MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 22, 1998
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REVIEW PROBLEMS FOR QUIZ 3

QUIZ DATE: Tuesday, April 28, 1998

COVERAGE: Lecture Notes 7 and 8; Problem Set 4; Weinberg, Chapters 6, 7, and 8; Silk, Chapters 6 and 7. The coverage will include the use of chemical potentials to describe equilibrium reactions, such as the discussion of hydrogren recombination in lecture, or the calculation of deuterium production in Problem 5 of Problem Set 4. It will also include the role of a possible cosmological constant, as discussed in lecture and used in Problem 3 of Problem Set 4. **One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the homework assignments, from this set of Review Problems, or from Quiz 3 of 1996.**

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They are all problems that I would consider fair for the quiz. I have included here all problems from the 1994 quizzes that are relevant to this quiz, and a number of problems from earlier years as well. (Problem 6 of Problem Set 4 was on the third and final quiz of 1994, counting 30% of the exam.) If a number of points is mentioned in this handout, it is based on 100 points for the full quiz. Problems from Quiz 3 of 1996 are not included here, but the quiz will be handed out separately.

INFORMATION TO BE GIVEN ON QUIZ:

The following material will be included on the quiz, so you need not memorize it. You should, however, make sure that you understand what these formulas mean, and how they can be applied.

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,$$

GEODESIC EQUATION:

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda}$$

or:
$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\rm vac} = -\rho_{\rm vac}c^2 \qquad \rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381\times 10^{-16}\,\text{erg}/^\circ\text{K}$$

$$= 8.617\times 10^{-5}\,\text{eV}/^\circ\text{K}\ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055\times 10^{-27}\ \text{erg-sec}$$

$$= 6.582\times 10^{-16}\ \text{eV-sec}\ ,$$

$$c = 2.998\times 10^{10}\ \text{cm/sec}$$

$$1\ \text{eV} = 1.602\times 10^{-12}\ \text{erg}\ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}$$

$$p = -\frac{1}{3}u \qquad \rho = u/c^2$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}$$

$$s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions ,} \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 \ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G} \right)^{1/4} \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860 \text{ MeV}}{\sqrt{t \text{ (in sec)}}}$$

CHEMICAL EQUILIBRIUM:

$$n_i = g_i \frac{(2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} \, e^{(\mu_i - m_i c^2)/kT} \ .$$

where n_i = number density of particle

g_i = number of spin states of particle

m_i = mass of particle

μ_i = chemical potential

For any reaction, the sum of the μ_i on the left-hand side of the reaction equation must equal the sum of the μ_i on the right-hand side. Formula assumes gas is nonrelativistic ($kT \ll m_i c^2$) and dilute ($n_i \ll (2\pi m_i kT)^{3/2}/(2\pi\hbar)^3$).

PROBLEM 1: DID YOU DO THE READING?

The following question was Problem 1(c) on Quiz 3, 1994, worth 5 points:

(a) Which of the following hypothetical observations would be considered evidence for the existence of cosmic strings? Indicate as many as apply.

(A) Double images of galaxies separated by a few arcseconds, especially if a line of such double images were observed.

(B) A linelike discontinuity in the temperature of the cosmic background radiation.

(C) Elongated galaxies with about 10^3 times more mass than ordinary galaxies.

(D) A background of gravitational waves, detectable through jitter in the time intervals of pulsar signals.

The following question was Problem 1(a) on Quiz 4, 1990, worth 5 points:

(b) In Chapter 6 of **The First Three Minutes**, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list:

(i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

(ii) There was a breakdown in communication between theorists and experimentalists.

(iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

(iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

(v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

(vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

PROBLEM 2: A NEW SPECIES OF LEPTON

The following problem was Problem 2, Quiz 3, 1992, worth 25 points.

Suppose the calculations describing the early universe were modified by including an additional, hypothetical lepton, called an 8.286ion. The 8.286ion has roughly the same properties as an electron, except that its mass is given by $mc^2 = 0.750$ MeV.

Parts (a)-(c) of this question require numerical answers, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included. (For example, if you were asked how many meters a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$.)

a) *(5 points)* What would be the number density of 8.286ions, in particles per cubic meter, when the temperature T was given by $kT = 3$ MeV?

b) *(5 points)* Assuming (as in the standard picture) that the early universe is accurately described by a flat, radiation-dominated model, what would be the value of the mass density at $t = .01$ sec? You may assume that 0.75 MeV $\ll kT \ll$ 100 MeV, so the particles contributing significantly to the black-body radiation include the photons, neutrinos, e^+-e^- pairs, and 8.286ion-anti8286ion pairs. Express your answer in the units of gm-cm^{-3}.

c) *(5 points)* Under the same assumptions as in (b), what would be the value of kT, in MeV, at $t = .01$ sec?

d) *(5 points)* When nucleosynthesis calculations are modified to include the effect of the 8.286ion, is the production of helium increased or decreased? Explain your answer in a few sentences.

e) *(5 points)* Suppose the neutrinos decouple while $kT \gg 0.75$ MeV. If the 8.286ions are included, what does one predict for the value of T_ν/T_γ today? (Here T_ν denotes the temperature of the neutrinos, and T_γ denotes the temperature of the cosmic background radiation photons.)

PROBLEM 3: FREEZE-OUT OF MUONS

The following problem was Problem 3, Quiz 3, 1990, where it was worth 30pts:

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) The black-body radiation formula, as shown at the front of this quiz, is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting R denote the Robertson-Walker scale factor, by what factor does the quantity RT increase when the muons disappear?

PROBLEM 4: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

Today the temperature of the cosmic microwave background radiation is 2.7°K. Calculate the number density of photons in this radiation. What is the number density of thermal neutrinos left over from the big bang?

PROBLEM 5: BIG BANG NUCLEOSYNTHESIS

The following problem was Problem 3, Quiz 4, 1990, where it was worth 20pts:

The calculations of big bang nucleosynthesis depend on a large number of measured parameters. Below you are asked to qualitatively describe the effects of changing some of these parameters. Include a sentence or two to explain each of your answers. Each part is worth 5 points.

(a) Suppose an extra neutrino species is added to the calculation. Would the predicted helium abundance go up or down?

(b) Suppose the weak interactions were stronger than they actually are, so that the thermal equilibrium distribution between neutrons and protons were maintained until $kT \approx 0.25$ MeV. Would the predicted helium abundance be larger or smaller than in the standard model?

(c) Suppose the proton-neutron mass difference were larger than the actual value of 1.29 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation?

(d) In the past few years some physicists have been exploring the consequences of inhomogeneous big-bang nucleosynthesis. They hypothesize that baryons became clumped during a phase transition at $t \approx 10^{-6}$ second. Does this clumping result in an increase or a decrease in the expected helium abundance?

SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) This subject is also discussed in Joseph Silk's book, on pp. 126–27. Statement (A) is true. A cosmic string can cause a double image, with one image visible on each side of the string. If the string crosses the line of sight of several distant galaxies, then these double images will form a line. Statement (B) is also true: a moving cosmic string actually causes a temperature differential between the radiation passing on either side of the string. Statement (C) is false: the galaxies that would be formed by cosmic string perturbations would look like normal galaxies, so it is conceivable that all the galaxies that we observe were in fact formed by cosmic strings. Statement (D) is true: cosmic strings actually dissipate most of their energy in gravitational radiation. Today these waves would distort the space through which the pulsar's radio waves are traveling, resulting in a nonuniformity in the time intervals between signals.

(b) The correct choices are (ii), (v), and (vi). *[Note: Statement (i) has no basis in fact, while statement (iii) is contradicted by Weinberg, who states that it was technologically possible to detect a 3° K background during the 1950s, and maybe even during the 1940s. As for statement (iv), Weinberg hardly mentions the steady state theory. Presumably the fact that one could not definitively choose between the steady state and big bang models contributed to the situation summarized by statement (v).]*

PROBLEM 2: A NEW SPECIES OF LEPTON

a) The number density is given by the formula at the start of the exam,

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3} \ .$$

Since the 8.286ion is like the electron, it has $g^* = 3$; there are 2 spin states for the particles and 2 for the antiparticles, giving 4, and then a factor of 3/4 because the particles are fermions. So

$$\begin{aligned} n &= 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3\ \cancel{\text{MeV}}}{6.582 \times 10^{-16}\ \cancel{\text{eV}}\text{-}\cancel{\text{sec}} \times 2.998 \times 10^{10}\ \cancel{\text{cm}}\text{-}\cancel{\text{sec}}^{-1}} \right)^3 \\ &\quad \times \left(\frac{10^6\ \cancel{\text{eV}}}{1\ \cancel{\text{MeV}}} \right)^3 \times \left(\frac{10^2\ \cancel{\text{cm}}}{1\ \text{m}} \right)^3 \\ &= 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}} \right)^3 \ \text{m}^{-3} \ . \end{aligned}$$

Then

$$\boxed{\text{Answer} = 3\frac{\zeta(3)}{\pi^2} \times \left(\frac{3 \times 10^6 \times 10^2}{6.582 \times 10^{-16} \times 2.998 \times 10^{10}}\right)^3 \ .}$$

You were not asked to evaluate this expression, but the answer is 1.29×10^{39}.

b) For a flat cosmology $\kappa = 0$ and one of the Einstein equations becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho \ .$$

During the radiation-dominated era $R(t) \propto t^{1/2}$, as claimed on the front cover of the exam. So,

$$\frac{\dot{R}}{R} = \frac{1}{2t} \ .$$

Using this in the above equation gives

$$\frac{1}{4t^2} = \frac{8\pi}{3} G\rho \ .$$

Solve this for ρ,

$$\rho = \frac{3}{32\pi G t^2} \ .$$

The question asks the value of ρ at $t = 0.01$ sec. With $G = 6.6732 \times 10^{-8}$ cm^3 sec^{-2} g^{-1}, then

$$\boxed{\rho = \frac{3}{32\pi \times 6.6732 \times 10^{-8} \times (0.01)^2}}$$

in units of g/cm^3. You weren't asked to put the numbers in, but, for reference, doing so gives $\rho = 4.47 \times 10^9$ g/cm^3.

c) The mass density $\rho = u/c^2$, where u is the energy density. The energy density for black-body radiation is given in the exam,

$$u = \rho c^2 = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \ .$$

We can use this information to solve for kT in terms of $\rho(t)$ which we found above in part (b). At a time of 0.01 sec, g has the following contributions:

Photons:	$g = 2$
e^+e^-:	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$
ν_e, ν_μ, ν_τ:	$g = 6 \times \frac{7}{8} = 5\frac{1}{4}$
8.286ion − anti8.286ion	$g = 4 \times \frac{7}{8} = 3\frac{1}{2}$

$$g_{\rm tot} = 14\frac{1}{4} \quad .$$

Solving for kT in terms of ρ gives

$$kT = \left[\frac{30}{\pi^2}\frac{1}{g_{\rm tot}}\hbar^3 c^5 \rho\right]^{1/4} \quad .$$

Using the result for ρ from part (b) as well as the list of fundamental constants from the cover sheet of the exam gives

$$\boxed{kT = \left[\frac{90 \times (1.055 \times 10^{-27})^3 \times (2.998 \times 10^{10})^5}{14.24 \times 32\pi^3 \times 6.6732 \times 10^{-8} \times (0.01)^2}\right]^{1/4} \times \frac{1}{1.602 \times 10^{-6}}}$$

where the answer is given in units of MeV. Putting in the numbers yields $kT = 8.02$ MeV.

d) The production of helium is increased. At any given temperature, the additional particle increases the energy density. Since $H \propto \rho^{1/2}$, the increased energy density speeds the expansion of the universe— the Hubble constant at any given temperature is higher if the additional particle exists, and the temperature falls faster. The weak interactions that interconvert protons and neutrons "freeze out" when they can no longer keep up with the rate of evolution of the universe. The reaction rates at a given temperature will be unaffected by the additional particle, but the higher value of H will mean that the temperature at which these rates can no longer keep pace with the universe will occur sooner. The freeze-out will therefore occur at a higher temperature. The equilibrium value of the ratio of neutron to proton densities is larger at higher temperatures: $n_n/n_p \propto \exp(-\Delta mc^2/kT)$, where n_n and n_p are the number densities of neutrons and protons, and Δm is the neutron-proton mass difference. Consequently, there are more neutrons present to combine with protons to build helium nuclei. In addition, the faster evolution rate implies that the temperature at which the deuterium bottleneck breaks is reached sooner. This implies that fewer neutrons will have a chance to decay, further increasing the helium production.

e) After the neutrinos decouple, the entropy in the neutrino bath is conserved separately from the entropy in the rest of the radiation bath. Just after neutrino decoupling, all of the particles in equilibrium are described by the same temperature which cools as $T \propto 1/R$. The entropy in the bath of particles still in equilibrium just after the neutrinos decouple is

$$S \propto g_{\text{rest}} T^3(t) R^3(t)$$

where $g_{\text{rest}} = g_{\text{tot}} - g_\nu = 9$. By today, the $e^+ - e^-$ pairs and the 8.286ion-anti8.286ion pairs have annihilated, thus transferring their entropy to the photon bath. As a result the temperature of the photon bath is increased relative to that of the neutrino bath. From conservation of entropy we have that the entropy after annihilations is equal to the entropy before annihilations

$$g_\gamma T_\gamma^3 R^3(t) = g_{\text{rest}} T^3(t) R^3(t) \quad .$$

So,

$$\frac{T_\gamma}{T(t)} = \left(\frac{g_{\text{rest}}}{g_\gamma}\right)^{1/3} \quad .$$

Since the neutrino temperature was equal to the temperature before annihilations, we have that

$$\boxed{\frac{T_\nu}{T_\gamma} = \left(\frac{2}{9}\right)^{1/3} \quad .}$$

PROBLEM 3: FREEZE-OUT OF MUONS

(a) The factors contributing to g from the muons are the following:

- 2 since there are two particles, the muon and the antimuon
- 2 since there are two spin states for each particle
- $\frac{7}{8}$ since the μ^- and the μ^+ are fermions

Thus

$$\boxed{g_\mu = 2 \times 2 \times \frac{7}{8} = \frac{7}{2} \,.}$$

(b) Besides the muons, the particles in thermal equilibrium when kT is just above 106 MeV are photons, neutrinos and electrons. As found in class

$$g_\gamma = 2$$

$$g_\nu = 3 \times 2 \times \frac{7}{8} = \frac{21}{4}$$

$$g_{e^-} = 2 \times 2 \times \frac{7}{8} = \frac{7}{2} \ .$$

So, for kT just above 106 MeV, g is the sum of all of these contributions:

$$\boxed{g = g_\mu + g_\gamma + g_\nu + g_{e^-} = \frac{57}{4} = 14.25 \ .}$$

(c) We know that entropy S is conserved and that it can be written as $S = R^3 \times s$ where s is the entropy density. The expression for the entropy density is given on the cover of the exam. It is

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} \ .$$

Therefore $S = R^3 \times s$ is given by

$$S = C \times g(T) R^3 T^3 \quad \text{where} \quad C = \text{constant.}$$

Let T_a denote the temperature of the universe when kT_a is just *above* 106 MeV. Let T_b denote the temperature of the universe when kT_b is just *below* 106 MeV. Since the entropy is constant $S(T_a) = S(T_b)$. Using the above expression for S we find

$$C \times g(T_a)(RT_a)^3 = C \times g(T_b)(RT_b)^3 \qquad \Longrightarrow$$

$$\frac{RT_b}{RT_a} = \left[\frac{g(T_a)}{g(T_b)}\right]^{1/3} \ .$$

We found $g(T_a)$ in part (b), $g(T_a) = 14.25$. After the muons disappear from the black body radiation they no longer contribute to the g in the expression for the entropy. Thus at temperatures below T_b, $g(T_b) = g_\gamma + g_\nu + g_{e^-} = 2 + \frac{21}{4} + \frac{7}{2} = \frac{43}{4} = 10.75$. Using these values in the expression above we obtain the increase in RT,

$$RT_b = \left(\frac{14.25}{10.75}\right)^{1/3} RT_a = \left(\frac{57}{43}\right)^{1/3} RT_a \qquad \Longrightarrow$$

$$\boxed{RT_b \approx (1.10) RT_a \ .}$$

PROBLEM 4: NUMBER DENSITIES IN THE COSMIC BACKGROUND RADIATION

In general, the number density of a particle in the black-body radiation is given by

$$n = g^* \frac{\xi(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$$

For photons, one has $g^* = 2$. Then

$$\left.\begin{aligned} k &= 1.381 \times 10^{-16} \text{erg}/^\circ\text{K} \\ T &= 2.7\,^\circ\text{K} \\ \hbar &= 1.055 \times 10^{-27} \text{erg-sec} \\ c &= 2.998 \times 10^{10} \text{cm}/\text{sec} \end{aligned}\right\} \implies \left(\frac{kT}{\hbar c}\right)^3 = 1.638 \times 10^3 \text{cm}^{-3} \ .$$

Then using $\xi(3) \simeq 1.202$, one finds

$$\boxed{n_\gamma = 399/\text{cm}^3 \ .}$$

For the neutrinos,

$$g_\nu^* = 2 \times \frac{3}{4} = \frac{3}{2} \quad \text{per species.}$$

The factor of 2 is to account for ν and $\bar{\nu}$, and the factor of $3/4$ arises from the Pauli exclusion principle. So for three species of neutrinos one has

$$g_\nu^* = \frac{9}{2} \ .$$

Using the result

$$T_\nu^3 = \frac{4}{11} T_\gamma^3$$

from Problem 4 of Problem Set 4, one finds

$$\begin{aligned} n_\nu &= \left(\frac{g_\nu^*}{g_\gamma^*}\right) \left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \\ &= \left(\frac{9}{4}\right) \left(\frac{4}{11}\right) 399 \text{cm}^{-3} \\ \implies & \boxed{n_\nu = 326/\text{cm}^3 \ \text{(for all three species combined).}} \end{aligned}$$

PROBLEM 5: BIG BANG NUCLEOSYNTHESIS

(a) It would go up. An extra neutrino species increases the value of g, and a look at the time-temperature relation shows that this means that the early universe evolves faster. The weak interactions would then freeze out earlier, resulting in a higher abundance of neutrons. The neutrons would also have less time to undergo free decay, and so more would survice.

(b) Smaller. Neutrons are heavier than protons, so in thermal equilibrium their abundance is suppressed relative to protons by the Boltzmann factor $\exp\{-\Delta mc^2/kT\}$. This factor gets bigger as T gets smaller, so the extension of thermal equilibrium to low temperatures implies that the density of neutrons is further reduced, so fewer are available for nucleosynthesis.

(c) Smaller. An increase in the mass difference would also enhance the Boltzmann factor mentioned in (b), so the the argument described in (b) would apply here also.

(d) The effect is expected to decrease the helium production. The reason hinges on the fact that the neutron has a much longer mean free than the proton, since the neutron is uncharged, while the proton interacts electromagnetically with the plasma that would fill the universe at this time. Thus the neutrons can diffuse away from the clumps of baryons that formed by the phase transition, resulting in a separation between protons and neutrons. Many neutrons, as a result, would decay without ever interacting.

Note, however, that the scenario described above assumes that the neutron diffusion length is larger than or comparable to the size of the clumps. If the diffusion length is smaller, on the other hand, than the effect of the inhomogeneity is to increase the effective density of baryons, and helium production would go up.