MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 9, 2004
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PROBLEM SET 3

DUE DATE: Thursday, April 1, 2004

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology*, Chapters 4 and 5.

PROBLEM 1: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY

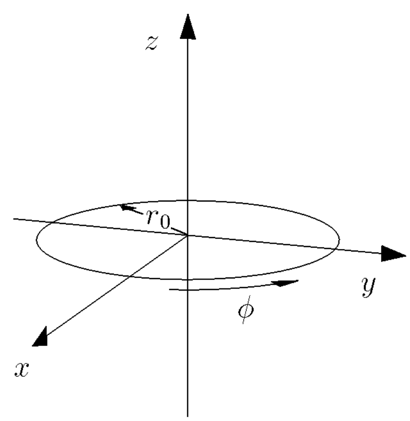
(5 points)

Consider a universe described by the Robertson-Walker metric, Eq. (6.21), which describes an open, closed, or flat universe, depending on the value of k:

$$ds^2 = R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\} \quad .$$

This problem will involve only the geometry of space at some fixed time, so we can ignore the dependence of R on t, and think of it as a constant. Consider a circle described by the equations

$$z = 0$$
$$x^2 + y^2 = r_0^2 \quad ,$$



or equivalently by the angular coordinates

$$r = r_0$$
$$\theta = \pi/2 \quad .$$

(a) Find the circumference S of this circle. Hint: break the circle into infinitesimal segments of angular size $d\phi$, calculate the arc length of such a segment, and integrate.

(b) Find the radius ρ of this circle. Note that ρ is the length of a line which runs from the origin to the circle ($r = r_0$), along a trajectory of $\theta = \pi/2$ and $\phi =$ constant. Hint: Break the line into infinitesimal segments of coordinate length dr, calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take $k = \pm 1$. (If you don't remember why we can take $k = \pm 1$, see the section called "Units" in Lecture Notes 4,). You will want the following integrals:

$$\int \frac{dr}{\sqrt{1 - r^2}} = \sin^{-1} r$$

and

$$\int \frac{dr}{\sqrt{1 + r^2}} = \sinh^{-1} r \quad .$$

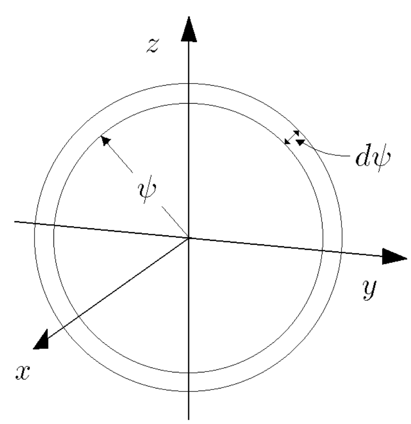
(c) Express the circumference S in terms of the radius ρ. This result is independent of the coordinate system which was used for the calculation, since S and ρ are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether S is larger or smaller than the value it would have for a Euclidean circle of radius ρ.

PROBLEM 2: VOLUME OF A CLOSED UNIVERSE *(5 points)*

Calculate the total volume of a closed universe. It will be easiest to use the metric in the form of Eq. (6.12):

$$ds^2 = a^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \quad .$$

We will continue to use the convention that $k = \pm 1$, so in this case $k = 1$ and $a = R$. Break the volume up into spherical shells of infinitesimal thickness, extending from ψ to $\psi + d\psi$:



By comparing Eqs. (6.8) and (6.12), one can see that as long as ψ is held fixed, the metric for varying θ and ϕ is the same as that for a spherical surface of radius $a \sin\psi$, and thus the area of the spherical surface is $4\pi a^2 \sin^2\psi$. Multiply this area by the thickness of the shell (which you can read off from the metric), and then integrate over the full range of ψ.

PROBLEM 3: CIRCULAR ORBITS IN A SCHWARZSCHILD METRIC *(10 points)*

The Schwarzschild metric, which describes the external gravitational field of any spherically symmetric distribution of mass, is given by

$$c^2 d\tau^2 = -ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \ ,$$

where M is the total mass of the object, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $\phi = 2\pi$ is identified with $\phi = 0$. We will be concerned only with motion outside the Schwarzschild horizon $R_{\rm Sch} = 2GM/c^2$, so we can take $r > R_{\rm Sch}$. (This restriction allows us to avoid the complications of understanding the effects of the singularity at $r = R_{\rm Sch}$.) In this problem we will use the geodesic equation to calculate the behavior of circular orbits in this metric. We will assume a perfectly circular orbit in the x-y plane: the radial coordinate r is fixed, $\theta = 90°$, and $\phi = \omega t$, for some angular velocity ω.

(a) Use the metric to find the proper time interval $d\tau$ for a segment of the path corresponding to a coordinate time interval dt. Note that $d\tau$ represents the time that would actually be measured by a clock moving with the orbiting body. Your result should show that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} \ .$$

Note that for $M = 0$ this reduces to the special relativistic relation $d\tau/dt = \sqrt{1 - v^2/c^2}$, but the extra term proportional to M describes an effect that is new with general relativity— the gravitational field causes clocks to slow down, just as motion does.

(b) Show that the geodesic equation of motion (Eq. (6.38)) for one of the coordinates takes the form

$$0 = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 \ .$$

(c) Show that the above equation implies

$$r\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^2}\left(\frac{dt}{d\tau}\right)^2 \ ,$$

which in turn implies that

$$r\omega^2 = \frac{GM}{r^2} \ .$$

Thus, the relation between r and ω is exactly the same as in Newtonian mechanics. *[Note, however, that this does not really mean that general relativity has no effect. First, ω has been defined by $d\phi/dt$, where t is a time coordinate which is not the same as the proper time τ that would be measured by a clock on the orbiting body. Second, r does not really have the same meaning as in the Newtonian calculation, since it is not the measured distance from the center of motion. Measured distances, you will recall, are calculated by integrating the metric, as for example in Problem 1. Since the angular ($d\theta^2$ and $d\phi^2$) terms in the Schwarzschild metric are unaffected by the mass, however, it can be seen that the circumference of the circle is equal to $2\pi r$, as in the Newtonian calculation.]*

PROBLEM 4: GEODESICS IN A FLAT UNIVERSE *(10 points)*

According to general relativity, in the absence of any non-gravitational forces a particle will travel along a spacetime geodesic. In this sense, gravity is reduced to a distortion in spacetime.

Consider the case of a flat (*i.e.*, $k = 0$) Robertson–Walker metric, which has the simple form

$$ds^2_{\rm ST} = -c^2 dt^2 + R^2(t)\left[dx^2 + dy^2 + dz^2\right] \ .$$

Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the x-axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (*i.e.*, $dx/d\tau$) falls off as $1/R^2(t)$.

(b) Use the expression for the spacetime metric to relate dx/dt to $dx/d\tau$.

(c) The physical velocity of the particle relative to the galaxies that it is passing is given by

$$v = R(t)\frac{dx}{dt} \ .$$

Show that the momentum of the particle, defined relativistically by

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

falls off as $1/R(t)$. (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength $\lambda = h/|\vec{p}\,|$, then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

PROBLEM 5: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE * *(15 points)*

The spacetime metric for a homogeneous, isotropic, open universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}\ ,$$

where I have taken $k = -1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sinh\psi\ .$$

Then

$$\frac{dr}{\sqrt{1+r^2}} = d\psi\ ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}\ .$$

The form of $R(t)$ depends on the nature of the matter in the universe, but for this problem you should consider $R(t)$ to be an arbitrary function. You should simplify your answers as far as it is possible without knowing the function $R(t)$.

a) Suppose that the Earth is at the origin of the coordinate system ($\psi = 0$), and that at the present time, t_0, we receive a light pulse from a distant galaxy G, located at $\psi = \psi_G$. Write down an equation which determines the time t_G at which the light pulse left the galaxy. (You may assume that the light pulse travels on a "null" trajectory, which means that $d\tau = 0$ for any segment of it. Since you don't know $R(t)$ you cannot solve this equation, so please do not try.)

b) What is the redshift z_G of the light from galaxy G? (Your answer may depend on t_G, as well as ψ_G or any property of the function $R(t)$.)

c) To estimate the number of galaxies that one expects to see in a given range of redshifts, it is necessary to know the volume of the region of space that corresponds to this range. Write an expression for the present value of the volume that corresponds to redshifts smaller than that of galaxy G. (You may

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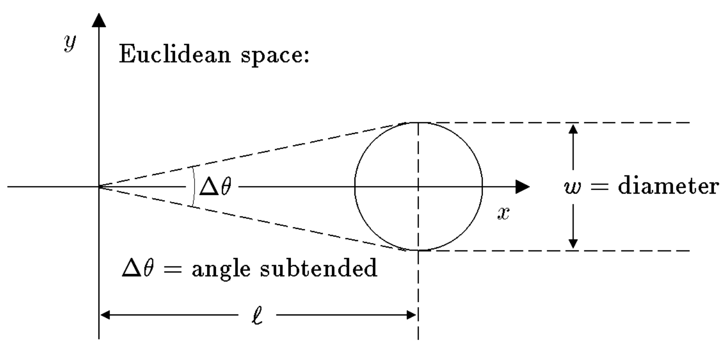
* Problem 5 of this problem set was taken from Quiz 2 of 1996, where it counted 50 points out of 100.

leave your answer in the form of a definite integral, which may be expressed in terms of ψ_G, t_G, z_G, or the function $R(t)$.)

d) There are a number of different ways of defining distances in cosmology, and generally they are not equal to each other. One choice is called ***proper distance***, which corresponds to the distance that one could in principle measure with rulers. The proper distance is defined as the total length of a network of rulers that are laid end to end from here to the distant galaxy. The rulers have different velocities, because each is at rest with respect to the matter in its own vicinity. They are arranged so that, at the present instant of time, each ruler just touches its neighbors on either side. Write down an expression for the proper distance ℓ_{prop} of galaxy G.

e) Another common definition of distance is ***angular size distance,*** determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter w at a distance ℓ will subtend an angle $\Delta\theta = w/\ell$:



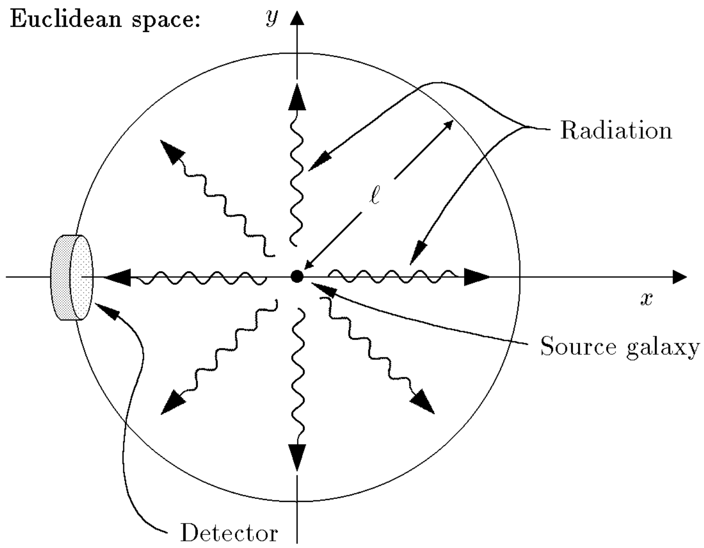
Motivated by this relation, cosmologists define the angular size distance ℓ_{ang} of an object by

$$\ell_{\text{ang}} \equiv \frac{w}{\Delta\theta} \ .$$

What is the angular size distance ℓ_{ang} of galaxy G?

f) A third common definition of distance is called ***luminosity distance***, which is determined by measuring the apparent brightness of an object for which the actual total power output is known. In a static, Euclidean space, the energy flux

J received from a source of power P at a distance ℓ is given by $J = P/(4\pi\ell^2)$:



Cosmologists therefore define the luminosity distance by

$$\ell_{\rm lum} \equiv \sqrt{\frac{P}{4\pi J}} \quad .$$

Find the luminosity distance $\ell_{\rm lum}$ of galaxy G. (Hint: the Robertson-Walker coordinates can be shifted so that the galaxy G is at the origin.)

PROBLEM 6: THE KLEIN DESCRIPTION OF THE G-B-L GEOMETRY

(This problem is not required, but can be done for 5 points extra credit.)

I stated in Lecture Notes 6 that the space invented by Klein, described by the distance relation

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2}\,\sqrt{1 - x_2^2 - y_2^2}} \quad ,$$

where

$$x^2 + y^2 < 1 \quad ,$$

is a two-dimensional space of constant negative curvature. In other words, this is just a two-dimensional Robertson–Walker metric, as would be described by a two-dimensional version of Eq. (6.21), with $k = -1$:

$$ds^2 = a^2 \left\{ \frac{dr^2}{1 + r^2} + r^2 d\theta^2 \right\} \quad .$$

The problem is to prove the equivalence.

(a) As a first step, show that if x and y are replaced by the polar coordinates defined by

$$x = u\cos\theta$$

$$y = u\sin\theta \quad ,$$

then the distance equation can be rewritten as

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - u_1 u_2 \cos(\theta_1 - \theta_2)}{\sqrt{1-u_1^2}\,\sqrt{1-u_2^2}} \quad .$$

(b) The next step is to derive the metric from the distance function above. Let

$$u_1 = u \qquad\qquad \theta_1 = \theta \quad ,$$

$$u_2 = u + du \qquad\qquad \theta_2 = \theta + d\theta \quad ,$$

and

$$d(1,2) = ds \quad .$$

Insert these expressions into the distance function, expand everything to second order in the infinitesimal quantities, and show that

$$ds^2 = a^2\left\{\frac{du^2}{(1-u^2)^2} + \frac{u^2 d\theta^2}{1-u^2}\right\} \quad .$$

(This part is rather messy, but you should be able to do it.)

(c) Now find the relationship between r and u and show that the two metric functions are identical. Hint: The coefficients of $d\theta^2$ must be the same in the two cases. Can you now see why Klein had to impose the condition $x^2 + y^2 < 1$?

Total points for Problem Set 3: 45, plus up to 5 points extra credit.