MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 31, 1998
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REVIEW PROBLEMS FOR QUIZ 2

QUIZ DATE: Tuesday, April 7, 1998

COVERAGE: Lecture Notes 4, 5, and 6; Problem Sets 2 and 3; Weinberg, Chapters 4-5; Silk, Chapters 1-5. **One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from the homework assignments, from this set of Review Problems, or from Quiz 1 or 2 of 1996.** Quiz 1 of 1996 was handed out earlier, but Problem 2 of that quiz would be a possible problem for the upcoming quiz. Quiz 2 of 1996 will be handed out with these Review Problems.

PURPOSE: These review problems are not to be handed in, but are being made available to help you study. They are all problems that I would consider fair for the upcoming quiz. I have included here all relevant problems from the 1994 quizzes, and a number of problems from earlier years as well. The 1996 quiz is being handed out separately. Whenever a number of points is mentioned in these problems, it is based on 100 points for the full quiz.

INFORMATION TO BE GIVEN ON QUIZ:

The following material will be included on the quiz, so you need not memorize it. You should, however, make sure that you understand what these formulas mean, and how they can be applied.

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$): $\quad R(t) \propto t^{2/3}$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):
$$ct = \alpha(\sinh\theta - \theta)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) \ ,$$
$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2} \ ,$$
$$\kappa \equiv -k \ .$$

COSMOLOGICAL REDSHIFT:
$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:
$$ds^2 = R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:
$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:
$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\frac{dx^\ell}{d\lambda}$$
or:
$$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

PROBLEM 1: THE DECELERATION PARAMETER *(10 points)*

The following problem was Problem 2, Quiz 2, 1992:

Many standard references in cosmology define a quantity called the ***deceleration parameter*** q, which is a direct measure of the slowing down of the cosmic expansion. The parameter is defined by
$$q \equiv -\ddot{R}(t)\frac{R(t)}{\dot{R}^2(t)} \ .$$
Find the relationship between q and Ω for a matter-dominated universe. *[In case you have forgotten, Ω is defined by*
$$\Omega = \rho/\rho_c \ ,$$
where ρ is the mass density and ρ_c is the critical mass density (i.e., that mass density which puts the universe just on the border between eternal expansion and eventual collapse).]

PROBLEM 2: DID YOU DO THE READING? *(20 points)*

The following problem was Problem 1, Quiz 2, 1994. Only 4 of the 5 parts are shown, because the last part involved material not included on this quiz.

The following questions are worth 5 points each. Where two items are requested, you will receive 3 points for getting one right.

a) Weinberg emphasizes that most of the detailed properties of the early universe are determined by the assumption that it was in a state of thermal equilibrium. Thermal equilibrium, however, cannot change a conserved quantity, so each conserved quantity must be specified. Weinberg mentions three conserved quantities whose densities must be specified in the recipe for the early universe. One is electric charge (which is specified to be zero or negligibly small). What are the other two?

b) An important number in cosmology is the ratio of baryon number to the number of photons. Is this ratio approximately 10^{-9}, 10^{-3}, 1, or 10^{6}?

c) At three minutes after the big bang, when the processes of nucleosynthesis were nearing completion, the energy density of the universe was dominated by two types of particles from the following list: pions, protons, neutrons, photons, neutrinos, electrons, positrons, muons, quarks, and kaons. What were these two types of particles?

d) Calculations of big bang nucleosynthesis were carried out as early as the 1940's by George Gamow and his collaborators Ralph Alpher and Robert Herman. They tried unsuccessfully to explain the abundances of all species of nuclei in terms of synthesis during the big bang. In contrast, scientists today believe that a) all elements other than hydrogen were synthesized primarily in stars; b) all elements other than hydrogen, helium, and perhaps lithium were synthesized primarily in stars; c) all elements heavier than calcium were synthesized in stars, while those lighter than calcium were synthesized mainly in the big bang; or d) all elements heavier than iron were synthesized in stars, while those lighter than iron were synthesized mainly in the big bang. Which choice is correct?

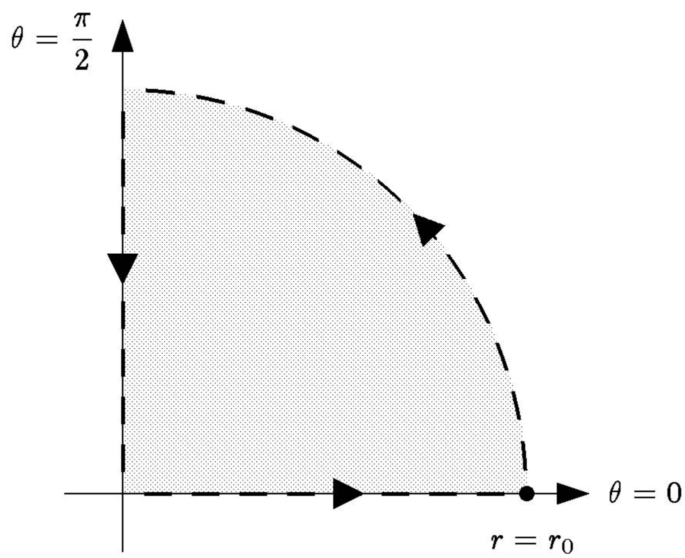
PROBLEM 3: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC *(25 points)*

The following problem was Problem 3, Quiz 2, 1994:

Suppose a two dimensional space, described in polar coordinates (r, θ), has a metric given by

$$ds^2 = (1 + ar)^2\, dr^2 + r^2(1 + br)^2\, d\theta^2 \ ,$$

where a and b are positive constants. Consider the path in this space which is formed by starting at the origin, moving along the $\theta = 0$ line to $r = r_0$, then moving at fixed r to $\theta = \pi/2$, and then moving back to the origin at fixed θ. The path is shown below:



a) *(10 points)* Find the total length of this path.

b) *(15 points)* Find the area enclosed by this path.

PROBLEM 4: SHORT ANSWERS *(10 points)*

The following questions were part of Problem 1, Quiz 3, 1994:

The following questions are each worth 5 points:

(a) The oldest rocks found on earth have been dated by radioactive elements, principally the decay of U^{238} to Pb^{206}. The age is estimated to be 2.1 billion years, 3.9 billion years, 6.3 billion years, or 9.7 billion years?

(b) When astronomers try to measure the distribution of radio galaxies in space, they find (A) that they appear to be uniformly distributed in space, (B) that there appear to be more nearby than far away, or (C) that there appear to be more far away than nearby?

PROBLEM 5: DID YOU DO THE READING? *(22 points)*

The following questions were part of Problem 1, Quiz 2, 1992:

a) *(8 points)* Penzias and Wilson discovered a diffuse background radiation in all directions over the sky. What, roughly, is the temperature in Kelvin characterizing the spectrum of the background radiation? What is the origin of this radiation? In the past, was the temperature of the microwave background higher or lower than its present temperature? In the future, will the temperature of the microwave background be higher or lower than its present temperature?

b) *(6 points)* Today, the universe is primarily composed of nonrelativistic matter. Which element has the largest primordial abundance by weight? Which element has the second largest primordial abundance by weight? At about what time were the nuclei of these elements synthesized?

c) *(6 points)* In the standard cosmological picture, what is the name given to the event that is believed to have initiated the expansion of the universe? *[Don't look for anything obscure— this one is a giveaway.]* Is the expansion slowing down or speeding up? Why?

d) *(2 points)* In addition to the microwave background, the standard model of the early universe predicts a neutrino background. Is the temperature of the neutrino background expected to be $10,000\,^\circ$K, $4000\,^\circ$K, $3\,^\circ$K, or $2\,^\circ$K.

PROBLEM 6: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(30 points)*

The following problem was Problem 3, Quiz 2, 1992:

The equations describing the evolution of an open, matter-dominated universe are shown on the first page of this quiz. The following mathematical identities, which you should know, may also prove useful:

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \ .$$

a) *(4 points)* Find the Hubble "constant" H as a function of α and θ.

b) *(4 points)* Find the mass density ρ as a function of α and θ.

c) *(4 points)* Find the mass density parameter Ω as a function of α and θ.

d) *(6 points)* Find the physical value of the horizon distance, $\ell_{p,\mathrm{horizon}}$, as a function of α and θ.

e) *(6 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then R as a function of t. Give the expression for $R(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

f) *(6 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

PROBLEM 7: DID YOU DO THE READING?

The following short answer question was worth 4 points on Quiz 2 of 1990:

> After the initial big bang the universe began to expand and cool. When the universe became sufficiently cool electrons and protons began to settle into neutral hydrogen. What name is often applied to this era? Did this take place around 1 second, 3 minutes, a few hundred thousand years, or a few million years after the big bang?

PROBLEM 8: TIME SCALES IN COSMOLOGY *(20 points)*

The following problem was Problem 2, Quiz 3, 1988:

In this problem you are asked to give the approximate times at which various important events in the history of the universe are believed to have taken place. The times are measured from the instant of the big bang. To avoid ambiguities, you are asked to choose the best answer from the following list:

10^{-43} sec.
10^{-35} sec.
10^{-12} sec.
10^{-5} sec.
1 sec.
4 mins.
10,000 – 1,000,000 years.
2 billion years.
5 billion years.
10 billion years.
13 billion years.
20 billion years.

For this problem it will be sufficient to state an answer from memory, without explanation. The events which must be placed are the following:

(a) the present time;

(b) the time at which the universe was half its present size (assuming a matter-dominated flat model);

(c) the time at which the universe ceased to be radiation-dominated and began to be matter-dominated.

Choosing from the same list of choices, state

(d) the lower limit on the Hubble time H_0^{-1};

(e) the upper limit on the Hubble time H_0^{-1}.

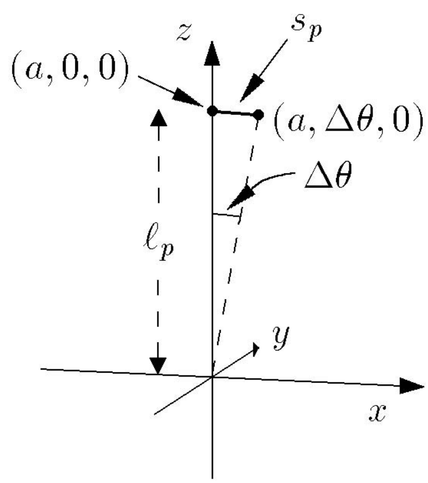
Since cosmology is fraught with uncertainty, in some cases more than one answer will be acceptable. You are asked, however, to give ***ONLY ONE*** of the acceptable answers.

PROBLEM 9: GEOMETRY IN A CLOSED UNIVERSE *(25 points)*

The following problem was Problem 4, Quiz 2, 1988:

Consider a universe described by the Robertson–Walker metric on the first page of the quiz, with $k = 1$. The questions below all pertain to some fixed time t, so the scale factor can be written simply as R, dropping its explicit t-dependence.

A small rod has one end at the point $(r = a,\ \theta = 0,\ \phi = 0)$ and the other end at the point $(r = a,\ \theta = \Delta\theta,\ \phi = 0)$. Assume that $\Delta\theta \ll 1$.



(a) Find the physical distance ℓ_p from the origin $(r = 0)$ to the first end $(a, 0, 0)$ of the rod. You may find one of the following integrals useful:

$$\int \frac{dr}{\sqrt{1 - r^2}} = \sin^{-1} r$$

$$\int \frac{dr}{1 - r^2} = \frac{1}{2} \ln\left(\frac{1 + r}{1 - r}\right) .$$

(b) Find the physical length s_p of the rod. Express your answer in terms of the scale factor R, and the coordinates a and $\Delta\theta$.

(c) Note that $\Delta\theta$ is the angle subtended by the rod, as seen from the origin. Write an expression for this angle in terms of the physical distance ℓ_p, the physical length s_p, and the scale factor R.

PROBLEM 10: THE GENERAL SPHERICALLY SYMMETRIC METRIC *(20 points)*

The following problem was Problem 3, Quiz 2, 1986:

The metric for a given space depends of course on the coordinate system which is used to describe it. It can be shown that for any three dimensional space which is spherically symmetric about a particular point, coordinates can be found so that the metric has the form

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

for some function $\rho(r)$. The coordinates θ and ϕ have their usual ranges: θ varies between 0 and π, and ϕ varies from 0 to 2π, where $\phi = 0$ and $\phi = 2\pi$ are identified. Given this metric, consider the sphere whose outer boundary is defined by $r = r_0$.

(a) Find the physical radius a of the sphere. (By "radius", I mean the physical length of a radial line which extends from the center to the boundary of the sphere.)

(b) Find the physical area of the surface of the sphere.

(c) Find an explicit expression for the volume of the sphere. Be sure to include the limits of integration for any integrals which occur in your answer.

(d) Suppose a new radial coordinate σ is introduced, where σ is related to r by

$$\sigma = r^2 \ .$$

Express the metric in terms of this new variable.

PROBLEM 11: VOLUMES IN A ROBERTSON-WALKER UNIVERSE *(20 points)*

The following problem was Problem 1, Quiz 3, 1990:

The metric for a Robertson-Walker universe is given by

$$ds^2 = R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \ .$$

Calculate the volume $V(r_{\rm max})$ of the sphere described by

$$r \le r_{\rm max} \ .$$

You should carry out any angular integrations that may be necessary, but you may leave your answer in the form of a radial integral which is not carried out. Be sure, however, to clearly indicate the limits of integration.

PROBLEM 12: THE SCHWARZSCHILD METRIC *(25 points)*

The follow problem was Problem 4, Quiz 3, 1992:

The space outside a spherically symmetric mass M is described by the Schwarzschild metric, given at the front of the exam. Two observers, designated A and B, are located along the same radial line, with values of the coordinate r given by r_A and r_B, respectively, with $r_A < r_B$. You should assume that both observers lie outside the Schwarzschild horizon.

a) *(5 points)* Write down the expression for the Schwarzschild horizon radius R_{Sch}, expressed in terms of M and fundamental constants.

b) *(5 points)* What is the proper distance between A and B? It is okay to leave the answer to this part in the form of an integral that you do not evaluate— but be sure to clearly indicate the limits of integration.

c) *(5 points)* Observer A has a clock that emits an evenly spaced sequence of ticks, with proper time separation $\Delta\tau_A$. What will be the coordinate time separation Δt_A between these ticks?

d) *(5 points)* At each tick of A's clock, a light pulse is transmitted. Observer B receives these pulses, and measures the time separation on his own clock. What is the time interval $\Delta\tau_B$ measured by B.

e) *(5 points)* Suppose that the object creating the gravitational field is a static black hole, so the Schwarzschild metric is valid for all r. Now suppose that one considers the case in which observer A lies on the Schwarzschild horizon, so $r_A \equiv R_{\text{Sch}}$. Is the proper distance between A and B finite for this case? Does the time interval of the pulses received by B, $\Delta\tau_B$, diverge in this case?

PROBLEM 13: DID YOU DO THE READING?

The first 3 parts of this question come from Quiz 2, 1988, and the 4th part comes from Quiz 3, 1986:

(a) Which of the following cosmologists were proponents of the steady-state theory: Fred Hoyle, Alexandre Friedmann, George Gamow, Herman Bondi, Georges Lemaître, Thomas Gold?

(b) The description of the early universe in Steven Weinberg's *The First Three Minutes* begins with a "frame" when $T = 10^{11}\,^\circ$K. At this time, what particles are believed to have dominated the mass density of the universe?

(c) Weinberg gives us the following description of the universe at $10^{9}\,^\circ$K: "The universe is now cool enough for tritium and helium three as well as ordinary helium nuclei to hold together, but the '_________ bottleneck' is still at work: nuclei of _________ do not hold together long enough to allow appreciable numbers of heavier nuclei to be built up." Both blanks are filled with the same word. What is it?

(d) A theory of big bang nucleosynthesis was first worked out in the late 1940's by George Gamow, Ralph Alpher, and Robert Herman. This theory differed from the currently accepted theory in at least four significant ways. Name one.

PROBLEM 14: GEODESICS *(20 points)*

The following problem was Problem 4, Quiz 2, 1986:

Ordinary Euclidean two-dimensional space can be described in polar coordinates by the metric

$$ds^2 = dr^2 + r^2\,d\theta^2 \ .$$

(a) Suppose that $r(\lambda)$ and $\theta(\lambda)$ describe a geodesic in this space, where the parameter λ is the arc length measured along the curve. Use the general formula on the front of the exam to obtain explicit differential equations which $r(\lambda)$ and $\theta(\lambda)$ must obey.

(b) Now introduce the usual Cartesian coordinates, defined by

$$x = r\cos\theta \ ,$$
$$y = r\sin\theta \ .$$

Use your answer to (a) to show that the line $y = 1$ is a geodesic curve.

PROBLEM 15: METRIC OF A STATIC GRAVITATIONAL FIELD *(30 points)*

The following problem was Problem 2, Quiz 3, 1990:

In this problem we will consider the metric

$$ds^2_{\mathrm{ST}} = -\left[c^2 + 2\phi(\vec{x})\right]\,dt^2 + \sum_{i=1}^{3}\left(dx^i\right)^2 \quad ,$$

which describes a static gravitational field. Here i runs from 1 to 3, with the identifications $x^1 \equiv x$, $x^2 \equiv y$, and $x^3 \equiv z$. The function $\phi(\vec{x})$ depends only on the spatial variables $\vec{x} \equiv (x^1, x^2, x^3)$, and not on the time coordinate t.

(a) Suppose that a radio transmitter, located at $\vec{x}_e$, emits a series of evenly spaced pulses. The pulses are separated by a proper time interval ΔT_e, as measured by a clock at the same location. What is the coordinate time interval Δt_e between the emission of the pulses? (I.e., Δt_e is the difference between the time coordinate t at the emission of one pulse and the time coordinate t at the emission of the next pulse.)

(b) The pulses are received by an observer at $\vec{x}_r$, who measures the time of arrival of each pulse. What is the ***coordinate*** time interval Δt_r between the reception of successive pulses?

(c) The observer uses his own clocks to measure the proper time interval ΔT_r between the reception of successive pulses. Find this time interval, and also the redshift Z, defined by

$$1 + Z = \frac{\Delta T_r}{\Delta T_e} \quad .$$

First compute an exact expression for Z, and then expand the answer to lowest order in $\phi(\vec{x})$ to obtain a weak-field approximation. (This weak-field approximation is in fact highly accurate in all terrestrial and solar system applications.)

(d) A freely falling particle travels on a spacetime geodesic $x^\mu(\tau)$, where τ is the proper time. (I.e., τ is the time that would be measured by a clock moving with the particle.) The trajectory is described by the geodesic equation

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \quad ,$$

where the Greek indices ($\mu, \nu, \lambda, \sigma$, etc.) run from 0 to 3, and are summed over when repeated. Calculate an explicit expression for

$$\frac{d^2x^i}{d\tau^2} \ ,$$

valid for $i = 1, 2$, or 3. (It is acceptable to leave quantities such as $dt/d\tau$ or $dx^i/d\tau$ in the answer.)

SOLUTIONS

PROBLEM 1: THE DECELERATION PARAMETER

From the front of the exam, we are reminded that

$$\ddot{R} = -\frac{4\pi}{3} G \rho R$$

and

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{R^2} \ ,$$

where a dot denotes a derivative with respect to time t. The critical mass density ρ_c is defined to be the mass density that corresponds to a flat ($k = 0$) universe, so from the equation above it follows that

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G \rho_c \ .$$

So, substituting into the definition of q, we find

$$\begin{aligned} q = -\ddot{R}(t) \frac{R(t)}{\dot{R}^2(t)} &= -\frac{\ddot{R}}{R} \left(\frac{R}{\dot{R}}\right)^2 \\ &= \left(\frac{4\pi}{3} G \rho\right) \left(\frac{3}{8\pi G \rho_c}\right) = \frac{1}{2} \frac{\rho}{\rho_c} = \boxed{\frac{1}{2} \Omega \ .} \end{aligned}$$

PROBLEM 2: DID YOU DO THE READING?

a) The other two conserved quantities are **baryon number** and **lepton number**. (Weinberg also mentions that the electron lepton number and the muon lepton number appear to be separately conserved. Today we would have to add tau lepton number to this list. These conservation laws are still consistent with all known experiments, but there are theoretical reasons for doubting their exactness. We will talk about this later in the course.)

b) It is approximately 10^{-9}.

c) Photons and neutrinos. (Protons and neutrons do not become an appreciable part of the mass density until about 100,000 years after the big bang.)

d) b: all elements other than hydrogen, helium, and perhaps lithium were synthesized primarily in stars.

PROBLEM 3: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1+ar)^2\,dr^2$, or $ds = (1+ar)\,dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1+ar)\,dr = r_0 + \frac{1}{2}ar_0^2 \ .$$

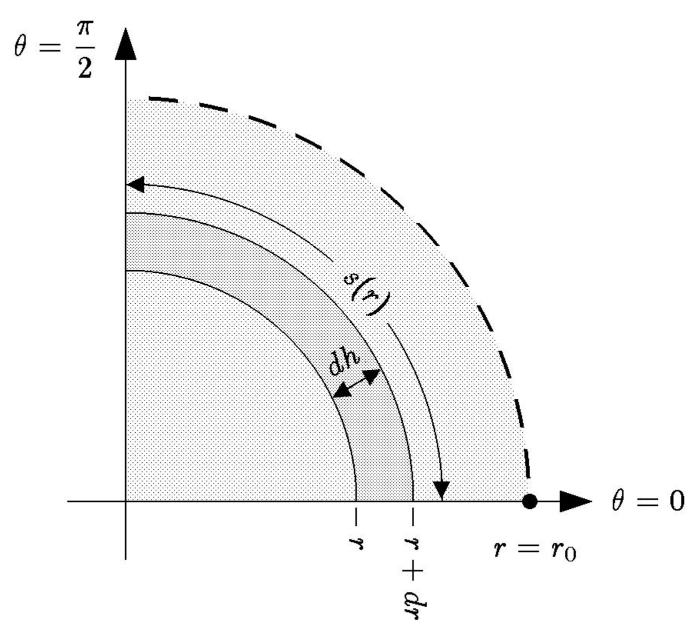
Along the second segment $dr = 0$, so $ds = r(1+br)\,d\theta$, where $r = r_0$. So the length of the second segment is

$$S_2 = \int_0^{\pi/2} r_0(1+br_0)\,d\theta = \frac{\pi}{2}r_0(1+br_0) \ .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$\begin{aligned} S = 2S_1 + S_2 &= 2\left(r_0 + \frac{1}{2}ar_0^2\right) + \frac{\pi}{2}r_0(1+br_0) \\ &= \boxed{\left(2+\frac{\pi}{2}\right)r_0 + \frac{1}{2}(2a+\pi b)r_0^2 \ .} \end{aligned}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1 + ar)\, dr \ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) \ .$$

The area is then

$$dA = s(r)\, dh \ ,$$

so

$$\begin{aligned} A &= \int_0^{r_0} s(r)\, dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1 + br)(1 + ar)\, dr \\ &= \frac{\pi}{2} \int_0^{r_0} \left[r + (a + b) r^2 + abr^3 \right] dr \\ &= \boxed{\frac{\pi}{2} \left[\frac{1}{2} r_0^2 + \frac{1}{3} (a + b) r_0^3 + \frac{1}{4} abr_0^4 \right]} \end{aligned}$$

PROBLEM 4: SHORT ANSWERS

a) This subject is discussed in **The Big Bang**, by Joseph Silk, on p. 70. The answer is 3.9 billion years.

b) This subject is also discussed in the book by Joseph Silk, on p. 81. The answer is (C): there appear to be more radio galaxies far away than nearby. It is believed that this is an evolutionary effect. When we look far away we are seeing into the past, since light travels at a finite velocity. Apparently radio galaxies were more intense in the past, so we see a larger fraction of them.

PROBLEM 5: DID YOU DO THE READING?

a) The temperature of the cosmic background radiation today is $\sim 3^\circ$ K (actually most recent observations find the temperature quite close to 2.726° K.)

The background radiation is a primordial relic of the hot radiation dominated era in the early history of the universe. As the universe cooled it became transparent to radiation. The primordial photons decoupled from matter and the thermal distribution of these photons continued to cool as the universe expanded. (In practice we accepted any answer to this question which referenced, however briefly, the hot early universe, decoupling, etc..)

The temperature of the cosmic background radiation was higher in the past. It will continue to cool as the universe expands. As some students noted, if the universe is closed, the expansion will cease and the universe may then contract, heating back up again as it does so.

b) H has the largest primordial abundance by weight while He has the second largest abundance. The nuclei of these elements were synthesized within roughly the first 3 minutes (the time of nucleosynthesis). Some students referred to recombination, at a time of roughly a hundred thousand years, as the time at which these elements were synthesized. Although the atoms may have been ionized until recombination when free electrons combined with the positively charged nuclei, it is the time of nucleosynthesis when these elements were actually synthesized and not recombination.

c) The Big Bang is the name given to the event which initiated the universe's expansion. The expansion is slowing at present due to the gravitational pull of the matter and energy in the universe. [However, in January and February of 1998, two groups of astronomers observing Type 1A supernovas up to redshifts just short of $z = 1$ reported evidence that the expansion of the universe is now accelerating. If verified, this would presumably mean that Einstein's cosmological constant is not zero after all, but has a positive value. One would still expect that the expansion slowed dramatically during the early history of the universe, but then somewhere in the middle of its history the expansion started to accelerate.]

d) The neutrino background is expected to be at a temperature of about 2° K. (Note: Some students suggested that the temperature of the neutrino background is lower than that of the photon background since the neutrinos decoupled earlier. Although it is true that the neutrinos decoupled earlier, this alone does not imply the neutrino background is cooler than the photon background. After the neutrinos decoupled, they continued to cool at the same rate as the photons, until electrons and positrons annihilated. Once electrons and positrons annihilated, energy released in these annihilations was transferred to the photon background which was still coupled to matter, thus increasing the photon temperature above the neutrino temperature.)

PROBLEM 6: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE

Note that parts (a), (b), and (c) of this problem are nearly identical to Problem Set 3, Problem 1 (1998). Nonetheless, many students seemed to have quite a bit of trouble with this one. For that reason, I have attempted to make the solutions as pedagogical as possible, even though they became quite long. You should not think, however, that you were expected to write this much explanation on your exam.

(a) The general formula for H is

$$H = \frac{1}{R}\frac{dR}{dt} \ ,$$

but in this case there is the complication that R is given as a function of θ rather than t. But θ depends on t, so one must apply the chain rule:

$$\frac{d}{dt}R(\theta) = \frac{dR}{d\theta}\frac{d\theta}{dt} \ .$$

The standard formula for the Hubble constant can then be rewritten as

$$H(\theta) = \frac{1}{R}\frac{dR}{d\theta}\frac{d\theta}{dt} \ .$$

On the front of the exam one finds the parametric equations for R and t:

$$ct = \alpha\,(\sinh\theta - \theta)$$
$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) \ .$$

Recall that the hyperbolic trigonometric functions are defined by

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \ ,$$
$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \ ,$$

and they are differentiated as

$$\frac{d}{d\theta}\sinh\theta = \cosh\theta \ ,$$
$$\frac{d}{d\theta}\cosh\theta = \sinh\theta \ .$$

So, differentiating the parametric equations,

$$\frac{dR}{d\theta} = \alpha\sqrt{k}\sinh\theta \ ,$$
$$\frac{dt}{d\theta} = \frac{\alpha}{c}(\cosh\theta - 1) \ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{\kappa}\alpha(\cosh\theta - 1)}\right]\left[\alpha\sqrt{\kappa}\sinh\theta\right]\left[\frac{c}{\alpha(\cosh\theta - 1)}\right]$$
$$= \boxed{\frac{c\sinh\theta}{\alpha(\cosh\theta - 1)^2} \ .}$$

(b) This problem can be attacked by at least three different methods. While you were expected to use only one, we will show all three.

(i) The equation from the front of the exam,

$$\alpha = \frac{4\pi}{3} \frac{G\rho R^3}{\kappa^{3/2} c^2} ,$$

can be solved for ρ to give

$$\rho = \frac{3}{4\pi} \frac{\alpha \kappa^{3/2} c^2}{G R^3} .$$

Then substitute the parametric equation for $R(\theta)$:

$$\begin{aligned} \rho &= \frac{3}{4\pi} \frac{\alpha \kappa^{3/2} c^2}{G} \frac{1}{\alpha^3 \kappa^{3/2} (\cosh\theta - 1)^3} \\ &= \boxed{\frac{3}{4\pi} \frac{c^2}{G\alpha^2 (\cosh\theta - 1)^3} .} \end{aligned}$$

(ii) Starting from

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} ,$$

one can write

$$\frac{8\pi}{3} G\rho = H^2 + \frac{kc^2}{R^2} .$$

Recalling that we described open universes by using $\kappa \equiv -k$, this can be rewritten as

$$\frac{8\pi}{3} G\rho = H^2 - \frac{\kappa c^2}{R^2} .$$

Replacing H by the answer in part (a) and R by its parametric equation, one finds

$$\begin{aligned} \frac{8\pi}{3} G\rho &= \frac{c^2 \sinh^2\theta}{\alpha^2 (\cosh\theta - 1)^4} - \frac{\kappa c^2}{\alpha^2 \kappa (\cosh\theta - 1)^2} \\ &= \frac{c^2}{\alpha^2 (\cosh\theta - 1)^4} \left[\sinh^2\theta - (\cosh\theta - 1)^2 \right] . \end{aligned}$$

Now make use of the hypertrigonometric identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to simplify:

$$\sinh^2\theta - (\cosh\theta - 1)^2 = \sinh^2\theta - \cosh^2\theta + 2\cosh\theta - 1 = 2(\cosh\theta - 1) \ ,$$

so

$$\frac{8\pi}{3} G\rho = \frac{2c^2}{\alpha^2(\cosh\theta - 1)^3} \ .$$

Dividing both sides of the equation by $(8\pi/3)G$, one duplicates the previous result.

(iii) ρ can also be found from $\ddot{R} = -(4\pi/3)G\rho R$. To evaluate $\ddot{R}$, again use the chain rule. Starting with $\dot{R}$,

$$\dot{R} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \alpha\sqrt{\kappa}\sinh\theta \frac{c}{\alpha(\cosh\theta - 1)} = \frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1} \ .$$

Then

$$\begin{aligned}
\ddot{R} &= \frac{d\dot{R}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left[\frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\right]\frac{c}{\alpha(\cosh\theta - 1)} \\
&= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)}\left[\frac{\cosh\theta}{\cosh\theta - 1} - \frac{\sinh^2\theta}{(\cosh\theta - 1)^2}\right] \\
&= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}\left[\cosh\theta(\cosh\theta - 1) - \sinh^2\theta\right] \\
&= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}(1 - \cosh\theta) = -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} \ .
\end{aligned}$$

So

$$\ddot{R} = -\frac{4\pi}{3}G\rho R \quad \Longrightarrow \quad -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} = -\frac{4\pi}{3}G\rho\alpha\sqrt{\kappa}(\cosh\theta - 1) \ ,$$

and

$$\rho = \frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta - 1)^3} \ .$$

(c) The critical mass density satisfies the cosmological evolution equations for $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c \ .$$

Then

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \ .$$

Now replace H by the answer to part (a), and ρ by the answer to part (b):

$$\begin{aligned}\Omega &= \frac{8\pi G}{3}\left[\frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta-1)^3}\right]\left[\frac{\alpha^2(\cosh\theta-1)^4}{c^2\sinh^2\theta}\right]\\ &= 2\frac{\cosh\theta-1}{\sinh^2\theta} = 2\frac{\cosh\theta-1}{\cosh^2\theta-1}\\ &= 2\frac{\cosh\theta-1}{(\cosh\theta+1)(\cosh\theta-1)} = \boxed{\frac{2}{\cosh\theta+1}\ .}\end{aligned}$$

The answer can be written even more compactly, if one wishes, by using a further hypertrigonometric identity:

$$\Omega = \frac{2}{\cosh\theta+1} = \frac{1}{\cosh^2\frac{1}{2}\theta} = \operatorname{sech}^2\frac{1}{2}\theta\ .$$

(d) The basic formula that determines the physical value of the horizon distance is given by Eq. (5.7) of the lecture notes:

$$\ell_{p,\mathrm{horizon}}(t) = R(t)\int_0^t \frac{c}{R(t')}dt'\ .$$

The complication here is that R is given as a function of θ, rather than t. The problem is handled, however, by a simple change of integration variables. One can change the integral over t' to an integral over θ', provided that one replaces

$$dt' \to \frac{dt'}{d\theta'}d\theta' = \frac{\alpha}{c}(\cosh\theta'-1)d\theta'\ .$$

One must also re-express the limits of integration in terms of θ. So

$$\begin{aligned}\ell_{p,\mathrm{horizon}}(\theta) &= R(\theta)\int_0^\theta \frac{c}{R(\theta')}\frac{dt'}{d\theta'}d\theta'\\ &= \alpha\sqrt{\kappa}(\cosh\theta-1)\int_0^\theta \frac{c}{\alpha\sqrt{\kappa}(\cosh\theta'-1)}\frac{\alpha}{c}(\cosh\theta'-1)\,d\theta'\ .\\ &= \alpha(\cosh\theta-1)\int_0^\theta d\theta' = \boxed{\alpha\,\theta(\cosh\theta-1)\ .}\end{aligned}$$

(e) The key to this problem is the ability to use a power series expansion, and I have to admit that I was very surprised to find that many of you seemed very inexperienced in this technique. It is a very useful method of approximation, however, so I strongly urge all of you to learn it if you don't know it already. In general, any sufficiently smooth function $f(x)$ can be expanded about the point x_0 by the series

$$f(x) = f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 + \ldots \; ,$$

where the prime is used to denote a derivative. In particular, the exponential, sinh, and cosh functions can be expanded about $\theta = 0$ by the formulas

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$$

$$\sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^5}{7!}\ldots$$

$$\cosh\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots \; .$$

For this problem, we expand the parametric equations for $R(\theta)$ and $t(\theta)$, keeping the first nonvanishing term in the power series expansions:

$$t = \frac{\alpha}{c}(\sinh\theta - \theta) = \frac{\alpha}{c}\left(\frac{\theta^3}{3!} + \ldots\right)$$

$$R = \alpha\sqrt{\kappa}(\cosh\theta - 1) = \alpha\sqrt{\kappa}\left(\frac{\theta^2}{2!} + \ldots\right) \; .$$

The first expression can be solved for θ, giving

$$\theta \approx \left(\frac{6ct}{\alpha}\right)^{1/3} \; ,$$

which can be substituted into the second expression to give

$$\boxed{R \approx \frac{1}{2}\alpha\sqrt{\kappa}\left(\frac{6ct}{\alpha}\right)^{2/3} \; .}$$

The power series expansions for the sinh and cosh are valid whenever the terms left out are much smaller than the last term kept, which happens when $\theta \ll 1$. Given the above relation between θ and t, this condition is equivalent to

$$t \ll \frac{\alpha}{6c} \; .$$

Thus,

$$\boxed{t^* \approx \frac{\alpha}{6c}, \text{ or } t^* \approx \frac{\alpha}{c}\ .}$$

Since there is no precise meaning to the statement that an approximation is valid, there is no precise value for t^*. Some students placed criteria on the size of the first omitted term in the series, and then derived a more precise value for t^*. These expressions for t^* were always in the form of a dimensionless constant times α/c. This approach is very good, but it was not required to get full credit for this problem.

(f) From part (c), the expression for Ω is given by

$$\Omega = \frac{2}{\cosh\theta + 1}\ .$$

So,

$$1 - \Omega = 1 - \frac{2}{\cosh\theta + 1} = \frac{\cosh\theta - 1}{\cosh\theta + 1}\ .$$

Expanding numerator and denominator in power series,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots}{2 + \frac{\theta^2}{2!} + \ldots}\ .$$

Keeping only the leading terms,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2}}{2} = \frac{1}{4}\theta^2\ ,$$

so

$$\boxed{1 - \Omega \approx \frac{1}{4}\left(\frac{6ct}{\alpha}\right)^{2/3}\ .}$$

This result shows that the deviation of Ω from 1 is amplified with time. This fact leads to a conundrum called the "flatness problem", which will be discussed later in the course.

A common mistake (very minor) was to keep extra terms, especially in the denominator. Keeping extra terms allows a higher degree of accuracy, so there is nothing wrong with it. However, one should always be sure to keep ***all*** terms of a given order, since keeping only a subset of terms may or may not increase the accuracy.

In this case, an extra term in the denominator can be rewritten as a term in the numerator:

$$\frac{\frac{\theta^2}{2!}}{2+\frac{\theta^2}{2!}} = \frac{1}{4}\frac{\theta^2}{1+\frac{\theta^2}{4}} = \frac{1}{4}\theta^2\left(1-\frac{\theta^2}{4}+\dots\right)$$
$$= \frac{1}{4}\theta^2 - \frac{1}{16}\theta^4 + \dots ,$$

where I used the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots .$$

Thus, the extra term in the denominator is equivalent to a term in the numerator of order θ^4, but other terms proportional to θ^4 have been dropped. So, it is not worthwhile to keep the 2nd term in the expansion of the denominator.

PROBLEM 7: DID YOU DO THE READING

The era when electrons and protons combined to form neutral hydrogen is known as recombination. This happened roughly a few hundred thousand years after the big bang.

PROBLEM 8: TIME SCALES IN COSMOLOGY

(a) 13 billion years. *[A shorter age is difficult to reconcile with the estimated age of globular clusters, while a longer age is difficult to reconcile with my belief in a flat, matter-dominated universe, which means that $t_0 = \frac{2}{3}H_0^{-1}$. 10 billion years and 20 billion years are also acceptable answers.]*

(b) 5 billion years. *[The scale factor in a matter-dominated flat model behaves as $R(t) \propto t^{2/3}$, so the universe was half its present size when it was $(1/2)^{3/2}$ times its present age. Taking the present age as 13 billion years, this gives 4.6 billion years.]*

(c) 10,000 – 1,000,000 years.

(d) 10 billion years.

(e) 20 billion years.

PROBLEM 9: GEOMETRY IN A CLOSED UNIVERSE

(a) As one moves along a line from the origin to $(a, 0, 0)$, there is no variation in θ or ϕ. So $d\theta = d\phi = 0$, and

$$ds = \frac{R\,dr}{\sqrt{1-r^2}} .$$

So

$$\boxed{\ell_p = \int_0^a \frac{R\,dr}{\sqrt{1-r^2}} = R\sin^{-1} a\ .}$$

(b) In this case it is only θ that varies, so $dr = d\phi = 0$. So

$$ds = Rr\,d\theta\ ,$$

so

$$\boxed{s_p = Ra\,\Delta\theta\ .}$$

(c) From part (a), one has

$$a = \sin(\ell_p/R)\ .$$

Inserting this expression into the answer to (b), and then solving for $\Delta\theta$, one has

$$\boxed{\Delta\theta = \frac{s_p}{R\sin(\ell_p/R)}\ .}$$

Note that as $R \to \infty$, this approaches the Euclidean result, $\Delta\theta = s_p/\ell_p$.

PROBLEM 10: THE GENERAL SPHERICALLY SYMMETRIC METRIC

(a) The metric is given by

$$ds^2 = dr^2 + \rho^2(r)\left[d\theta^2 + \sin^2\theta\,d\phi^2\right]\ .$$

The radius a is defined as the physical length of a radial line which extends from the center to the boundary of the sphere. The length of a path is just the integral of ds, so

$$a = \int_{\substack{\text{radial path from}\\ \text{origin to } r_0}} ds\ .$$

The radial path is at a constant value of θ and ϕ, so $d\theta = d\phi = 0$, and then $ds = dr$. So

$$a = \int_0^{r_0} dr = \boxed{r_0\ .}$$

(b) On the surface $r = r_0$, so $dr \equiv 0$. Then

$$ds^2 = \rho^2(r_0)\left[d\theta^2 + \sin^2\theta\,d\phi^2\right]\ .$$

To find the area element, consider first a path obtained by varying only θ. Then $ds = \rho(r_0)\,d\theta$. Similarly, a path obtained by varying only ϕ has length $ds = \rho(r_0)\sin\theta\,d\phi$. Furthermore, these two paths are perpendicular to each other, a fact that is incorporated into the metric by the absence of a $dr\,d\theta$ term. Thus, the area of a small rectangle constructed from these two paths is given by the product of their lengths, so

$$dA = \rho^2(r_0)\sin\theta\,d\theta\,d\phi\ .$$

The area is then obtained by integrating over the range of the coordinate variables:

$$\begin{aligned} A &= \rho^2(r_0)\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \\ &= \rho^2(r_0)(2\pi)\left(-\cos\theta\Big|_0^{\pi}\right) \end{aligned}$$

$$\Longrightarrow \quad \boxed{A = 4\pi\rho^2(r_0)\ .}$$

As a check, notice that if $\rho(r) = r$, then the metric becomes the metric of Euclidean space, in spherical polar coordinates. In this case the answer above becomes the well-known formula for the area of a Euclidean sphere, $4\pi r^2$.

(c) As in Problem 4 of Problem Set 3 (1998), we can imagine breaking up the volume into spherical shells of infinitesimal thickness, with a given shell extending from r to $r + dr$. By the previous calculation, the area of such a shell is $A(r) = 4\pi\rho^2(r)$. (In the previous part we considered only the case $r = r_0$, but the same argument applies for any value of r.) The thickness of the shell is just the path length ds of a radial path corresponding to the coordinate interval dr. For radial paths the metric reduces to $ds^2 = dr^2$, so the thickness of the shell is $ds = dr$. The volume of the shell is then

$$dV = 4\pi\rho^2(r)\,dr\ .$$

The total volume is then obtained by integration:

$$\boxed{V = 4\pi\int_0^{r_0} \rho^2(r)\,dr\ .}$$

Checking the answer for the Euclidean case, $\rho(r) = r$, one sees that it gives $V = (4\pi/3)r_0^3$, as expected.

(d) If r is replaced by a new coordinate $\sigma \equiv r^2$, then the infinitesimal variations of the two coordinates are related by

$$\frac{d\sigma}{dr} = 2r = 2\sqrt{\sigma}\ ,$$

so

$$dr^2 = \frac{d\sigma^2}{4\sigma} .$$

The function $\rho(r)$ can then be written as $\rho(\sqrt{\sigma})$, so

$$\boxed{ds^2 = \frac{d\sigma^2}{4\sigma} + \rho^2(\sqrt{\sigma})\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] .}$$

PROBLEM 11: VOLUMES IN A ROBERTSON-WALKER UNIVERSE

The product of differential length elements corresponding to infinitesimal changes in the coordinates r, θ and ϕ equals the differential volume element dV. Therefore

$$dV = R(t)\frac{dr}{\sqrt{1-kr^2}} \times R(t) r d\theta \times R(t) r \sin\theta d\phi$$

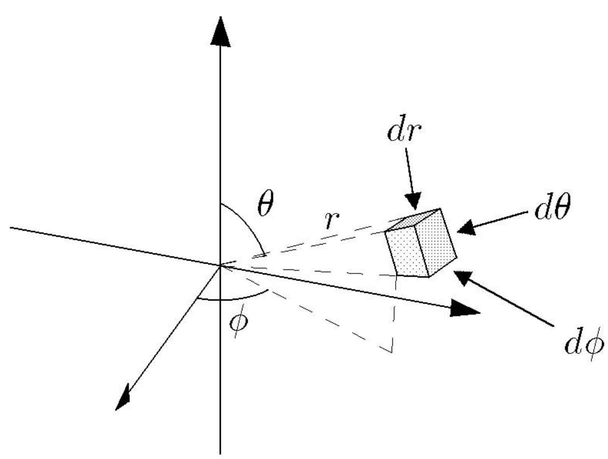
The total volume is then

$$V = \int dV = R^3(t)\int_0^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1-kr^2}}$$

We can do the angular integrations immediately:

$$\boxed{V = 4\pi R^3(t)\int_0^{r_{max}} \frac{r^2 dr}{\sqrt{1-kr^2}} .}$$

[Pedagogical Note: If you don't see through the solutions above, then note that the volume of the sphere can be determined by integration, after first breaking the volume into infinitesimal cells. A generic cell is shown in the diagram below:



The cell includes the volume lying between r and $r + dr$, between θ and $\theta + d\theta$, and between ϕ and $\phi + d\phi$. In the limit as dr, $d\theta$, and $d\phi$ all approach zero, the cell approaches a rectangular solid with sides of length:

$$\begin{aligned} ds_1 &= R(t)\frac{dr}{\sqrt{1-kr^2}} \\ ds_2 &= R(t) r\, d\theta \\ ds_3 &= R(t) r \sin\theta\, d\theta\ . \end{aligned}$$

Here each ds is calculated by using the metric to find ds^2, in each case allowing only one of the quantities dr, $d\theta$, or $d\phi$ to be nonzero. The infinitesimal volume element is then $dV = ds_1 ds_2 ds_3$, resulting in the answer above. The derivation relies on the orthogonality of the dr, $d\theta$, and $d\phi$ directions; the orthogonality is implied by the metric, which otherwise would contain cross terms such as $dr\, d\theta$.]

[Extension: The integral can in fact be carried out, using the substitution

$$\begin{aligned} \sqrt{k}\, r &= \sin\psi \quad (\text{if } k > 0) \\ \sqrt{-k}\, r &= \sinh\psi \quad (\text{if } k > 0). \end{aligned}$$

The answer is

$$V = \begin{cases} 2\pi R^3(t)\left[\dfrac{\sin^{-1}\left(\sqrt{k}\, r_{\rm max}\right)}{k^{3/2}} - \dfrac{\sqrt{1-kr_{\rm max}^2}}{k}\right] & (\text{if } k > 0) \\[2ex] 2\pi R^3(t)\left[\dfrac{\sqrt{1-kr_{\rm max}^2}}{(-k)} - \dfrac{\sinh^{-1}\left(\sqrt{-k}\, r_{\rm max}\right)}{(-k)^{3/2}}\right] & (\text{if } k < 0) \quad .] \end{cases}$$

PROBLEM 12: THE SCHWARZSCHILD METRIC

a) The Schwarzschild horizon is the value of r for which the metric becomes singular. Since the metric contains the factor

$$\left(1 - \frac{2GM}{rc^2}\right) ,$$

it becomes singular at

$$\boxed{R_{\rm Sch} = \frac{2GM}{c^2}\ .}$$

b) The separation between A and B is purely in the radial direction, so the proper length of a segment along the path joining them is given by

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 \ ,$$

so

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \ .$$

The proper distance from A to B is obtained by adding the proper lengths of all the segments along the path, so

$$\boxed{s_{AB} = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \ .}$$

EXTENSION: The integration can be carried out explicitly. First use the expression for the Schwarzschild radius to rewrite the expression for s_{AB} as

$$s_{AB} = \int_{r_A}^{r_B} \frac{\sqrt{r}\,dr}{\sqrt{r - R_{\text{Sch}}}} \ .$$

Then introduce the hyperbolic trigonometric substitution

$$r = R_{\text{Sch}} \cosh^2 u \ .$$

One then has

$$\begin{aligned}
\sqrt{r - R_{\text{Sch}}} &= \sqrt{R_{\text{Sch}}}\, \sinh u \\
dr &= 2R_{\text{Sch}} \cosh u \sinh u \, du \ ,
\end{aligned}$$

and the indefinite integral becomes

$$\begin{aligned}
\int \frac{\sqrt{r}\,dr}{\sqrt{r - R_{\text{Sch}}}} &= 2R_{\text{Sch}} \int \cosh^2 u \, du \\
&= R_{\text{Sch}} \int (1 + \cosh 2u) du \\
&= R_{\text{Sch}} \left(u + \frac{1}{2} \sinh 2u\right) \\
&= R_{\text{Sch}} (u + \sinh u \ \cosh u) \\
&= R_{\text{Sch}} \sinh^{-1}\left(\sqrt{\frac{r}{R_{\text{Sch}}} - 1}\right) + \sqrt{r(r - R_{\text{Sch}})} \ .
\end{aligned}$$

Thus,

$$s_{AB} = R_{\text{Sch}} \left[\sinh^{-1}\left(\sqrt{\frac{r_B}{R_{\text{Sch}}} - 1}\right) - \sinh^{-1}\left(\sqrt{\frac{r_A}{R_{\text{Sch}}} - 1}\right) \right] + \sqrt{r_B(r_B - R_{\text{Sch}})} - \sqrt{r_A(r_A - R_{\text{Sch}})} \ .$$

c) A tick of the clock and the following tick are two events that differ only in their time coordinates. Thus, the metric reduces to

$$-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 \ ,$$

so

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}}\, dt \ .$$

The reading on the observer's clock corresponds to the proper time interval $d\tau$, so the corresponding interval of the coordinate t is given by

$$\boxed{\Delta t_A = \frac{\Delta \tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}} \ .}$$

d) Since the Schwarzschild metric does not change with time, each pulse leaving A will take the same length of time to reach B. Thus, the pulses emitted by A will arrive at B with a time coordinate spacing

$$\Delta t_B = \Delta t_A = \frac{\Delta \tau_A}{\sqrt{1 - \frac{2GM}{r_A c^2}}} \ .$$

The clock at B, however, will read the proper time and not the coordinate time. Thus,

$$\Delta \tau_B = \sqrt{1 - \frac{2GM}{r_B c^2}}\, \Delta t_B$$

$$= \boxed{\sqrt{\frac{1 - \frac{2GM}{r_B c^2}}{1 - \frac{2GM}{r_A c^2}}}\, \Delta \tau_A \ .}$$

e) From parts (a) and (b), the proper distance between A and B can be rewritten as

$$s_{AB} = \int_{R_{\rm Sch}}^{r_B} \frac{\sqrt{r}\,dr}{\sqrt{r - R_{\rm Sch}}} \ .$$

The potentially divergent part of the integral comes from the range of integration in the immediate vicinity of $r = R_{\rm Sch}$, say $R_{\rm Sch} < r < R_{\rm Sch} + \epsilon$. For this range the quantity $\sqrt{r}$ in the numerator can be approximated by $\sqrt{R_{\rm Sch}}$, so the contribution has the form

$$\sqrt{R_{\rm Sch}} \int_{R_{\rm Sch}}^{R_{\rm Sch}+\epsilon} \frac{dr}{\sqrt{r - R_{\rm Sch}}} \ .$$

Changing the integration variable to $u \equiv r - R_{\rm Sch}$, the contribution can be easily evaluated:

$$\sqrt{R_{\rm Sch}} \int_{R_{\rm Sch}}^{R_{\rm Sch}+\epsilon} \frac{dr}{\sqrt{r - R_{\rm Sch}}} = \sqrt{R_{\rm Sch}} \int_0^{\epsilon} \frac{du}{\sqrt{u}} = 2\sqrt{R_{\rm Sch}\epsilon} < \infty \ .$$

So, although the integrand is infinite at $r = R_{\rm Sch}$, the integral is still finite.

The proper distance between A and B does not diverge.

Looking at the answer to part (d), however, one can see that when $r_A = R_{\rm Sch}$,

The time interval $\Delta\tau_B$ diverges.

PROBLEM 13: DID YOU DO THE READING?

(a) Fred Hoyle, Herman Bondi, and Thomas Gold.

(b) Mainly photons, e^+-e^- pairs, and neutrino-antineutrino pairs. There were trace amounts of protons and neutrons, which need not even be mentioned.

(c) Deuterium.

(d) (1) They assumed that the universe began in a state of all neutrons, rather the thermal equilibrium mix assumed in modern calculations.

(2) They took into account the conversion of neutrons to protons only by free decay of the neutrons. They ignored the reactions

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e$$

$$n + \nu_e \longleftrightarrow p + e^- \ ,$$

which play a very important role in modern calculations.

(3) They attempted (unsuccessfully) to account for all of nucleosynthesis — they did not realize that the nucleosynthesis of heavier elements takes place primarily in the interior of stars.

(4) They used fewer than the presently accepted number of neutrinos.

PROBLEM 14: GEODESICS

The geodesic equation for a curve $x^i(\lambda)$, where the parameter λ is the arc length along the curve, can be written as

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\frac{dx^\ell}{d\lambda} \; .$$

Here the indices j, k, and ℓ are summed from 1 to the dimension of the space, so there is one equation for each value of i.

(a) The metric is given by

$$ds^2 = g_{ij}\,dx^i dx^j = dr^2 + r^2\,d\theta^2 \; ,$$

so

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2 \; , \qquad g_{r\theta} = g_{\theta r} = 0 \; .$$

First taking $i = r$, the nonvanishing terms in the geodesic equation become

$$\frac{d}{d\lambda}\left\{g_{rr}\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r g_{\theta\theta}\right)\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda} \; ,$$

which can be written explicitly as

$$\frac{d}{d\lambda}\left\{\frac{dr}{d\lambda}\right\} = \frac{1}{2}\left(\partial_r r^2\right)\left(\frac{d\theta}{d\lambda}\right)^2 \; ,$$

or

$$\boxed{\frac{d^2 r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \; .}$$

For $i = \theta$, one has the simplification that g_{ij} is independent of θ for all (i, j). So

$$\boxed{\frac{d}{d\lambda}\left\{r^2\frac{d\theta}{d\lambda}\right\} = 0 \; .}$$

(b) The first step is to parameterize the curve, which means to imagine moving along the curve, and expressing the coordinates as a function of the distance traveled. (I am calling the locus $y = 1$ a curve rather than a line, since the techniques that are used here are usually applied to curves. Since a line is a special case of a curve, there is nothing wrong with treating the line as a curve.) In Cartesian coordinates, the curve $y = 1$ can be parameterized as

$$x(\lambda) = \lambda \ , \qquad y(\lambda) = 1 \ .$$

(The parameterization is not unique, because one can choose $\lambda = 0$ to represent any point along the curve.) Converting to the desired polar coordinates,

$$r(\lambda) = \sqrt{x^2(\lambda) + y^2(\lambda)} = \sqrt{\lambda^2 + 1} \ ,$$

$$\theta(\lambda) = \tan^{-1} \frac{y(\lambda)}{x(\lambda)} = \tan^{-1}(1/\lambda) \ .$$

Calculating the needed derivatives,*

$$\begin{aligned}
\frac{dr}{d\lambda} &= \frac{\lambda}{\sqrt{\lambda^2+1}} \\
\frac{d^2 r}{d\lambda^2} &= \frac{1}{\sqrt{\lambda^2+1}} - \frac{\lambda^2}{(\lambda^2+1)^{3/2}} = \frac{1}{(\lambda^2+1)^{3/2}} = \frac{1}{r^3} \\
\frac{d\theta}{d\lambda} &= -\frac{1}{1+\left(\frac{1}{\lambda}\right)^2} \frac{1}{\lambda^2} = -\frac{1}{r^2} \ .
\end{aligned}$$

Then, substituting into the geodesic equation for $i = r$,

$$\frac{d^2 r}{d\lambda^2} = r \left(\frac{d\theta}{d\lambda}\right)^2 \Longleftrightarrow \frac{1}{r^3} = r \left(-\frac{1}{r^2}\right)^2 \ ,$$

which checks. Substituting into the geodesic equation for $i = \theta$,

$$\frac{d}{d\lambda}\left\{r^2 \frac{d\theta}{d\lambda}\right\} = 0 \Longleftrightarrow \frac{d}{d\lambda}\left\{r^2 \left(-\frac{1}{r^2}\right)\right\} = 0 \ ,$$

which also checks.

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* If you do not remember how to differentiate $\phi = \tan^{-1}(z)$, then you should know how to derive it. Write $z = \tan\phi = \sin\phi / \cos\phi$, so

$$dz = \left(\frac{\cos\phi}{\cos\phi} + \frac{\sin^2\phi}{\cos^2\phi}\right) d\phi = (1 + \tan^2\phi) d\phi \ .$$

Then

$$\frac{d\phi}{dz} = \frac{1}{1+\tan^2\phi} = \frac{1}{1+z^2} \ .$$

PROBLEM 15: METRIC OF A STATIC GRAVITATIONAL FIELD

(a) $ds^2_{\rm ST}$ is the invariant separation between the event at (x^i, t) and the event at $(x^i + dx^i, t + dt)$. Here x^i and t are arbitrary coordinates that are connected to measurements only through the metric. $ds^2_{\rm ST}$ is defined to equal

$$-c^2 dT^2 + d\vec{r}^{\,2} \ ,$$

where $d\vec{r}$ and dT denote the space and time separation as it would be measured by a freely falling observer. Taking the transmitter as the freely falling observer* and taking the emission of two successive pulses as the two events, one has

$$ds^2_{\rm ST} = -c^2(\Delta T_e)^2 \ .$$

To connect with the metric, note that the successive emissions have a separation in the time coordinate of Δt_e, and a separation of space coordinates $dx^i = 0$. So

$$ds^2_{ST} = -\left[c^2 + 2\phi(\vec{x}_e)\right](\Delta t_e)^2 \ ,$$

and then

$$-c^2(\Delta T_e)^2 = -[c^2 + 2\phi(\vec{x}_e)](\Delta t_e)^2 \quad \Longrightarrow$$

$$\boxed{\Delta t_e = \frac{\Delta T_e}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \ .}$$

(b) Since the metric is independent of t, each pulse follows a trajectory identical to the previous pulse, but delayed in t. Thus each pulse requires the same time interval Δt to travel from emitter to receiver, so the pulses arrive with the same t-separation as they have at emission:

$$\boxed{\Delta t_r = \Delta t_e \ .}$$

(c) This is similar to part (a), but in this case we consider the two events corresponding to the reception of two successive pulses. $ds^2_{\rm ST}$ is related to the physical measurement ΔT_r by

$$ds^2_{\rm ST} = -c^2(\Delta T_r)^2 \ .$$

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* The transmitter is not really a freely falling observer, but is presumably held at rest in this coordinate system. Thus gravity is acting on the clock, and could in principle affect its speed. It is standard, however, to assume that such effects are negligible. That is, one assumes that the clock is ideal, meaning that it ticks at the same rate as a freely falling clock that is instantaneously moving with the same velocity.

It is connected to the coordinate separation Δt_r through the metric, where again we use the fact that the two events have zero separation in their space coordinates—i.e., $dx^i = 0$. So

$$ds^2_{ST} = -[c^2 + 2\phi(\vec{x}_r)](\Delta t_r)^2 \ .$$

Then

$$-c^2(\Delta T_r)^2 = -[c^2 + 2\phi(\vec{x}_r)](\Delta t_e)^2 \quad \Longrightarrow$$

$$\Delta T_r = \sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}\,\Delta t_e \ .$$

We can cast this into a more useful form for the problem by using the solution for Δt_e found in part (c). This gives

$$\Delta T_r = \left[\frac{\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}}\right] \Delta T_e \ .$$

Substitute this result for ΔT_r directly into the definition for Z to obtain the exact expression for the redshift,

$$\boxed{1 + Z = \frac{\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \ .}$$

Remember that $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for small x. For weak fields, that is, for small values of $\phi(\vec{x})$, we can expand our result to lowest order in $\phi(\vec{x})$. Expanding the numerator we have

$$\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}} \approx 1 + \frac{\phi(\vec{x}_r)}{c^2} \ .$$

Similarly we find for

$$\frac{1}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \approx 1 - \frac{\phi(\vec{x}_e)}{c^2} \ .$$

Putting these approximations into our exact expression for $1 + Z$ we obtain

$$1 + Z \approx \left(1 + \frac{\phi(\vec{x}_r)}{c^2}\right)\left(1 - \frac{\phi(\vec{x}_e)}{c^2}\right) \approx 1 + \frac{\phi(\vec{x}_r)}{c^2} - \frac{\phi(\vec{x}_e)}{c^2} \ ,$$

where we dropped terms in $\phi(\vec{x}_e)\phi(\vec{x}_r)$. Finally,

$$\boxed{Z \approx \frac{\phi(\vec{x}_r) - \phi(\vec{x}_e)}{c^2} \ .}$$

(d) For the metric at hand we know $g_{00} = -[c^2 + 2\phi(\vec{x})]$, $g_{k0} = 0$ and $g_{ik} = g_{ki} = \delta_{ik}$. It is useful to notice that only g_{00} depends on $\vec{x}$ and thus $\partial_i g_{km} = 0$. The geodesic equation corresponding to $\mu = i$, where i runs from 1 to 3, is

$$\frac{d}{d\tau}\left(g_{ik}\frac{dx^k}{d\tau}\right) = \frac{1}{2}(\partial_i g_{\lambda\sigma})\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \quad \Longrightarrow$$

$$\delta_{ik}\frac{d^2x^k}{d\tau^2} = \frac{1}{2}(\partial_i g_{00})\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} .$$

Using $x^0 \equiv t$, $\delta_{ik}y^k = y^i$ and

$$\partial_i g_{00} = -\partial_i(c^2 + 2\phi(\vec{x})) = -\frac{2}{c^2}\partial_i\phi(\vec{x})$$

we find

$$\boxed{\frac{d^2x^i}{d^2\tau} = -\partial_i\phi(\vec{x})\left(\frac{dt}{d\tau}\right)^2 .}$$

[Pedagogical Note: You might prefer to use the notation $x^0 \equiv ct$, which is also a very common choice. In that case the metric is rewritten as

$$ds^2_{\mathrm{ST}} = -\left[1 + \frac{2\phi(\vec{x})}{c^2}\right]\left(dx^0\right)^2 + \sum_{i=1}^{3}\left(dx^i\right)^2 ,$$

so one takes $g_{00} = -\left[1 + (2\phi(\vec{x})/c^2)\right]$. In the end one finds the same answer as the boxed equation above.

Note also that when ϕ is small and velocities are nonrelativistic, then $dt/d\tau \approx 1$. Thus one has $d^2x^i/d^2t \approx -\partial_i\phi(\vec{x})$, so $\phi(\vec{x})$ can be identified with the Newtonian gravitational potential. In the context of general relativity, Newtonian gravity is a distortion of the metric in the time-direction.]