbf{d}^1 = d^1$ and $\mathbf{d}^2 = d^2$ such that $d^1 > d^2$ and determine the relationship between $v^1(t)$ and $v^2(t)$ and then between $p^1(t)$ and $p^2(t)$, the velocity and the position at time t corresponding to d^1 and d^2, respectively.

Case (1): If $b^2 + 4a > 0$, then λ_1 and λ_2 in (15) are real numbers. The solution of (14) then takes the form

$$v(t) = \frac{1}{\lambda_2 - \lambda_1}\left((v_0(\lambda_2 - b) - ap_0 - d)\, e^{\lambda_1 t} - (v_0(\lambda_1 - b) - ap_0 - d)\, e^{\lambda_2 t} \right). \tag{16}$$

If we replace d in equation (16) with d^1 and d^2 in order to obtain their corresponding velocities at time t, represented by $v^1(t)$ and $v^2(t)$, respectively, we have

$$v^1(t) - v^2(t) = \frac{d^1 - d^2}{\lambda_2 - \lambda_1}\left(e^{\lambda_2 t} - e^{\lambda_1 t} \right). \tag{17}$$

Equation (17) can become zero only when $t = 0$. Therefore because of the continuity of flow of the system with respect to time, for all $t \in \mathbb{R}_+$, either $v^1(t) - v^2(t) > 0$ or $v^1(t) - v^2(t) < 0$. To determine which of these two cases holds, we note that in general for any $x \in \mathbb{R} - \{0\}$ we have that if $x > 0$, then $e^x - 1 > 0$ and if $x < 0$, then $e^x - 1 < 0$. These two statements together imply that $\frac{e^x - 1}{x} > 0$. Since in Case (1) $\lambda_2 - \lambda_1 \neq 0$ and we are considering $t \in \mathbb{R}_+$, then $(\lambda_2 - \lambda_1)t \neq 0$. Therefore we can replace x with $(\lambda_2 - \lambda_1)t$ in order to obtain

$$\frac{e^{(\lambda_2 - \lambda_1)t} - 1}{(\lambda_2 - \lambda_1)t} > 0 \Rightarrow te^{\lambda_1 t}\left[\frac{e^{(\lambda_2 - \lambda_1)t} - 1}{(\lambda_2 - \lambda_1)t} \right] > 0 \Rightarrow \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} > 0 \Rightarrow$$
$$\frac{d^1 - d^2}{\lambda_2 - \lambda_1}\left(e^{\lambda_2 t} - e^{\lambda_1 t} \right) > 0 \Rightarrow v^1(t) > v^2(t), \tag{18}$$

where we have used the facts that $t \in \mathbb{R}_+$, $e^{\lambda_1 t} > 0$ and $d^1 - d^2 > 0$. By integrating both sides of (18) to determine the position at time t, we obtain

$$\int_0^t v^1(s)ds > \int_0^t v^2(s)ds \Rightarrow$$

$$p_0 + \int_0^t v^1(s)ds > p_0 + \int_0^t v^2(s)ds \Rightarrow p^1(t) > p^2(t). \tag{19}$$

Case (2): If $b^2 + 4a < 0$, then λ_1 and λ_2 in (15) are complex numbers. The solution of (14) then takes the form

$$v(t) = e^{\alpha t}\left(\frac{ap_0 + (b-\alpha)v_0 + d}{\beta}\sin\beta t + v_0\cos\beta t\right),$$

$$\text{with } \alpha = 0.5b, \text{ and } \beta = 0.5\sqrt{-(b^2+4a)}. \tag{20}$$

If we replace d in equation (20) with d^1 and d^2 in order to obtain their corresponding velocities at time t, represented by $v^1(t)$ and $v^2(t)$, respectively, we have

$$v^1(t) - v^2(t) = \frac{d^1 - d^2}{\beta}e^{\alpha t}\sin\beta t. \tag{21}$$

We observe that in Case (2), unlike Case (1), we cannot guarantee that for all $t \in \mathbb{R}_+,\ v^1(t) - v^2(t) \neq 0$. Note that $v^1(t) - v^2(t) = 0$ for all t such that $\sin\beta t = 0$ or alternatively, $\beta t = k\pi$, for all $k \in \mathbb{Z}$. The smallest $t \in \mathbb{R}_+$ that satisfies $\sin\beta t = 0$ is $t^* = \frac{\pi}{\beta}$. The velocity at time t^* corresponding to d^1 and d^2, based on equation (20), is given by

$$v^i(t^*) = v_0 e^{\alpha t^*}\cos\frac{\beta\pi}{\beta} = -v_0 e^{\alpha t^*} < 0 \text{ for } i \in \{1, 2\}. \tag{22}$$

Since for all $t \in \mathbb{R}_+,\ v(t) \geq 0$, we must have $v^i(t^*) = 0$, or in other words, for all $t \in [0, t^*]$ we have either $v^1(t) - v^2(t) \geq 0$ or $v^1(t) - v^2(t) \leq 0$. In order to determine which case holds, we note that for all $t \in [0, t^*]$ we have $\beta > 0$ and $0 \leq \sin\beta t \leq 1$. Therefore in any case, for all $t \in [0, t^*]$ we have $\frac{\sin\beta t}{\beta} \geq 0$. Also $e^{\alpha t}(d^1 - d^2) > 0$. These two statements along with (21) imply that $v^1(t) - v^2(t) \geq 0$. Since in (20) we have $v^2(0) = v_0 > 0$ and $v^2(t^*) = -v_0 e^{\alpha t^*} < 0$, then because of the continuity of flow of the system with respect to time, there is a $\bar{t} \in (0, t^*)$ such that $\bar{t} := \min\{t \in (0, t^*) \mid v^2(t) = 0\}$. Then we have for all $t \in (0, \bar{t}),\ v^1(t) - v^2(t) > 0$. For a $t \in (0, \bar{t})$, we have

$$p^1(t) - p^2(t) = \int_0^t (v^1(s) - v^2(s))ds > 0; \tag{23}$$

for a $t \in [\bar{t}, t^*)$ we have

$$p^1(t) - p^2(t) = \int_0^{\bar{t}} (v^1(s) - v^2(s))ds + \int_{\bar{t}}^t (v^1(s) - v^2(s))ds >$$

$$0 + \int_{\bar{t}}^t (v^1(s) - v^2(s))ds \geq 0 \Rightarrow p^1(t) - p^2(t) > 0; \tag{24}$$

and for a $t \in [t^*, \infty)$, we have

$$p^1(t) - p^2(t) = \int_0^{t^*} (v^1(s) - v^2(s))ds + \int_{t^*}^t (v^1(s) - v^2(s))ds =$$

$$\int_0^{t^*} (v^1(s) - v^2(s))ds + 0 > 0 \Rightarrow p^1(t) - p^2(t) > 0. \tag{25}$$

Case (3): If $b^2 + 4a = 0$, then $\lambda_1 = \lambda_2 = \lambda$, which is also a real number. The solution of (14) then takes the form

$$v(t) = e^{\lambda t} \left[v_0 + (ap_0 + (b - \lambda)v_0 + d)\, t \right], \tag{26}$$

and for $v^1(t) - v^2(t)$ we have

$$v^1(t) - v^2(t) = t(d^1 - d^2)e^{\lambda t} > 0, \tag{27}$$

which implies

$$p^1(t) - p^2(t) = \int_0^t (v^1(s) - v^2(s))ds > 0. \tag{28}$$

We had assumed initially that $v^1(t) > 0$ and $v^2(t) > 0$. In general, we may have a time t^* such that $v^1(t^*) = 0$ and $v^2(t^*) = 0$. In this case, because of the continuity of flow of the system with respect to time, there are times $\bar{t}_1$ and $\bar{t}_2$ such that $\bar{t}_1 = \sup\{t \in (0, t^*) \mid v^1(t) > 0\}$ and $\bar{t}_2 = \sup\{t \in (0, t^*) \mid v^2(t) > 0\}$. Since we have proved through Cases (1)-(3) that as long as $v^1(t) > 0$ and $v^2(t) > 0$ we have $v^1(t) - v^2(t) > 0$, then $\bar{t}_2 \leq \bar{t}_1$. For an arbitrary time $\tau \in (0, \bar{t}_2)$ Cases (1)-(3) imply that $p^1(\tau) - p^2(\tau) > 0$; if $\tau \in [\bar{t}_2, \bar{t}_1)$, then

$$p^1(\tau) - p^2(\tau) = \int_0^{\bar{t}_2} (v^1(s) - v^2(s))ds + \int_{\bar{t}_2}^{\tau} (v^1(s) - 0)ds > 0; \tag{29}$$

and if $\tau \in [\bar{t}_1, \infty)$, then

$$p^1(\tau) - p^2(\tau) = \int_0^{\bar{t}_2} (v^1(s) - v^2(s))ds + \int_{\bar{t}_2}^{\bar{t}_1} (v^1(s) - 0)ds + \int_{\bar{t}_1}^{\tau} (0 - 0)ds > 0, \tag{30}$$

and the proof is complete. □