MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe May 25, 1998
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QUIZ 4 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

(a) "The answer to the ancient question 'Why is there something rather than nothing?' would then be that 'nothing' is $\boxed{\text{unstable}}$."

(b) This issue is discussed in *"The Cosmic Asymmetry Between Matter and Antimatter"*, by Frank Wilczek (**Scientific American**, December 1980). Statement (A) is false: polarization can provide evidence of magnetic fields, but it does not distinguish between matter and antimatter. Statement (B) is also false: a photon is its own antiparticle, so there is no such thing as an antiphoton. $\boxed{\text{Statement (C) is true.}}$ It may not sound like a completely compelling reason, but it is the best that we have. Statement (D) is false. In principle, an anti-star can be distinguished from a star by the fact that it emits mainly antineutrinos rather than neutrinos. Unfortunately, however, we can detect neither neutrinos nor antineutrinos from any star other than our sun. [Bonus point: in statement (D), the word "receive" was changed to "detect," to make the sentence unambiguously false. We presumably do *receive* neutrinos and not antineutrinos from stars in distant galaxies, although we have no way of detecting them.]

(c) The most plausible hot dark matter candidate is the $\boxed{\text{neutrino}}$.

PROBLEM 2: THE HORIZON PROBLEM AT $t = 10^{-6}$ SECOND

(a) For a flat, radiation-dominated universe, the scale factor $R(t) \propto t^{1/2}$. The physical horizon distance is then

$$\ell_{p,\text{horizon}} = R(t) \int_0^t \frac{c\,dt'}{R(t')} = t^{1/2} \int_0^t \frac{c\,dt'}{t'^{1/2}} = 2ct\ .$$

So, for $t = 10^{-6}$ second, $\boxed{\ell_{p,\text{horizon}} = 2 \times 10^{-6}\text{ light-second.}}$

(b) To within the order-of-magnitude accuracy requested in the statement of the problem, we can use the relations $RT \approx const$ and

$$kT \approx \frac{1\text{ MeV}}{\sqrt{t\text{ (in sec)}}}\ ,$$

where the latter formula is an approximate form of the relation on the front of the exam. Thus, for $t = 10^{-6}$ second, $kT \approx 1$ GeV, where 1 GeV $= 10^3$ MeV $= 10^9$ eV.

[*Discussion* (not needed for full credit): The formula on the front of the exam is valid for 106 MeV $\gg kT \gg$ 0.511 MeV, but the changes needed to apply the formula to earlier times come only from the factor g. To extend the formula to 1 GeV, we would have to add the muon and antimuon, with $g = 7/2$, the light quarks (up, down, strange), with $g = 3 \times 3 \times 2 \times 2 \times \frac{7}{8} = 63/2$ (3 flavors $\times$ 3 colors $\times$ 2 spin states $\times$ 2 charge states [i.e., particle/antiparticle] $\times \frac{7}{8}$ fermion factor), and the gluons with $g = 8 \times 2 = 16$ (8 types of gluons $\times$ 2 spin states [like photons]). The total additional contribution to g is then 51. Since $kT \propto g^{-1/4}$, this change decreases the value of kT by a factor $[(51 + 10.75)/10.75]^{1/4} = 1.55$, which does not matter for an order-of-magnitude estimate.]

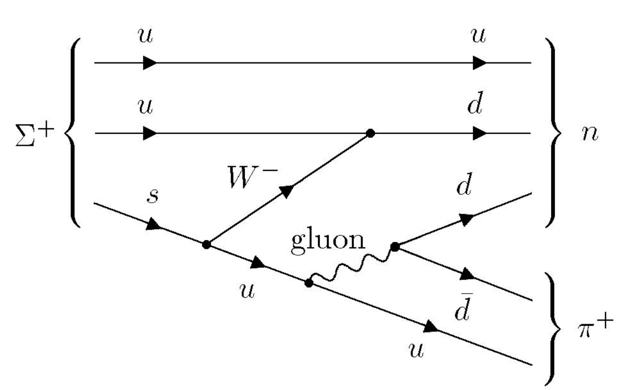
Then

$$
\begin{aligned}
r(10^{-6}\text{ sec}) &= \frac{R(10^{-6}\text{ sec})}{R(t_0)}\,(3 \times 10^{10}\text{ light-yr}) \\
&= \frac{kT(t_0)}{kT(10^{-6}\text{ sec})}\,(3 \times 10^{10}\text{ light-yr}) \\
&= \frac{10^{-4}\text{ eV/K} \times 3\,\text{K}}{10^{9}\text{ eV}}\,(3 \times 10^{10}\text{ light-yr}) \\
&\approx \boxed{10^{-2}\text{ light-yr.}}
\end{aligned}
$$

[*Discussion:* Using the fact that 1 yr $= 3.16 \times 10^7$ sec, one concludes that, at the time of the quark phase transition, the radius of the region that evolved to become the presently observed universe was about 10^{11} times larger than the horizon distance.]

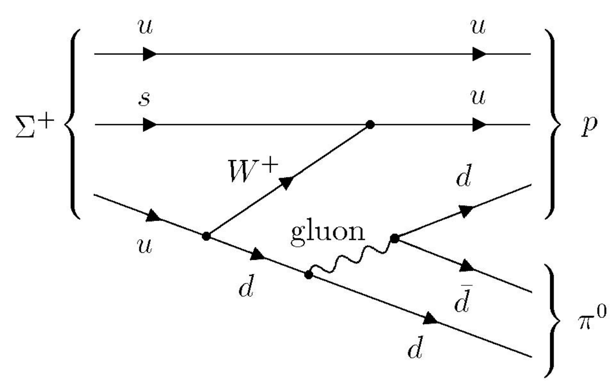
PROBLEM 3: QUARK DIAGRAMS FOR THE WEAK AND STRONG INTERACTIONS

(a)

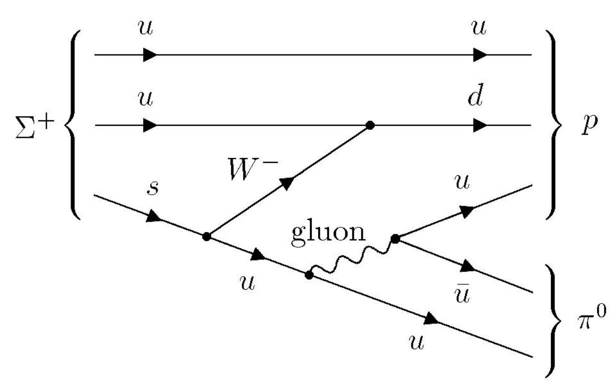


There are typically many gluon exchanges, since gluons (unlike W's) have a high probability of emission and absorption. I have drawn only the one gluon line that is absolutely necessary to create the $d\bar{d}$ pair — one end of this gluon line must be at the $d\bar{d}$ vertex, but the other end could attach to any of the quark lines (but not the W^- line). There are also diagrams in which the W^- line rejoins the same quark line from which it began, producing a d quark.

(b)

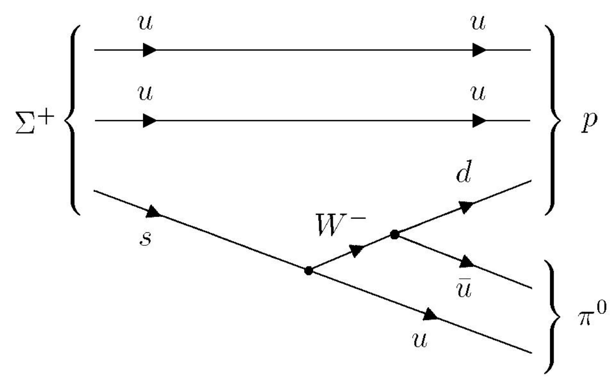


or



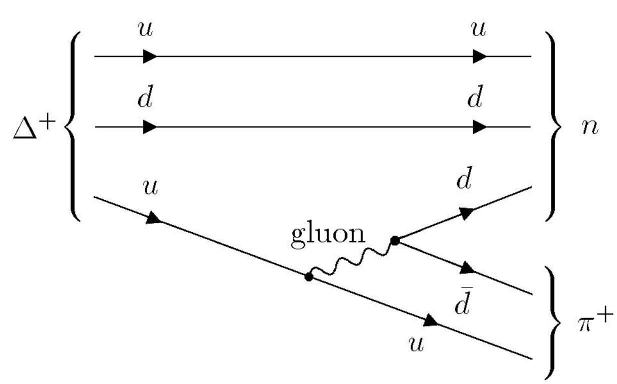
(While we are using these diagrams only to see what reactions can happen, they can be used to actually compute the quantum probability amplitudes for these processes. In that case the contributions from these two diagrams must be subtracted, since $\pi^0 = u\bar{u} - d\bar{d}$.)

The above diagrams are not unique, since the gluon line can begin on any of the quark lines. In addition, there are diagrams in which a W^- particle is emitted and then reabsorbed by the s quark line, turning it into a d quark. In each of the diagrams shown, the W line travelling from the bottom quark line to the middle quark line can be replaced by a W particle of the opposite charge travelling from the middle quark line to the bottom quark line. There are also diagrams such as the following, which do not require a gluon, which were found by a number of students:



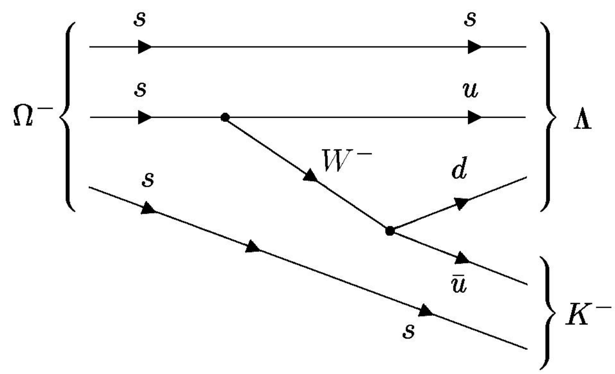
Several students constructed diagrams in which a gluon line produced either the $u\bar{u}$ or $d\bar{d}$ pair that constituted the pion. I gave full credit, since the issues are subtle and were not discussed in the course, but in fact the conservation of angular momentum implies that such diagrams do not exist. The gluon is a spin-1 particle, and it cannot turn into a spin-0 pion.

(c)

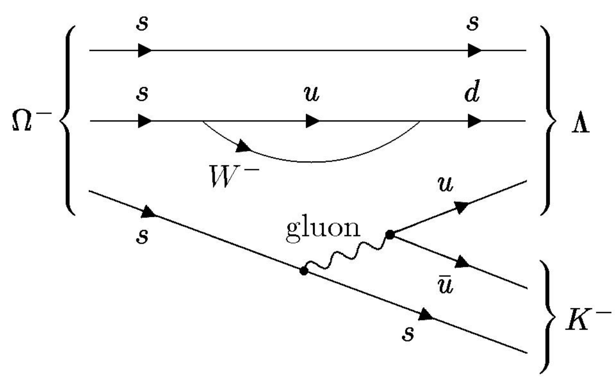


Again the diagram is not unique, since the gluon line can originate on any of the quark lines.

(d)



or



(e) (a) and (b) are weak interactions because they require W exchanges, while (c) is strong because it can occur with only gluon exchanges. (d) is a weak interaction, because it requires a W^- exchange.

PROBLEM 4: NEUTRINO NUMBER AND THE NEUTRON/ PROTON EQUILIBRIUM

(a) From the chemical equilibrium equation on the front of the exam, the number densities of neutrons and protons can be written as

$$n_n = g_n \frac{(2\pi m_n kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_n - m_n c^2)/kT}$$

$$n_p = g_p \frac{(2\pi m_p kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_p - m_p c^2)/kT} ,$$

where $g_n = g_p = 2$. Dividing,

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(\Delta E + \mu_p - \mu_n)/kT} \ ,$$

where $\Delta E = (m_n - m_p)c^2$ is the proton-neutron mass-energy difference. Approximating $m_n/m_p \approx 1$, one has

$$\boxed{\frac{n_n}{n_p} = e^{-(\Delta E + \mu_p - \mu_n)/kT} \ .}$$

The approximation $m_n/m_p \approx 1$ is very accurate (0.14%), but is clearly not necessary. Full credit was given whether or not this approximation was used.

(b) For any allowed chemical reaction, the sum of the chemical potentials on the two sides must be equal. So, from

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e \ ,$$

we can infer that

$$-\mu_e + \mu_n = \mu_p - \mu_\nu \ ,$$

which implies that

$$\boxed{\mu_n - \mu_p = \mu_e - \mu_\nu \ .}$$

(c) Applying the formula given in the problem to the number densities of electron neutrinos and the corresponding antineutrinos,

$$n_\nu = g_\nu^* \frac{\varsigma(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{\mu_\nu/kT}$$
$$\bar{n}_\nu = g_\nu^* \frac{\varsigma(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} e^{-\mu_\nu/kT} \ ,$$

since the chemical potential for the antineutrinos ($\bar{\nu}$) is the negative of the chemical potential for neutrinos. A neutrino has only one spin state, so $g_\nu = 3/4$, where the factor of 3/4 arises because neutrinos are fermions. Setting

$$x \equiv e^{-\mu_\nu/kT}$$

and

$$A \equiv \frac{3}{4} \frac{\varsigma(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ ,$$

the number density equations can be written compactly as

$$n_\nu = \frac{A}{x} \;, \qquad \bar{n}_\nu = xA \;.$$

The quantity x can then be determined from

$$\Delta n = \bar{n}_\nu - n_\nu = xA - \frac{A}{x} \;.$$

Rewriting the above formula as an explicit quadratic,

$$Ax^2 - \Delta n\, x - A = 0 \;,$$

one finds

$$x = \frac{\Delta n \pm \sqrt{\Delta n^2 + 4A^2}}{2A} \;.$$

Since the definition of x implies $x > 0$, only the positive root is relevant. Since the number of electrons is still assumed to be equal to the number of positrons, $\mu_e = 0$, so the answer to (b) reduces to $\mu_n - \mu_p = -\mu_\nu$. From (a),

$$\begin{aligned}\frac{n_n}{n_p} &= e^{-(\Delta E + \mu_p - \mu_n)/kT} \\ &= e^{-(\Delta E + \mu_\nu)/kT} \\ &= x e^{-\Delta E/kT} \\ &= \boxed{\frac{\sqrt{\Delta n^2 + 4A^2} + \Delta n}{2A}\, e^{-\Delta E/kT} \;,}\end{aligned}$$

where

$$\boxed{A \equiv \frac{3}{4}\frac{\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} \;.}$$

(d) For $\Delta n > 0$, the answer to (c) implies that the ratio n_n/n_p would be larger than in the usual case ($\Delta n = 0$). This is consistent with the expectation that an excess of antineutrinos will tend to cause p's to turn into n's according to the reaction

$$p + \bar{\nu}_e \longrightarrow e^+ + n \;.$$

Since the amount of helium produced is proportional to the number of neutrons that survive until the breaking of the deuterium bottleneck, starting with a higher equilibrium abundance of neutrons will $\boxed{\text{increase}}$ the production of helium.