

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Physics Department

Physics 8.286: The Early Universe  
Prof. Alan Guth

April 23, 2004

**PROBLEM SET 5**

Corrected Version: May 1, 2004

**DUE DATE:** Tuesday, May 4, 2004

**READING ASSIGNMENT:** Barbara Ryden, *Introduction to Cosmology*, Chapters 8 and 9, and the 2-page Epilogue. You may find Chapters 6 and 7 useful for helping you to understand the lecture notes, but you are not otherwise responsible for these chapters.

**PROBLEM 1: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT** (*10 points*)

In Lecture Notes 8, we derived the relation between the power output  $P$  of a source and the energy flux  $J$ , for the case of a closed universe:

$$J = \frac{PH_0^2|\Omega_{k,0}|}{4\pi(1+z_S)^2c^2\sin^2\psi_D} ,$$

where

$$\psi_D = \sqrt{|\Omega_{k,0}|} \int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} .$$

Here  $z_S$  denotes the observed redshift,  $H_0$  denotes the present value of the Hubble constant,  $\Omega_{m,0}$ ,  $\Omega_{\text{rad},0}$ , and  $\Omega_{\text{vac},0}$  denote the present contributions to  $\Omega$  from nonrelativistic matter, radiation, and vacuum energy, respectively, and  $\Omega_{k,0} \equiv 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0}$ .

- (a) Derive the corresponding formula for the case of an open universe. You can of course follow the same logic as the derivation in the lecture notes, but the solution you write should be complete and self-contained. (I.e., you should **NOT** say “the derivation is the same as the lecture notes except for ... .”)
- (b) Derive the corresponding formula for the case of a flat universe. Here there is of course no need to repeat anything that you have already done in part (a).

**PROBLEM 2: MASS DENSITY OF VACUUM FLUCTUATIONS** (10 points)

The energy density of vacuum fluctuations will be discussed qualitatively in lecture. In this problem we will calculate in detail the energy density associated with quantum fluctuations of the electromagnetic field. To keep the problem finite, we will not consider all of space at once, but instead we will consider the electromagnetic field inside a cube of side  $L$ , defined by coordinates

$$0 \leq x \leq L ,$$

$$0 \leq y \leq L ,$$

$$0 \leq z \leq L .$$

Our goal, however, will be to compute the energy density in the limit as the size of the box is taken to infinity.

- (a) The electromagnetic waves inside the box can be decomposed into a Fourier sum of sinusoidal normal modes. Suppose we consider only modes that extend up to a maximum wave number  $k_{\max}$ , or equivalently modes that extend down to a minimum wavelength  $\lambda_{\min}$ , where

$$k_{\max} = \frac{2\pi}{\lambda_{\min}} .$$

How many such modes are there? I do not expect an exact answer, but your approximations should become arbitrarily accurate when  $\lambda_{\min} \ll L$ . (These mode counting techniques are probably familiar to many of you, but in case they are not I have attached an extended hint after part (c).)

- (b) When the electromagnetic field is described quantum mechanically, each normal mode behaves exactly as a harmonic oscillator: if the angular frequency of the mode is  $\omega$ , then the quantized energy levels have energies given by

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega ,$$

where  $\hbar$  is Planck's original constant divided by  $2\pi$ , and  $n$  is an integer. The integer  $n$  is called the "occupation number," and is interpreted as the number of photons in the specified mode. The minimum energy is not zero, but instead is  $\frac{1}{2}\hbar\omega$ , which is the energy of the quantum fluctuations of the electromagnetic field. Assuming that the mode sum is cut off at  $\lambda_{\min}$  equal to the Planck length (as defined in the Lecture Notes), what is the total mass density of these quantum fluctuations?

- (c) How does the mass density of the quantum fluctuations of the electromagnetic field compare with the critical density of our universe?

*Extended Hint:*

The electromagnetic fields inside a closed box can be expanded as the sum of modes, each of which has a sinusoidally varying time dependence, but the precise form of these modes depends on the nature of the boundary conditions on the walls of the box. Physically reasonable boundary conditions, such as total reflection, are in fact difficult to use in calculations. However, it is known that in the limit of an infinite-sized box, the nature of the boundary conditions will not make any difference. We are therefore free to choose the simplest boundary conditions that we can imagine, and for this purpose we will choose periodic boundary conditions. That is, we will assume that the fields and their normal derivatives on each wall are fixed to precisely match the fields and their normal derivatives on the opposite wall.

To begin, we consider a wave in one dimension, moving at the speed of light. Such waves are most easily described in terms of complex exponentials. If  $A(x, t)$  represents the amplitude of the wave, then a sinusoidal wave moving in the positive  $x$ -direction can be written as

$$A(x, t) = \text{Re} \left[ B e^{ik(x-ct)} \right] ,$$

where  $B$  is a complex constant and  $k$  is a real constant. Defining  $\omega = c|k|$ , waves in either direction can be written as

$$A(x, t) = \text{Re} \left[ B e^{i(kx-\omega t)} \right] ,$$

where the sign of  $k$  determines the direction. To be periodic with period  $L$ , the parameter  $k$  must satisfy

$$kL = 2\pi n ,$$

where  $n$  is an integer. So the spacing between modes is  $\Delta k = 2\pi/L$ . The density of modes  $dN/dk$  (i.e., the number of modes per interval of  $k$ ) is then one divided by the spacing, or  $1/\Delta k$ , so

$$\frac{dN}{dk} = \frac{L}{2\pi} \quad (\text{one dimension}) .$$

In three dimensions, a sinusoidal wave can be written as

$$A(\vec{x}, t) = \text{Re} \left[ B e^{i(\vec{k}\cdot\vec{x}-\omega t)} \right] ,$$

where  $\omega = c|\vec{k}|$ , and

$$k_x L = 2\pi n_x , \quad k_y L = 2\pi n_y , \quad k_z L = 2\pi n_z ,$$

where  $n_x$ ,  $n_y$ , and  $n_z$  are integers. Thus, in three-dimensional  $\vec{k}$ -space the allowed values of  $\vec{k}$  lie on a cubical lattice, with spacing  $2\pi/L$ . In counting the modes, one should also remember that for photons there is an extra factor of 2 associated with the fact that electromagnetic waves have two possible polarizations for each allowed value of  $\vec{k}$ .

**PROBLEM 3: THE HORIZON PROBLEM** (8 points)

The success of the big bang predictions for the abundances of the light elements suggests that the universe was already in thermal equilibrium at one second after the big bang. At this time, the region which later evolves to become the observed universe was many horizon distances across. Try to estimate how many. You may assume that the universe is flat.

**PROBLEM 4: THE FLATNESS PROBLEM** (7 points)

Suppose that today  $\Omega_0 = 0.1$ . It then follows that at  $10^{-37}$  second after the big bang (about the time of the grand unified theory phase transition),  $\Omega$  must have been extraordinarily close to one. Writing  $\Omega = 1 - \delta$ , estimate the value of  $\delta$  at  $t = 10^{-37}$  sec (using the standard cosmological model).

**PROBLEM 5: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT — NUMERICAL INTEGRATION** (EXTRA CREDIT, 8 pts)

Calculate numerically the result from Problem 1 for the case of a flat universe in which the critical density is comprised of nonrelativistic matter and vacuum energy (cosmological constant). Specifically, calculate numerical values for  $J/(PH_0^2)$  as a function of  $z$ , for  $\Omega_{m,0} = 0.3$  and  $\Omega_{\text{vac},0} = 0.7$ . Compute a table of values for  $z = 0.1, 0.2, 0.3, \dots, 1.5$ . Feel free to attach a computer printout of these results, but be sure that it is labeled well enough to be readable to someone other than yourself. (If you are not confident in the expression that you obtained in Problem 1 for the flat universe case, you can for equal credit do this problem for an open universe, with  $\Omega_{m,0} = 0.3$  and  $\Omega_{\text{vac},0} = 0.6$ .) For pedagogical purposes you are asked to compute these numbers to 5 significant figures, although one does not need nearly so much accuracy to compare with data. For the fun of it, the solutions will be written to 15 significant figures. Note that the speed of light is now *defined* to be 299,792,458 m/s.

**PROBLEM 6: PLOTTING THE SUPERNOVA DATA** (EXTRA CREDIT, 7 pts)

The original data on the Hubble diagram based on Type Ia supernovae are found in two papers. One paper is authored by the High Z Supernova Search Team,\* and the other is by the Supernova Cosmology Project.† You are asked

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\* <http://arXiv.org/abs/astro-ph/9805201>, later published as Riess *et al.*, *Astronomical Journal* **116**, 1009 (1998).

† <http://arXiv.org/abs/astro-ph/9812133>, later published as Perlmutter *et al.*, *Astrophysical Journal* **517**:565–586 (1999).

to plot this data for one of these two papers, and to include on the graph the theoretical predictions for several cosmological models. If you would prefer to use the most recent data, which includes many more data points, you can find it in Riess *et al.*, <http://arXiv.org/abs/astro-ph/0402512>. (By the way, the lead author Adam Riess was an MIT undergraduate physics major about 10 years ago.)

The plot will be similar to the plots contained in these papers, and to the plot on p. 121 of Ryden's book, showing a graph of (corrected) magnitude  $m$  vs. redshift  $z$ . Your graph should include the error bars. The magnitude is related to the flux  $J$  of the observed radiation by  $m = -\frac{5}{2} \log_{10}(J) + \text{const}$ . The value of the constant in this expression will not be needed. The word "corrected" refers both to corrections related to the spectral sensitivity of the detectors and to the brightness of the supernova explosions themselves. That is, the supernova at various distances are observed with different redshifts, and hence one must apply corrections if the detectors used to measure the radiation do not have the same sensitivity at all wavelengths. In addition, to improve the uniformity of the supernova as standard candles, the astronomers apply a correction based on the duration of the light output. Note that our ignorance of the absolute brightness of the supernova, of the precise value of the Hubble constant, and of the constant that appears in the definition of magnitude all combine to give an unknown but constant contribution to the predicted magnitudes. The consequence is that you will be able to move your predicted curves up or down (i.e., translate them by a fixed distance along the  $m$  axis). You should choose the vertical positioning of your curve to optimize your fit, either by eyeball or by some more systematic method.

For your convenience, the magnitudes and redshifts for the Supernova Cosmology Project paper, from Tables 1 and 2, are summarized in a text file on the server. The data from the most recent Riess *et al.* paper, mentioned above, will also appear on the server shortly.

For the cosmological models to plot, you should include:

- (i) A matter-dominated universe with  $\Omega_m = 1$ .
- (ii) An open universe, with  $\Omega_{m,0} = 0.3$ .
- (iii) A universe with  $\Omega_{m,0} = 0.3$  and a cosmological constant, with  $\Omega_{\text{vac},0} = 0.7$ . (If you prefer to avoid the flat case, you can use  $\Omega_{\text{vac},0} = 0.6$ .)

You may include any other models if they interest you. You can draw the plot with either a linear or a logarithmic scale in  $z$ . I would recommend extending your theoretical plot to  $z = 3$ , if you do it logarithmically, or  $z = 2$  if you do it linearly, even though the data does not go out that far. That way you can see what possible knowledge can be gained by data at higher redshift.

**Total points for Problem Set 5: 35, plus up to 15 points extra credit.**