MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 10, 2000
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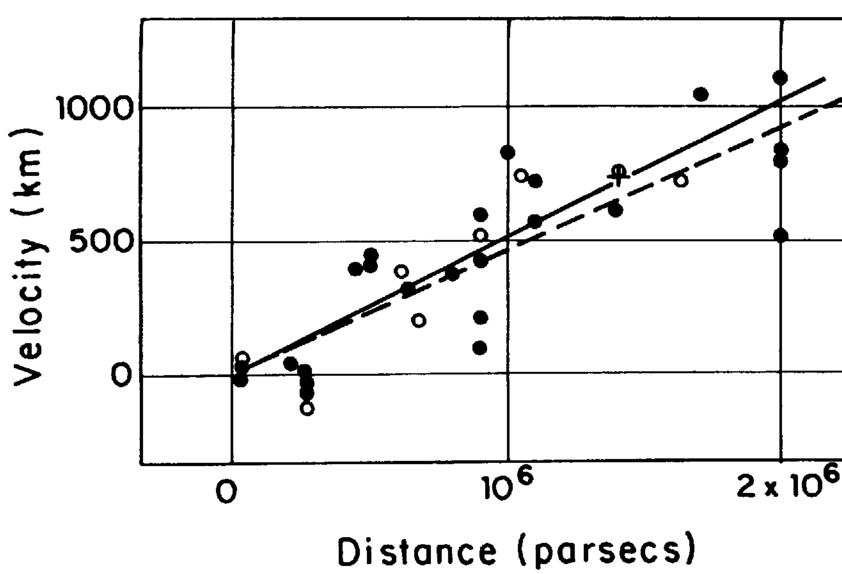
QUIZ 1 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING? *(35 points)*

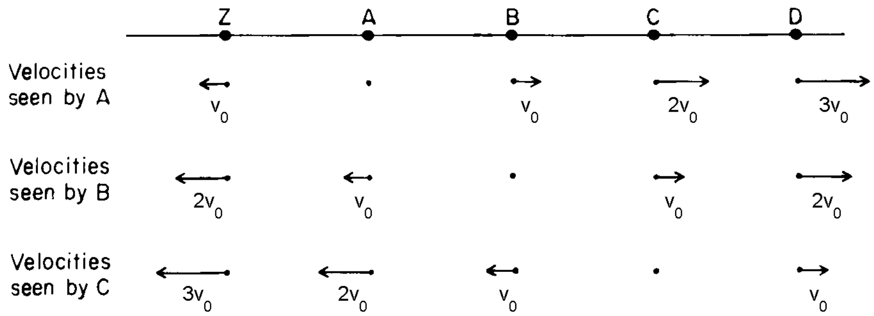
a) Doppler predicted the Doppler effect in **1842.**

b) Most of the stars of our galaxy, including our sun, lie in a flat disk. We therefore see much more light when we look out from earth along the plane of the disk than when we look in any other direction.

c) Hubble's original paper on the expansion of the universe was based on a study of only **18 galaxies.** Well, at least Weinberg's book says 18 galaxies. For my own book I made a copy of Hubble's original graph, which seems to show 24 black dots, each of which represents a galaxy, as reproduced below. The vertical axis shows the recession velocity, in kilometers per second. The solid line shows the best fit to the black dots, each of which represents a galaxy. Each open circle represents a group of the galaxies shown as black dots, selected by their proximity in direction and distance; the broken line is the best fit to these points. The cross shows a statistical analysis of 22 galaxies for which individual distance measurements were not available. I am not sure why Weinberg refers to 18 galaxies, but it is possible that the text of Hubble's article indicated that 18 of these galaxies were measured with more reliability than the rest.



d)



e) During a time interval in which the linear size of the universe grows by 1%, the horizon distance grows by **more than 1%.** To see why, note that the horizon distance is equal to the scale factor times the comoving horizon distance. The scale factor grows by 1% during this time interval, but the comoving horizon distance also grows, since light from the distant galaxies has had more time to reach us.

f) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

g) (i) the average distance between photons: **proportional to the size of the universe** (Photons are neither created nor destroyed, so the only effect is that the average distance between them is stretched with the expansion. Since the universe expands uniformly, all distances grow by the same factor.)

(ii) the typical wavelength of the radiation: **proportional to the size of the universe** (See Lecture Notes 3.)

(iii) the number density of photons in the radiation: **inversely proportional to the cube of the size of the universe** (From (i), the average distance between photons grows in proportion to the size of the universe. Since the volume of a cube is proportional to the cube of the length of a side, the average volume occupied by a photon grows as the cube of the size of the universe. The number density is the inverse of the average volume occupied by a photon.)

(iv) the energy density of the radiation: **inversely proportional to the fourth power of the size of the universe** (The energy of each photon is proportional to its frequency, and hence inversely proportional to its wavelength. So from (ii) the energy of each photon is inversely proportional to the size of the universe, and from (iii) the number density is inversely proportional to the cube of the size.)

(v) the temperature of the radiation: **inversely proportional to the size of the universe** (The temperature is directly proportional to the average energy of a photon, which according to (iv) is inversely proportional to the size of the universe.)

PROBLEM 2: THE STEADY-STATE UNIVERSE THEORY *(25 points)*

a) *(10 points)* According to Eq. (3.7),

$$H(t) = \frac{1}{R(t)} \frac{dR}{dt} .$$

So in this case

$$\frac{1}{R(t)} \frac{dR}{dt} = H_0 ,$$

which can be rewritten as

$$\frac{dR}{R} = H_0 \, dt .$$

Integrating,

$$\ln R = H_0 \, t + c ,$$

where c is a constant of integration. Exponentiating,

$$\boxed{R = b e^{H_0 \, t} ,}$$

where $b = e^c$ is an arbitrary constant.

b) *(15 points)* Consider a cube of side ℓ_c drawn on the comoving coordinate system diagram. The physical length of each side is then $R(t)\,\ell_c$, so the physical volume is

$$V(t) = R^3(t)\,\ell_c^3 .$$

Since the mass density is fixed at $\rho = \rho_0$, the total mass inside this cube at any given time is given by

$$M(t) = R^3(t)\,\ell_c^3\,\rho_0 .$$

In the absence of matter creation the total mass within a comoving volume would not change, so the increase in mass described by the above equation must be attributed to matter creation. The rate of matter creation per unit time per unit volume is then given by

$$\begin{aligned}\text{Rate} &= \frac{1}{V(t)}\frac{dM}{dt} \\ &= \frac{1}{R^3(t)\,\ell_c^3}\,3R^2(t)\frac{dR}{dt}\,\ell_c^3\,\rho_0 \\ &= \frac{3}{R}\frac{dR}{dt}\,\rho_0 \\ &= \boxed{3H_0\,\rho_0\ .}\end{aligned}$$

You were not asked to insert numbers, but it is worthwhile to consider the numerical value after the exam, to see what this answer is telling us. Suppose we take $H_0 = 70$ km-sec^{-1}-Mpc^{-1}, and take ρ_0 to be the critical density, $\rho_c = 3H_0^2/8\pi G$. Then

$$\begin{aligned}\text{Rate} &= \frac{9H_0^3}{8\pi G} \\ &= \frac{9\times(70\ \text{km-s}^{-1}\text{-Mpc}^{-1})^3}{8\pi\times 6.67260\times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}} \\ &= \frac{9\times(70\ \cancel{\text{km}}\text{-s}^{-1}\text{-}\cancel{\text{Mpc}}^{-1})^3}{8\pi\times 6.67260\times 10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{-s}^{-2}} \\ &\qquad \times\left(\frac{1\ \cancel{\text{Mpc}}}{3.086\times 10^{22}\ \cancel{\text{m}}}\right)^3\times\left(\frac{10^3\ \cancel{\text{m}}}{\cancel{\text{km}}}\right)^3 \\ &= 6.26\times 10^{-44}\ \text{kg-m}^{-3}\text{-s}^{-1}\ .\end{aligned}$$

To put this number into more meaningful terms, note that the mass of a hydrogen atom is 1.67×10^{-27} kg, and that 1 year $= 3.156\times 10^7$ s. The rate of matter production required for the steady-state universe theory can then be expressed as roughly one hydrogen atom per cubic meter per billion years! Needless to say, such a rate of matter production is totally undetectable, so the steady-state theory cannot be ruled out by the failure to detect matter production.

PROBLEM 3: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION *(40 points)*

a) *(5 points)* The cosmological redshift is given by the usual form,

$$1+z = \frac{R(t_0)}{R(t_e)}\quad .$$

For light emitted by an object at time t_e, the redshift of the received light is

$$1 + z = \frac{R(t_0)}{R(t_e)} = \left(\frac{t_0}{t_e}\right)^\gamma \quad .$$

So,

$$\boxed{z = \left(\frac{t_0}{t_e}\right)^\gamma - 1 \quad .}$$

b) *(5 points)* The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0 \left(1 - \frac{t_e}{t_0}\right) \quad .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0} = (1 + z)^{-1/\gamma} \quad .$$

Therefore,

$$\boxed{t_0 - t_e = t_0 \left[1 - (1 + z)^{-1/\gamma}\right] \quad .}$$

c) *(10 points)* The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = R(t_0) \int_{t_e}^{t_0} \frac{c}{R(t)} dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{ct_0^\gamma}{1 - \gamma} \left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Factoring $t_0^{\gamma-1}$ out of the parentheses gives

$$\ell_p(t_0) = \frac{ct_0}{1 - \gamma} \left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

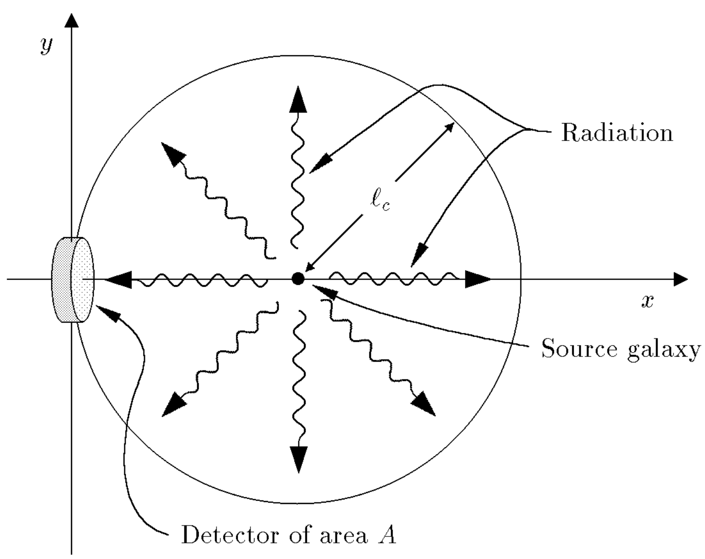
$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\ell_p(t_0) = cH_0^{-1} \frac{\gamma}{1-\gamma} \left[1 - (1+z)^{\frac{\gamma-1}{\gamma}}\right] \quad .$$

d) *(10 points)* A nearly identical problem was worked through in Problem 8 of Problem Set 1.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/R(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/R(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/R(t_0)^2)}{4\pi\ell_c^2} \frac{P}{(1+z)^2} \quad .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2} \frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = R(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, z, and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \ .}$$

e) *(10 points)* Following the solution of Problem 1 of Problem Set 1, we can introduce a fictitious relay station that is at rest relative to the galaxy, but located just next to the jet, between the jet and Earth. As in the previous solution, the relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but allows us to divide the problem into two simple parts.

The distance between the jet and the relay station is very short compared to cosmological scales, so the effect of the expansion of the universe is negligible. For this part of the problem we can use special relativity, which says that the period with which the relay station measures the received radiation is given by

$$\Delta t_{\text{relay station}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \times \Delta t_{\text{source}} \ .$$

Note that I have used the formula from the front of the exam, but I have changed the size of v, since the source in this case is moving toward the relay station, so the light is blue-shifted. To observers on Earth, the relay station is just a source at rest in the comoving coordinate system, so

$$\Delta t_{\text{observed}} = (1+z)\Delta t_{\text{relay station}} \ .$$

Thus,

$$\begin{aligned} 1 + z_J \equiv \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{source}}} &= \frac{\Delta t_{\text{observed}}}{\Delta t_{\text{relay station}}} \frac{\Delta t_{\text{relay station}}}{\Delta t_{\text{source}}} \\ &= (1+z)\big|_{\text{cosmological}} \ \times \ (1+z)\big|_{\text{special relativity}} \\ &= (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \ . \end{aligned}$$

Thus,

$$\boxed{z_J = (1+z)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} - 1 \ .}$$

Note added: In looking over the solutions to this problem, I found that a substantial number of students wrote solutions based on the incorrect assumption that the Doppler shift could be treated as if it were entirely due to motion. These students used the special relativity Doppler shift formula to convert the redshift z of the galaxy to a velocity of recession, then subtracted from this the speed v of the jet, and then again used the special relativity Doppler shift formula to find the Doppler shift corresponding to this composite velocity. However, as discussed at the end of Lecture Notes 3, the cosmological Doppler shift is given by

$$1 + z \equiv \frac{\Delta t_o}{\Delta t_e} = \frac{R(t_o)}{R(t_e)} \;, \tag{3.11}$$

and is not purely an effect caused by motion. It is really the combined effect of the motion of the distant galaxies and the gravitational field that exists between the galaxies, so the special relativity formula relating z to v does not apply.