MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 7, 1998
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# QUIZ 2

## USEFUL INFORMATION:

### EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta) ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2} ,$$

$$\kappa \equiv -k .$$

### COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

### ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

**SCHWARZSCHILD METRIC:**

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ ,$$

**GEODESIC EQUATION:**

$$\frac{d}{ds}\left\{ g_{ij} \frac{dx^j}{ds} \right\} = \frac{1}{2} \left(\partial_i g_{k\ell}\right) \frac{dx^k}{ds} \frac{dx^\ell}{ds}$$

or:
$$\frac{d}{d\tau}\left\{ g_{\mu\nu} \frac{dx^\nu}{d\tau} \right\} = \frac{1}{2} \left(\partial_\mu g_{\lambda\sigma}\right) \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}$$

**NOTE:** Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question.

**PROBLEM 1: DID YOU DO THE READING?** *(20 points)*

The following questions are worth 5 points each.

(a) If the universe is closed, then the net electric charge of the universe must

1) be equal to the number of protons;
2) be equal to the number of neutrinos;
3) be exactly zero;
4) be equal to the number of neutrinos minus the number of electrons.

Circle the number of the correct statement.

(b) William Herschel (1738-1822), the British astronomer, made two discoveries that tended to confirm the astronomical validity of Newton's law of gravity. Name one of them.

*The following two parts were on the Review Problems for Quiz 2:*

(c) The oldest rocks found on earth have been dated by radioactive elements, principally the decay of $U^{238}$ to $Pb^{206}$. Is the age estimated to be 2.1 billion years, 3.9 billion years, 6.3 billion years, or 9.7 billion years?

(d) The description of the early universe in Steven Weinberg's *The First Three Minutes* begins with a "frame" when $T = 10^{11}\,^\circ\mathrm{K}$. At this time, what particles are believed to have dominated the energy density of the universe? [Hint: the expected list includes 5 items, if particles and antiparticles are counted separately. Depending on how specific you are in your choice of names, your list could correctly include more than 5 items.]

**REMINDER:** Any answer to any problem may be expressed in terms of symbols representing the answers to previous parts of the same question.

## PROBLEM 2: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)*

*The following problem was Problem 1 of Problem Set 3:*

It was shown in Lecture Notes 5 that the evolution of a closed, matter-dominated universe can be described by introducing the time-parameter $\theta$, with

$$ct = \alpha(\theta - \sin\theta) ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

where $\alpha$ is a constant with the units of length.

(a) *(10 points)* Use these expressions to find $H(\theta)$, the Hubble "constant" as a function of $\theta$. *(Hint: You can use the first of the equations above to calculate $d\theta/dt$.)*

(b) *(10 points)* Find $\rho(\theta)$, the mass density as a function of $\theta$.

(c) *(5 points)* Find $\Omega(\theta)$, the value of $\Omega \equiv \rho/\rho_c$ as a function of $\theta$.

**PROBLEM 3: TRACING LIGHT RAYS IN A CLOSED, MATTER-DOMINATED UNIVERSE** *(30 points)*

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 \left( d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate $\psi$, related to $r$ by

$$r = \sin\psi \ .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi \ ,$$

so the metric simplifies to

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + R^2(t) \left\{ d\psi^2 + \sin^2\psi \left( d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} \ .$$

(a) *(7 points)* A light pulse travels on a null trajectory, which means that $d\tau = 0$ for each segment of the trajectory. Consider a light pulse that moves along a radial line, so $\theta = \phi =$ constant. Find an expression for $d\psi/dt$ in terms of quantities that appear in the metric.

(b) *(8 points)* Write an expression for the physical horizon distance $\ell_{\text{phys}}$ at time $t$. You should leave your answer in the form of a definite integral.

The form of $R(t)$ depends on the content of the universe. If the universe is matter-dominated (*i.e.*, dominated by nonrelativistic matter), then $R(t)$ is described by the parametric equations

$$ct = \alpha(\theta - \sin\theta) \ ,$$

$$R = \alpha(1 - \cos\theta) \ ,$$

where

$$\alpha \equiv \frac{4\pi}{3} \frac{G\rho R^3}{c^2} \ .$$

These equations are identical to those on the front of the exam, except that I have chosen $k = 1$.

(c) *(10 points)* Consider a radial light-ray moving through a matter-dominated closed universe, as described by the equations above. Find an expression for $d\psi/d\theta$, where $\theta$ is the parameter used to describe the evolution.

(d) *(5 points)* Suppose that a photon leaves the origin of the coordinate system ($\psi = 0$) at $t = 0$. How long will it take for the photon to return to its starting place? Express your answer as a fraction of the full lifetime of the universe, from big bang to big crunch.

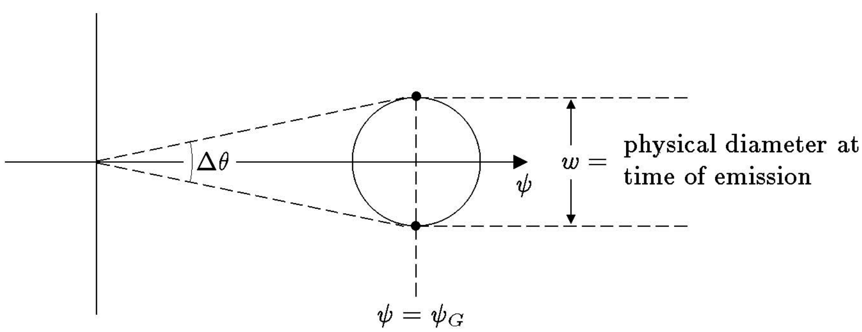
**PROBLEM 4: BRIGHTNESS AND ANGULAR SIZE IN A CLOSED UNIVERSE** *(25 points)*

Consider a homogeneous, isotropic, closed universe, which as in Problem 3 we describe with the spacetime metric

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

Here $\psi$ is related to the usual Robertson-Walker coordinate $r$ by $r = \sin\psi$ .

(a) *(10 points)* Suppose that the Earth is at the center of these coordinates, and that we observe a spherical galaxy that is located at $\psi = \psi_G$. The light that we see was emitted from the galaxy at time $t_G$, and is being received today, at a time that we call $t_0$. The physical diameter of the galaxy at time $t_G$ was $w$. Find the apparent angular size $\Delta\theta$ (measured from one edge to the other) of the galaxy as it would be observed from Earth today.




(b) *(10 points)* At the time of emission, the galaxy had a power output $P$ (which could be measured, for example, in ergs/sec). The power was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux $J$ from this galaxy at the earth today? Energy flux (which might be measured in ergs-cm$^{-2}$-sec$^{-1}$) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of the energy flow. [Hint: it is easiest to use a comoving coordinate system with the radiating galaxy at the origin.]

(c) *(5 points)* The *surface brightness* $\sigma$ of the distant galaxy is defined to be the energy flux $J$ per solid angle subtended by the galaxy.* Calculate the surface brightness $\sigma$ of the galaxy described in parts (a) and (b). [Hint: if you have the right answer, it can be written in terms of $P$, $w$, and the redshift $z$, without any reference to $\psi_G$. The rapid decrease in $\sigma$ with $z$ means that high-$z$ galaxies are difficult to distinguish from the night sky.]

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* *Definition of solid angle:* To define the solid angle subtended by the galaxy, imagine surrounding the observer by a sphere of arbitrary radius $r$. The sphere should be small compared to cosmological distances, so that Euclidean geometry is valid within the sphere. If a picture of the galaxy is painted on the surface of the sphere so that it just covers the real image, then the solid angle, in steradians, is the area of the picture on the sphere, divided by $r^2$.