MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe
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QUIZ 4

USEFUL INFORMATION:

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$R(t) \propto t^{2/3}$ (matter-dominated)

$R(t) \propto t^{1/2}$ (radiation-dominated)

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{d\lambda} \left\{ g_{ij} \frac{dx^j}{d\lambda} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell} \right) \frac{dx^k}{d\lambda} \frac{dx^\ell}{d\lambda}$$

or: $$\frac{d}{d\tau} \left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma} \right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\text{vac}} = -\rho_{\text{vac}} c^2 \qquad \rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$G = 6.672 \times 10^{-11}\ \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$$

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\, \text{erg/K}$$

$$= 8.617 \times 10^{-5}\, \text{eV/K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-sec}$$

$$= 6.582 \times 10^{-16}\ \text{eV-sec} \ ,$$

$$c = 2.998 \times 10^{10}\ \text{cm/sec}$$

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg} \ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad p = \frac{1}{3}u \qquad \rho = u/c^2$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4}\frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t\ (\text{in sec})}}$$

CHEMICAL EQUILIBRIUM:

$$n_i = g_i\frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3}\, e^{(\mu_i - m_i c^2)/kT}\ .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_ic^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

PARTICLE PROPERTIES:

While working on this exam you may refer to any of the tables in Lecture Notes 10.

PROBLEM 1: DID YOU DO THE READING? *(15 points)*

(a) *(5 points)* At the very end of the *Scientific American* article by Frank Wilczek, he said: "The answer to the ancient question 'Why is there something rather than nothing?' would then be that 'nothing' is ____________ . What word belongs in this blank? (The precise quotation will be given full credit, and reasonable synonyms will receive 4 points.)

The following question was part of Problem 1 on the Review Problems for Quiz 4, and appeared originally on Quiz 3, 1994. (One word has been changed to improve the clarity— 1 bonus point for anyone who can identify the altered word and say what it was in the original version.)

(b) *(5 points)* Most cosmologists believe that the entire visible universe, like the planet Earth, is composed of matter rather than antimatter. Which of the following are valid reasons for this belief? List as many as apply.

(A) The polarization of the photons from distant galaxies confirms that they are composed of matter and not antimatter.

(B) The fact that we receive photons and not antiphotons from distant galaxies implies that they are composed of matter and not antimatter.

(C) The fact that gamma rays associated with matter-antimatter annihilation are not detected suggests that nearby clusters of galaxies must be entirely matter or else entirely antimatter. If both matter and antimatter have significant abundances in the visible universe, it is difficult to imagine how they became segregated into distinct regions.

(D) The fact that we detect neutrinos and not antineutrinos from stars in distant galaxies implies that they are composed of matter rather than antimatter.

(c) *(5 points)* Dark matter particles are classified as either cold or hot, depending on whether the random velocities of the particles are small or large at the onset of matter domination. There is one prime candidate for hot dark matter, which is mentioned in Silk's book. What is it?

PROBLEM 2: THE HORIZON PROBLEM AT $t = 10^{-6}$ SECOND *(15 points)*

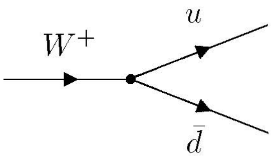
(a) *(5 points)* According to the standard cosmological model, as we have discussed in this class, the phase transition at which the quarks became bound inside strongly interacting particles (hadrons) took place at a cosmic time of $t \approx 10^{-6}$ second. What was the horizon distance at this time? Express your answer in light-seconds. You may assume that the universe was flat and radiation-dominated.

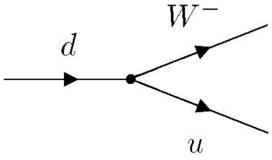
(b) *(10 points)* At this time ($t \approx 10^{-6}$) second, what approximately was the radius of the region that later evolved to become the presently observed universe? Take the radius of the observed universe today as 30 billion light-years. Express your answer in light-years, and be sure to show how you obtained your estimate. Anything within an order of magnitude of my own estimate will be given full credit.

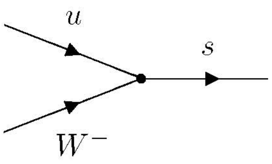
PROBLEM 3: QUARK DIAGRAMS FOR THE WEAK AND STRONG INTERACTIONS *(35 points)*

Parts (a)-(c) of the following problem were Problem 4 of Problem Set 5. The preamble has been expanded.

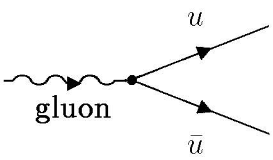
The diagrams which describe the weak interactions of the quarks allow couplings between any two quarks and any one of the charged intermediate vector bosons (W^+ or W^-), provided that electric charge and baryon number are conserved. (Recall that each quark has a baryon number of 1/3, and each antiquark has a baryon number of -1/3. The W has baryon number zero.) For example, the following couplings are allowed:

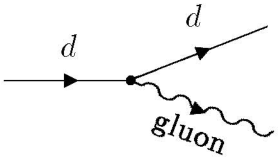


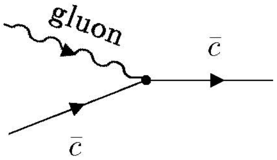




The probability associated with these couplings is largest when the two quarks belong to the same generation, but diagrams which mix generations also exist. The gluons, the carriers of the strong interactions, couple only to pairs of quarks with the same flavor, as in the diagrams below:







The neutral intermediate vector boson (Z^0), the remaining carrier of the weak interactions, attaches to quark lines exactly as gluons do, but it can also interact with leptons. The (strong interaction) gluon diagrams will always dominate over the (weak interaction) Z^0 diagrams when only quarks are involved, but the ability of the Z^0 to interact with leptons allows it to appear in certain diagrams for which gluons would not be allowed.

Using these rules, construct quark diagrams for the following four decays:

(a) *(7 points)* $\Sigma^+ \to n\pi^+$.

(b) *(7 points)* $\Sigma^+ \to p\pi^0$.

(c) *(7 points)* $\Delta^+ \to n\pi^+$.

(d) *(7 points)* $\Omega^- \to \Lambda K^-$.

(e) *(7 points)* Explain why reaction (c) is a strong interaction, while (a) and (b) are weak interactions. Is (d) a strong interaction or a weak interaction?

PROBLEM 4: NEUTRINO NUMBER AND THE NEUTRON/PROTON EQUILIBRIUM *(35 points)*

In the standard treatment of big bang nucleosynthesis it is assumed that at early times the ratio of neutrons to protons is given by the Boltzmann formula,

$$\frac{n_n}{n_p} = e^{-\Delta E/kT} \ , \tag{1}$$

where k is Boltzmann's constant, T is the temperature, and $\Delta E = 1.29$ MeV is the proton-neutron mass-energy difference. This formula is believed to be very accurate, but it assumes that the chemical potential for neutrons μ_n is the same as the chemical potential for protons μ_p.

(a) *(10 points)* Give the correct version of Eq. (1), allowing for the possibility that $\mu_n \neq \mu_p$.

The equilibrium between protons and neutrons in the early universe is sustained mainly by the following reactions:

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e$$

$$\nu_e + n \longleftrightarrow p + e^- \ .$$

Let μ_e and μ_ν denote the chemical potentials for the electrons (e^-) and the electron neutrinos (ν_e) respectively. The chemical potentials for the positrons (e^+) and the anti-electron neutrinos ($\bar{\nu}_e$) are then $-\mu_e$ and $-\mu_\nu$, respectively, since the chemical potential of a particle is always the negative of the chemical potential for the antiparticle.*

(b) *(10 points)* Express the neutron/proton chemical potential difference $\mu_n - \mu_p$ in terms of μ_e and μ_ν.

The black-body radiation formulas at the beginning of the quiz did not allow for the possibility of a chemical potential, but they can easily be generalized. For example, the formula for the number density n_i (of particles of type i) becomes

$$n_i = g_i^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_i/kT} \ .$$

(c) *(10 points)* Suppose that the density of anti-electron neutrinos $\bar{n}_\nu$ in the early universe was higher than the density of electron neutrinos n_ν. Express the thermal equilibrium value of the ratio n_n/n_p in terms of ΔE, T, and the antineutrino excess $\Delta n = \bar{n}_\nu - n_\nu$. (Your answer may also contain fundamental constants, such as k, $\hbar$, and c.)

(d) *(5 points)* Would an excess of anti-electron neutrinos, as considered in part (c), increase or decrease the amount of helium that would be produced in the early universe? Explain your answer.

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* This fact is a consequence of the principle that the chemical potential of a particle is the sum of the chemical potentials associated with its conserved quantities, while particle and antiparticle always have the opposite values of all conserved quantities.