ON THE SPECTRUM OF THE METAPLECTIC GROUP

WITH APPLICATIONS TO DEDEKIND SUMS

by

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ABSTRACT

The subject of this thesis is the theory of non-holomorphic modular forms of non-integral weight, and its applications to arithmetical functions involving Dedekind sums and Kloosterman sums.

As was discovered by Andre Weil, automorphic forms of non-integral weight correspond to invariant funtions on Metaplectic groups. We thus give an explicit description of Meptaplectic groups corresponding to rational weight automorphic forms and explain this correspondence.

We also describe the spectral decomposition of automorphic forms, and use this to find the spectral decomposition of a class of automorphic forms: the Poincare series.

The applications center around the fact that for congruence subgroups of SL(2,Z), the Dedekind η function can be used to define multiplier systems of arbitrary weight, and these involve the Dedekind sum. We can use the general theory to bound sums of Kloosterman sums which involve the above multiplier systems, and therefore Dedekind sums. From this follows results about the distribution of values of the Dedekind sum.

PREFACE

The subject of this thesis is the theory of non-holomorphic modular forms of non-integral weight, and its applications to Number Theory.

As was discovered by Andre Weil, modular forms of non-integral weight correspond to invariant functions on Metaplectic groups. This explains the title of the thesis.

Most of the work on modular forms of non-integral weight has been centered on the case of half-integral weight. In this work, however, arbitrary rational weight is considered, yealding some interesting results. Thus the results of §4.2 seem to be new, including the main theorem: Theorem 4.2.4.

Chapter II is a description of the Metaplectic group, and the exposition has been modelled on the one given by Gelbart in [Gel]. However, it would seem that our decomposition of a more general Metaplectic group has not been given in the form of Theorem 2.2.1 .

The material of Chapter III has been well known since Selberg's paper [Sel], however the explicit spectral decomposition of the Poincare series has never been published. Also, the computation of the classical integral of Lemma 3.1.1 is new.

Acknowledgments

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TABLE OF CONTENTS

NOTATIONS

R	The set of real numbers
C	The set of complex numbers
Z	The set of integers
$\mathbb{N}$	The set of positive integers
Q	The set of rational numbers
$\exp(z)$	Denotes e^z, $z \in C$.
$e(z)$	Denotes $e^{2\pi i z}$, $z \in C$.
$[x]$	The integral part of x, $x \in R$. $[x] = \operatorname{Max}_{n \in Z} \{ n \leq x \}$
δ_x^y	The Kronecker symbol: $\delta_x^y = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases}$

Let f,g be complex valued functions defined for all sufficiently large $x \in R$, or $n \in \mathbb{N}$. We write:

$f = O(g)$, $f \ll g$	If g is a positive function, $K>0$ a constant with: $\lvert f(x) \rvert < K\, g(x)$ for all sufficiently large x, or $\lvert f(n) \rvert < K\, g(n)$ for all sufficiently large n.
$f = o(g)$	If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, or $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$
$f \sim g$	If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$, or $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$
$\log z$	$\log(\lvert z \rvert) + i \cdot \arg(z)$: $-\pi < \arg(z) \leq \pi$, $z \neq 0$. $z = \lvert z \rvert \cdot \exp(i \cdot \arg(z))$. The <u>Principal branch of log</u>.

INTRODUCTION

The subject of this thesis is the theory of modular forms of non-integral weight, and its applications to arithmetical functions involving Dedekind sums and Kloosterman sums.

We recall that the Dedekind sum is defined for integers c,d $s(d,c)=\sum_{j=1}^{c}((j/c))\cdot((jd/c))$, where $((x))=x-[x]-\frac{1}{2}$ for x a real number, and $[x]$ denotes the integral part of x.

The Dedekind sum plays an important role in Number Theory and has been extensively studied, the work of Rademacher being outstanding. Very little, however, is known about the distribution of values of the Dedekind sum. In [Rad-Gro] p.28, Rademacher and Grosswald ask if the values of $s(d,c)$, as d and c vary over the integers, are dense on the real line.

In this thesis, the following related result is proved:

Theorem A (§4.2, p.59)

Let $\{x\}=x-[x]$ denote the fractional part of x, for x real.

Let h and k be integers, then the sequence of fractional parts:

$$\left\{ \left\{ \frac{hs(d,c)}{k} \right\} \right\}_{\substack{c>0 \\ c\equiv 0 \pmod{12k} \\ 0<d<12kc \\ (d,c)=1,\, d\equiv 1 \pmod{12k}}}$$

is equidistributed on [0,1).

We now recall Weyl's criterion for equidistribution (§4.2,p.58) which states that a sequence $\{\xi_j\}_{j=1}^{\infty}$ of complex numbers of absolute value 1 is equidistributed iff for every positive integer m, we have:

$$\sum_{j<x} \xi_j^m = o(x) \qquad \text{as} \qquad x \to \infty.$$

Since the study of the fractional part of x on [0,1) is equivalent to considering $e(x)=e^{2\pi ix}$ on the unit circle, we see by Weyl's criterion that Theorem A amounts to giving non-trivial estimates of the sum:

$$(1) \qquad \sum_{\substack{0<c<x \\ c\equiv 0 \pmod{12k}}} \quad \sum_{\substack{0<d<12kc \\ (d,c)=1 \\ d\equiv 1 \pmod{12k}}} e\left(\frac{mhs(d,c)}{k} \right)$$

However it turns out that this sum appears naturally in the theory of modular forms, as a sum of Kloosterman sums.

This occurs as $\sum_{\substack{0<d<12kc \\ (d,c)=1 \\ d\equiv 1 \pmod{12k}}} e\left(\frac{mhs(d,c)}{k}\right)$, for $c\equiv 0 \pmod{12k}$

can be expressed as a generalized Kloosterman sum defined below.

The classical Kloosterman sum is given by:

$$S(m,n,c)=\sum_{\substack{0<d<c \\ (d,c)=1 \\ a\cdot d\equiv 1 \pmod{c}}} e\left(\frac{am+dn}{c}\right), \text{ for integers } m,n,c.$$

There has been much work on the Magnitude of $S(m,n,c)$ culminating in Weil's estimate [Weil]:

$S(m,n,c) \ll c^{\frac{1}{2}+\varepsilon}$, for m,n fixed and any $\varepsilon>0$; and this is the best possible estimate.

In another direction Kuznetsov [Kuzn] has shown:

$$\sum_{0<c<x} \frac{S(m,n,c)}{c} \ll x^{1/6+\varepsilon}, \text{ for } m,n \text{ fixed and any } \varepsilon>0.$$

Though Selberg has conjectured that this also holds with $x^{1/6+\varepsilon}$ replaced by x^{ε}.

In [Sel] Selberg generalized the Kloosterman sum, however, in order to explain this, we will first have to recall the basic notions of the theory of modular forms.

It is well known that $SL(2,R)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \varepsilon R, ad-bc=1\}$ acts on the <u>complex upper half plane</u> $H=\{x+iy | x,y \varepsilon R, y>0\}$ by: $z \to \frac{az+b}{cz+d}$, for $z \varepsilon H$. It is known that $\frac{dxdy}{y^2}$ is a <u>volume element invariant under this action</u>.

We recall that a subgroup G of $SL(2,Z)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,R) | a,b,c,d \varepsilon Z\}$ is a <u>congruence subgroup</u> if $\Gamma(N) \subseteq G$ for some $N \varepsilon \mathbb{N}$, where $\Gamma(N)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,Z) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } N)\}$

We say that z_1, z_2 H are <u>G-equivalent</u> if there is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon G$ with $\frac{az_1+b}{cz_2+d} = z_2$. Now a <u>Fundamental domain</u> F for G, is an open set $F \subseteq H$ satisfying:

a) If $z_1, z_2 \varepsilon F$, then z_1 is not G-equivalent to z_2.

b) Every $z \varepsilon H$ is G-equivalent to a $z' \varepsilon \overline{F}$, the topological closure of F.

It is known that every congruence subgroup G has a fundamental domain F which has <u>finite invariant volume</u>. That is $\iint_F \frac{dxdy}{y^2} < \infty$

For example: if G=SL(2,Z),F looks like:

We let G be a congruence subgroup of $SL(2,Z)$, r a real number, $\chi: G \to T=\{z\varepsilon\mathbf{C} \mid |z|^2=1\}$, and define $z^r=\exp(r \log(z))$ where we have chosen the principal branch of log .

We now say that $f: H \to \mathbf{C}$ is an <u>Automorphic form of weight r and multiplier χ for G</u> if f satisfies:

a) $$f\left(\frac{az+b}{cz+d}\right)=\chi(g)\left(\frac{cz+d}{|cz+d|}\right)^r f(z), \text{ for } z\varepsilon H,\ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varepsilon G.$$

b) $$\iint_F |f(z)|^2 \frac{dxdy}{y^2} < \infty$$

We denote by $M(r,\chi,G)$ the space of such functions.

It turns out that there is a correspondence between automorphic forms of weight r and multiplier χ and invariant functions on the <u>Metaplectic group</u>, which is a <u>Covering group</u> of $SL(2,R)$. This is the reason for the title of this thesis.

We now say that $u\varepsilon M(r,\chi,G)$ is a <u>Modular form</u> if u is a C^∞ function of x and y, and there is a $\lambda\varepsilon R$ such that:

(*) $$\Delta_r u + \lambda u = 0, \quad \text{where} \quad \Delta_r = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iry\frac{\partial}{\partial x}$$ is the invariant Laplacian for $M(r,\chi,G)$.

It is known that (*) has an orthonormal set of solutions

$\{u_j\}_{j=-\nu}^{\kappa}, u_j \varepsilon M(r,\chi,G)$, with eigenvalues $\{\lambda_{-\nu}<\ldots<0=\lambda_0<\lambda_1\ldots\lambda_j\}_{j=-\nu}^{\kappa}$

$\lambda_j \to \infty$ as $j \to \infty$, if $\kappa=\infty$.

The finite set $\Lambda=\{\lambda_j \mid 0<\lambda_j<\frac{1}{4}\}$ is called the set of

Exceptional eigenvalues.

The set of eigenvalues has received much attention and is completely unknown except for a few cases corresponding to zeta functions of Quadratic fields.

There has been much work on the problem of whether $\lambda_1 > \frac{1}{4}$,

that is whether the set of exceptional eigenvalues is empty. This result would have many applications, for example when r=0, there is [Iwan-Des]. There is a survey article on this problem in [Vigneras].

For our purposes, however, the trivial bound $\lambda>0$ suffices.

We can now define the Generalized Kloosterman sum:

$$S(m,n,c,\chi,G)=\sum_{\substack{0<d<qc \\ g=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \varepsilon G}} e\left(\frac{a(m-\alpha)+d(n-\alpha)}{qc} \right)\overline{\chi(g)}$$

Where $q>0$ is such that $G_\infty=\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \varepsilon G\}=\{\begin{pmatrix}1 & nq\\ 0 & 1\end{pmatrix} n \varepsilon Z\}$, $e(-\alpha)=\chi(\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix})$

$0 \leqslant \alpha < 1$.

This generalizes the classical Kloosterman sum, for it is seen that: $S(m,n,c)=S(m,n,c,1,SL(2,Z))$.

Kuznetsov's result (1) generalizes to:

<u>Theorem B. (Kuznetsov-Goldfeld-Sarnak-Proskurin)</u>

$$\sum_{0<c<x} \frac{S(m,n,c,\chi,G)}{c} = \sum_{\lambda_j \in \Lambda} A_j\, x^{\tau_j} + \mathcal{O}(x^{\beta/3+\varepsilon}), \text{ for any } \varepsilon>0.$$

Where $\tau_j = 2\sqrt{\tfrac{1}{4}-\lambda_j}$, the A_j's are constants, and

$$\beta = \operatorname*{Inf}_{\delta>0} \left\{ \sum_{c>0} \frac{|S(m,n,c,\chi,G)|}{c^{1+\delta}} < \infty \right\}$$

The relation between the generalized Kloosterman sum and Dedekind sum arises from the transformation law of the Dedekind η-function: $\eta(z)=e(z/24)\prod_{n=1}^{\infty}(1-e(nz))$. This is:

$$\log \eta\left(\frac{az+b}{cz+d}\right) = \begin{cases} \log \eta(z)+\tfrac{1}{2}[\log\left(\frac{cz+d}{i}\right)+2\pi i\left(\frac{a+d}{12c}-s(d,c)\right)], & \text{for } c\neq 0. \\ \log \eta(z)+\tfrac{1}{2}\left(\frac{2\pi i b}{12}\right), & \text{for } c=0. \end{cases}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,Z)$.

This gives that $\eta^{2r}(z)$ satisfies an automorphy condition:

$$\eta^{2r}\left(\frac{az+b}{cz+d}\right) = \chi_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(cz+d)^r\, \eta^{2r}(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,Z), \qquad \text{where:}$$

χ_r is therefore a multiplier system of weight r for SL(2,Z), and therefore for all congruence subgroups.

We can explicitly calculate:

$$\chi_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} e[r(\frac{a+d}{12c} - s(d,c) - \frac{1}{4})], & \text{if } c>0 \\ e(rb/12), & \text{if } c=0 \end{cases}$$

From this, it follows that for h,m,k positive integers we have:

$$e(\frac{-hm}{4k}) S(hm,hm,c,\chi_{hm/k},\Gamma(12k)) = \sum_{\substack{0<d<12kc \\ (d,c)=1 \\ d\equiv 1 \pmod{12k}}} e\left(\frac{hms(d,c)}{k}\right), \text{ and } c\equiv 0 \pmod{12k}.$$

And it is now clear how Theorem A follows from Theorem B using partial summation to give non-trivial estimates of:

$$\sum_{0<c<x} S(hm,hm,c,\chi_{hm/k},\Gamma(12k))$$

OUTLINE OF THESIS

Chapter I is an outline of the prerequisites to understanding the following chapters. In §1.1 we recall the action of SL(2,R) on the complex upper half plane, and also describe some properties of the discrete action of congruence subgroups on H. In §1.2 we define our space of automorphic forms, while §1.3 gives us the Fourier expansions of automorphic forms at the cusps. In §1.4 we recall some theorems about Kloosterman sums.

Chapter II is independent of the others, and explains how automorphic forms of non-integral weight correspond to invariant functions on the Metaplectic group $\overline{SL(2,R)}$, which is a covering group of SL(2,R). §2.1 proves some technical facts about multiplier systems. In §2.2 we construct the Metaplectic group, and find an explicit decomposition of it. We also give the correspondence between automorphic forms and invariant forms on the Metaplectic group.

In §2.3 we take the approach of Representation Theory to define modular forms. We then review some facts about the eigenvalues corresponding to modular forms.

<u>Chapter III</u> deals with the spectral decomposition of automorphic forms. In §3.1 we introduce the Eisenstein series which are a fundamental tool in the theory of automorphic forms. In §3.2 we describe Selberg's spectral decomposition of the space of automorphic forms. In §3.3 we find the explicit spectral decomposition of a class of automorphic forms: the Poincare series.

<u>Chapter IV</u> is an application of the theory developped in the previous chapter to a special case involving rational weight automorphic forms and special congruence subgroups. In §4.1 we show how the Dedekind η function can be used to construct multiplier systems of arbitrary weight. In §4.2 we restrict ourselves to rational weight mh/k, and to the congruence subgroup $\Gamma(12k)$, and show how

the generalized Kloosterman sum reduces to a simple sum involving Dedekind sums. This fact, combined with a theorem of Kuznetsov, Proskurin, Goldfeld and Sarnak, allows us to prove the main theorem of this work: Theorem 4.2.4.

CHAPTER I

BASIC DEFINITIONS

1.1 BASIC DEFINITIONS

Let R be the set of real numbers, we denote by $H=\{x+iy\,|\,y>0\}$ the complex upper half plane. If we write $\infty=\frac{1}{0}$, then it is well known that $SL(2,R)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,|\,a,b,c,d\varepsilon R, ad-bc=1\}$ acts on H and on $R\cup\{\infty\}$ by

$$z \mapsto \frac{az+b}{cz+d} \qquad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\varepsilon SL(2,R),\ z\varepsilon H \text{ or } z\varepsilon R\cup\{\infty\}$$

Note that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ have the same action, and the group of transformations is $PSL(2,R)=SL(2,R)/\pm 1$. However we shall denote a transformation $g\varepsilon PSL(2,R)$ by one of its associated matrices. We thus write:

$$\sigma z=\frac{az+b}{cz+d} \qquad \text{for } \sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varepsilon SL(2,R)$$

The classical theory gives that $\frac{dx^2+dy^2}{y^2}$ is an SL(2,R) invariant metric on H, and the invariant volume element is given by $\frac{dxdy}{y^2}$. Thus considered, the complex upper half plane becomes a Riemann surface of constant negative curvature, and is called the Poincare or Lobachevski plane.

Let G be a subgroup of SL(2,R), we say that $z_1,z_2\varepsilon H$ are G-equivalent if there is a $g\varepsilon G$ such that $z_1=gz_2$.

We say that G is a <u>discrete subgroup</u> of SL(2,R) if G is a subgroup and for any $z\varepsilon H$, the set $\{gz | z\varepsilon G\}$ has no limit point in H.

Let Z be the set of integers and $SL(2,Z)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d\varepsilon Z, ad-bc=1\}$. It is well known that SL(2,Z) is a discrete subgroup of SL(2,R).

Now for N a positive integer we define $\Gamma(N)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,Z) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } N)\}$, where the congruence is defined componentwise. We call $\Gamma(N)$ the <u>Principal congruence subgroup of level N</u>.

It can be shown that $\Gamma(N)$ has finite index in SL(2,Z), in fact: $[\Gamma(N):SL(2,Z)] = N^3 \prod_{p|N} (1-\frac{1}{p^2})$

Defining a subgroup $G \subseteq SL(2,Z)$ to be a <u>Congruence subgroup</u> of SL(2,Z) if $\Gamma(N) \subseteq G$ for some $N>0$, we conclude that: G is a discrete subgroup of SL(2,R) and G has finite index in SL(2,Z).

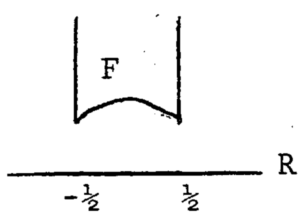
We will restrict ourselves to G a congruence subgroup of SL(2,Z), so in this section <u>G will aways denote a congruence subgroup of SL(2,Z)</u>

A <u>Fundamental domain</u> F for G is an open set $F \subseteq H$ satisfying:

a) If $z_1, z_2 \varepsilon F$; $z_1 \neq z_2$ implies that z_1, z_2 are not G-equivalent.

b) Every $z \varepsilon H$ is G-equivalent to a $z_0 \varepsilon \overline{F}$, the topological closure of F.

It is well known that a fundamental domain for SL(2,Z) can be given by $F=\{x+iy \mid x^2+y^2>1,\ -\tfrac{1}{2}<x<\tfrac{1}{2}\}$



and an easy calculation shows that

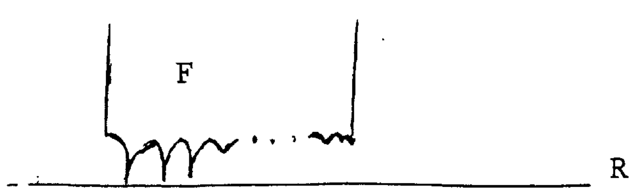
$\iint_F \frac{dxdy}{y^2} = \frac{\pi}{3}$, thus $\mathrm{Vol}(SL(2,Z) \backslash H) < \infty$

Therefore G has finite index in SL(2,Z) implies:

a) The fundamental domain F of G is a finite union of translates of a fundamental domain of SL(2,Z), and can thus be chosen to be a simply connected set.

b) $\mathrm{Vol}(G \backslash H) = [G:SL(2,Z)]\mathrm{Vol}(SL(2,Z) \backslash H) < \infty$

The fundamental domain for G looks like:



We say that an element $g\in SL(2,R)$ is <u>parabolic</u> if $|a+d|=2$, where $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let G be as before, then we say that $\kappa\in H\cup R\cup\{\infty\}$ is a <u>cusp</u> of G, if κ is left fixed by a parabolic element in G. It turns out that all cusps are in $R\cup\{\infty\}$.

Let $N=\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x\in R\}$ then it is clear that

a) N consists of parabolic elements.

b) N is the subgroup of SL(2,R) of elements fixing ∞ .

c) ∞ is a cusp of G iff $G_\infty = G\cap N \neq 0$

Further, as G is a congruence subgroup, there is an M such that $\Gamma(M)\subseteq G$. So as $\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix}\in\Gamma(M)$, we have $G_\infty \neq 0$, and ∞ is always a cusp of G.

We note that $SL(2,Z)_\infty = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n\in Z\} \cong Z$ is a cyclic group and $G_\infty \subseteq SL(2,Z)_\infty$. Thus G has a unique generator $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, $q>0$, and we denote this by $q=q(G)$.

It can be shown that $\{\infty\}$ is a complete set of inequivalent cusps of SL(2,Z). So as G has finite index in SL(2,Z), we have that G has a finite set of inequivalent cusps:

$\infty = \kappa_1, \ldots, \kappa_h$ and there are $\sigma_1, \ldots, \sigma_h \varepsilon\ SL(2,Z)$, $\sigma_1 = id$ such that $\sigma_j(\infty) = \kappa_j$.

1.2 AUTOMORPHIC FORMS

Let G be any subgroup of $SL(2,R)$ then we say that

$j: G \times H \to \mathbf{C}$ is a Factor of automorphy for G

if for $g_1, g_2 \varepsilon G$, $z \varepsilon H$: $\quad j(g_1 g_2, z) = j(g_1, g_2 z) j(g_2, z)$

Direct computation immediately gives that $j(g,z) = cz+d$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ SL(2,R)$, is a factor of automorphy for $SL(2,R)$. From this it follows that $|j(g,z)|^r$ is a factor of automorphy for $SL(2,R)$, for r any real number.

Let $\log z$ be the principal branch of $\log$ and let $z^r = \exp(r \log z)$ we define: $\quad j_r(g,z) = (cz+d)^r$,

$J_r(g,z) = \left(\frac{cz+d}{|cz+d|}\right)^r$; where $r \varepsilon R$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,R)$.

We recall that $T = \{t \varepsilon \mathbf{C} \mid |t|^2 = 1\}$ be the multiplicative group of the circle. If G is a subgroup of $SL(2,R)$, we say that $\chi: G \to T$ is a Factor of automorphy of weight r for G if for

$g_1, g_2 \varepsilon G$, $z \varepsilon H$: $\quad \dfrac{\chi(g_1 g_2)}{\chi(g_1)\chi(g_2)} = \dfrac{j_r(g_1, g_2 z) j_r(g_2, z)}{j_r(g_1 g_2, z)}$

this can also be written:

$$\text{(1.2.1)} \qquad \frac{\chi(g_1g_2)}{\chi(g_1)\chi(g_2)} = \frac{J_r(g_1,g_2z)J_r(g_2,z)}{J_r(g_1g_2,z)}$$

as $|j_r(g,z)|^r$ is a factor of automorphy.

We now define our space of automorphic forms for G a congruence subgroup: Let r be a real number and χ a multiplier system of weight r for G. We define the space $M'(r,\chi,G)$ of functions $f:H\to \mathbf{C}$ satisfying:

a) For $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon G$ we have

$f(gz)=\chi(g)j_r(g,z)f(z)=\chi(g)(cz+d)^r f(z)$

b) $\iint_F y^r |f(z)|^2 \frac{dxdy}{y^2} < \infty$

where F is a fundamental domain for G.

We will find it more natural to consider the space $M(r,\chi,G)$ of functions of the form $y^{r/2}f(z)$ where $f\varepsilon M'(r,\chi,G)$. If we let $\text{Im}(z)= y$, where $z=x+iy$; $x,y\varepsilon R$, then a simple computation gives $\text{Im}(gz)= \frac{y}{|cz+d|^2}$, $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varepsilon SL(2,R)$. Using this fact, it becomes clear that $f\varepsilon M(r,\chi,G)$ iff:

a) $f(gz)=\chi(g)J_r(g,z)f(z)=\chi(g)\left(\frac{cz+d}{|cz+d|}\right)^r f(z)$, $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varepsilon G$

b) $\iint_F |f(z)|^2 \frac{dxdy}{y^2} < \infty$

For example: if $f(z)=y^{r/2} f_1(z)$, $f_1 \varepsilon M'(r,\chi,G)$; then

$f(z)= \mathrm{Im}(z)^{r/2} f_1(z)$ and if $g\varepsilon G$

$$f(gz)= \mathrm{Im}(gz)^{r/2} f_1(gz) = \left(\frac{y}{|cz+d|^2}\right)^{r/2} \chi(g)(cz+d)^{r/2} f_1(z)$$

$$= \chi(g)\left(\frac{cz+d}{|cz+d|}\right)^r y^{r/2} f_1(z).$$

It is seen from a) that $|f|^2$ is G-invariant, as $|\chi(g) J_r(g,z)|=1$. Thus b) makes sense. We also note that $\mathrm{Vol}(F)< \infty$ implies that the constant functions are in $M(0,\chi,G)$. Condition b) gives that $M(r,\chi,G)$ is a Hilbert space with inner product $\langle f,g\rangle = \iint_F f\bar{g} \frac{dxdy}{y^2}$, $f,g\varepsilon M(r,\chi,G)$.

We call this the Petersson Inner Product.

1.3 FOURIER EXPANSIONS

Let G be a congruence subgroup, we recall that $G_\infty =\{g \ \varepsilon\ G | g\infty =g\}$, $q=q(G) >0$ is such that $G_\infty=\{\begin{pmatrix}1 & nq\\ o & 1\end{pmatrix} | n\varepsilon\ Z\}$. Now let χ be a multiplier system of weight r for G, $r\varepsilon$ R and $\chi\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}= e(-\alpha)$, $0 \leq \alpha<1$. We write $\alpha=\alpha(\chi,G)$.

If $f\varepsilon\, M(r,\chi,G)$ we see that $f(z+q)=\chi\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}(0\cdot z+1)^r f(z)=e(-\alpha)f(z)$.

So $f(x+iy)$ has a <u>Fourier expansion at ∞</u>:

$$f(x+iy)=\sum_{n=-\infty}^{n=\infty} a_n(y)\ e\left(\frac{(n-\alpha)}{q}\, x\right)$$

Similarily if κ is any cusp of G, with $\sigma\varepsilon\, SL(2,Z)$, $\sigma(\infty)=\kappa$.

We let $G_\kappa=\{g\varepsilon\, G|g\kappa=\kappa\}$, so $\sigma^{-1}G_\kappa\sigma = G_\infty$ We then have that

$h(z)=J_r(\sigma\ ,z)^{-1}\ f(z)$ satisfies $h(z+q)=e(-\alpha)f(z)$;

We thus have the <u>Fourier expansion at the cusp κ</u>:

(1.3.1) $$J_r(\sigma\ ,z)^{-1}\ f(\sigma\ z)=\sum_{n=-\infty}^{n=\infty} a_{n,\kappa}(y)\ \ e\left(\frac{(n-\alpha)}{q}\, x\right).$$

We define the space of <u>Cusp forms</u> $S(r,\chi,G)$ to be the subspace of functions $f\varepsilon\, M(r,\chi,G)$ such that for any cusp κ of G, then $a_{0,\kappa}(y)\equiv 0$, where $a_{0,\kappa}(y)$ is the zero'th Fourier coefficient at the cusp κ, $a_{n,\kappa}(y)$ as in (1.3.1).

<u>Remark</u>: In this work, we will be working exclusively with Fourier expansions at ∞.

1.4 KLOOSTERMAN SUMS

The classical <u>Kloosterman sum</u> is defined for integers m,n,c and is given by: $S(m,n,c)= \sum_{\substack{d \pmod c \\ (d,c)=1 \\ a\cdot d\equiv 1 \pmod c}} e\left(\frac{ma+nd}{c} \right)$

The Kloosterman has been extensively studied, with much work on finding upper bounds on $|S(m,n,c)|$ for m,n fixed.

The trivial estimate is $|S(m,n,c)|<c$. Estermann and Salie obtained $S(m,n,c)\ll c^{3/4+\varepsilon}$, for any $\varepsilon>0$, m,n fixed. Davenport improved this to $S(m,n,c) \ll c^{2/3+\varepsilon}$, any $\varepsilon>0$. Finally. Andre Weil found that $S(m,n,c)\ll c^{\frac{1}{2}+\varepsilon}$, for $\varepsilon>0$, and m,n fixed; and this is also the best possible.

In another direction Selberg studied the sum

(1.4.1) $$\sum_{0<c<x} \frac{S(m,n,c)}{c}$$ and conjectured that

$$\sum_{0<c<x} \frac{S(m,n,c)}{c} \ll x^{\varepsilon} \quad \text{for any } \varepsilon>0$$

The best result known, however, is due to Kuznetsov and gives:

(1.4.2) $$\sum_{0<c<x} \frac{S(m,n,c)}{c} \ll x^{1/6+\varepsilon} \qquad \text{for any } \varepsilon>0.$$

In his paper on the Fourier coefficients of modular forms [Selberg 1], Selberg showed that the estimation of the sum (1.4.1) is inextricably related to the theory of modular forms.

Selberg also showed how the Kloosterman sum could be generalized to a congruence subgroup G of SL(2,Z), and a multiplier system χ of weight r for G, $r \varepsilon R$, by:

$$S(m,n,c,\chi,G) = \sum_{g \varepsilon G_\infty \backslash G / G_\infty} \overline{\chi(g)}\; e\left(\frac{(m-\alpha)a + (n-\alpha)d}{qc} \right) \quad , \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \sum_{\substack{d (\mathrm{mod}\ c) \\ 0<d<qc \\ g=\begin{pmatrix} * & * \\ c & d \end{pmatrix} \varepsilon G,\ ad \equiv 1 (\mathrm{mod}\ c)}} \overline{\chi(g)}\; e\left(\frac{(m-\alpha)a + (n-\alpha)d}{qc} \right)$$

where G_∞ , $q=q(G)$, α are as in section 1.3.

The sum (1.4.1) can be generalized to:

$\sum_{c<x} \frac{S(m,n,c,\chi,G)}{c}$, and the following result has been proved:

Theorem 1.4 (Kuznetsov, Goldfeld, Sarnak, Proskurin)

$$\sum_{c<x} \frac{S(m,n,c,\chi,G)}{c} = \sum_{j=1}^{h} A_j\, x^{\tau_j} + \mathcal{O}(x^{\beta/3+\varepsilon}), \text{ for any } \varepsilon>0.$$

Where A_j are constants, $\tau_j = 2\sqrt{\frac{1}{4}-\lambda_j}$, where $0<\lambda_j<\frac{1}{4}$ belong to the set Λ of exceptional eigenvalues defined in section 2.3, and $\beta = \lim_{\delta} \inf \{ \sum_{0<c} \frac{|S(m,n,\chi,G)|}{c^{1+\delta}} < \infty \}$.

In the case $G=SL(2,Z)$, $r=0, \chi=1$ it turns out that $S(m,n,c,\chi,G) = S(m,n,c)$. Also there are no exceptional eigenvalues, and $\beta=\frac{1}{2}$ by Weil's estimate. Thus Kuznetsov result (1.4.2) follows.

CHAPTER II

METAPLECTIC GROUPS

2.1 FURTHER PROPERTIES OF MULTIPLIER SYSTEMS

We first characterize factors of automorphy and multiplier systems.

Proposition 2.1.1

Let G be a subgroup of SL(2,R) and $j: G \times H \to C$, then the existence of a non-vanishing $f: H \to C$ satisfying

$$f(gz) = j(g,z)f(z) \quad , \quad g \in G$$

implies that $j(g,z)$ is a factor of automorphy for G.

Proof: Let $g_1, g_2 \in G$ then

$$f(g_1 g_2 z) = j(g_1 g_2, z)f(z) = f(g_1(g_2 z)) = j(g_1, g_2 z)f(g_2 z)$$

$$= j(g_1, g_2 z) j(g_2, z) f(z)$$ and $f(z) \neq 0$ give the result //

We next have.

Lemma 2.1.1

Let G be a subgroup of SL(2,R) and $\chi: G \to T$, $r \in R$, then χ is a multiplier system of weight r for G iff $\chi(g) j_r(g,z)$ and $\chi(g) J_r(g,z)$ are factors of automorphy for G.

Proof: The proof follows by cross multiplying in

$$(2.1.1)\quad \frac{\chi(g_1g_2)}{\chi(g_1)\chi(g_2)} = \frac{j_r(g_1,g_2z)j_r(g_2,z)}{j_r(g_1g_2,z)} = \frac{J_r(g_1,g_2z)J_r(g_2,z)}{J_r(g_1g_2,z)}, \quad g_1,g_2\varepsilon G$$

where the last equality is a consequence of the fact that $|j_r(g,z)|$ was shown in section 1.2 to be a factor of automorphy for SL(2,R), and in fact is true for *any* $g_1,g_2\varepsilon SL(2,R)$ //

Corollary 2.1.1

Let G be a subgroup of SL(2,R), and $\chi:G\to T$. Then the existence of a novanishing function $f:H\to C$ satisfying

$f(gz)=\chi(g)j_r(g,z)f(z)$, $g\varepsilon G$

implies that χ is a multiplier system of weight r for G.

Proof: Follows from above. //

Definition 2.1.1

We will say that a multiplier system of weight r for G is *non-trivial* if there exists a function f satisfying the conditions of Corollary 2.1.1.

We will restrict ourselves to such multiplier systems, and *χ is always assumed to be non-trivial*.

Proposition 2.1.2

Let χ_1, χ_2 be multiplier systems for G of weight r_1, r_2 respectively, then $\chi_1\chi_2$ is a multiplier system of weight r_1+r_2 for G.

Proof: Follows if we note that for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$j_{r_1}(g,z)j_{r_2}(g,z)=(cz+d)^{r_1}(cz+d)^{r_1}=(cz+d)^{(r_1+r_2)}=j_{(r_1+r_2)}(g,z) \quad //$$

We define $\chi : G \to T$ to be an Abelian character of G if:

a) $\chi(g_1g_2)=\chi(g_1)\chi(g_2) \qquad g_1,g_2 \varepsilon G$

b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \varepsilon\ G$ implies $\chi\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}=1$

Proposition 2.1.3

Let $n \varepsilon Z$, then a multiplier system χ of weight n for G is an abelian character of G.

Proof: As in section 1.2 $j_1(g,z)=cz+d$, $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a factor of automorphy for SL(2,R). Thus, so is $j_1(g,z)$, hence

$$\frac{\chi(g_1g_2)}{\chi(g_1)\chi(g_2)} = \frac{j_n(g_1,g_2z)j_n(g_2,z)}{j_n(g_1g_2,z)} = 1$$

Also if $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \varepsilon\ G$, then letting $f:H \to C$ be as in Definition 2.1.1 gives $f(z)=f(\frac{-z+0}{0+-1})=\chi(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})\ f(z)$, so $\chi(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})=1$

//

Proposition 2.1.4

Let r be a real number, and χ be a multiplier system of weight r for G, then any other multiplier system χ' of weight r for G is of the form $\chi'=\chi_0\chi$, where χ_0 is and Abelian character of G.

Proof: It follows from Proposition 2.1.2 that χ/χ' is a multiplier system of weight zero, which is an Abelian character by Proposition 2.1.3. //

We now recall a Theorem of Maass.

Theorem 2.1. (Maass)

Let G be a congruence subgroup of $SL(2,Z)$, then the group of abelian characters of G is isomorphic to G/K^*, where K^* is the group generated by the commutator of G and by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover this group is finite of order $n(G)$.

Proof: [Maass 1] page 117. //

Corollary 2.1.2

Let G be a congruence subgroup, r a real number, then there are exactly n(G) multiplier systems of weight r for G.

proof: By theorem 2.1 and proposition 2.1.4, the existence of one multiplier system of weight r for G implies that there are exactly n(G) such. But in Chapter IV (Corollary 4.1.2) we will construct a multiplier system of weight r for any $r \in R$, and any congruence subgroup.

//

Corollary 2.1.3

Let $r=m/k$ be a rational number with $m,k \in Z$. If χ is a multiplier system of weight r for G for the congruence subgroup G, then $\chi(g)$ is aways a $k \cdot n(G)$ root of unity.

Proof: By proposition 2.1.4 χ^k is a multiplier system of weight m, and is thus an abelian character of G, and so is an n(G) root of unity by theorem 2.1

//

Let $T_n=\{t \in C \mid t^n=1\}$ be the multiplicative group of n^{th} roots of unity, then we can express the result of Corollary 2.1.3 as $\chi: G \to T_n$, $n=k \cdot n(G)$.

We will require the following result:

Proposition 2.1.5

Let $r=m/k\varepsilon Q$; $m,k\varepsilon Z$, then for $g_1, g_2 \varepsilon SL(2,R)$

$\dfrac{j_r(g_1, g_2 z) j_r(g_2, z)}{j_r(g_1 g_2, z)}$ is a k^{th} root of unity.

Proof: as in Proposition 2.1.2

$$\left(\frac{j_{m/k}(g_1, g_2 z) j_{m/k}(g_2, z)}{j_{m/k}(g_1 g_2, z)}\right)^k = \frac{j_m(g_1, g_2 z) j_m(g_2, z)}{j_m(g_1 g_2, z)} = 1$$

as in the proof of proposition 2.1.3 //

Corollary 2.1.4

With notation as in proposition 2.1.5, we have that

$\dfrac{J_{m/k}(g_1, g_2 z) J_{m/k}(g_2, z)}{J_{m/k}(g_1 g_2, z)}$ is aways a k^{th} root of unity.

Proof: Follows as above //

2.2 METAPLECTIC GROUPS

In this section we fix a rational number $r=m/k$; $m,k\in Z$ and fix $n\in Z$ with $k|n$. Also for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g\in SL(2,R)$ we will write $J(g,z)$ for $J_{m/k}(g,z)=\left(\frac{cz+d}{|cz+d|}\right)^{m/k}$, $z\in H$.

Definition 2.2.1

We define the Metaplectic Group $\overline{SL(2,R)}_{m,k,n}$ to be the set of pairs $\{(g,t)|g\in SL(2,R),\ t\in T_n\}$ with multiplication law

(2.2.1) $$(g_1,t_1)(g_1,t_2)=(g_1g_2,\alpha(g_1,g_2)t_1t_2)$$

(2.2.2) $$\alpha(g_1,g_2)=\frac{J(g_1,g_2z)J(g_2,z)}{J(g_1g_2,z)}$$

We note that, by corollary 2.1.4, $\alpha(g_1,g_2)\in T_k\subseteq T_n$, as $k|n$, so (2.2.1) is well defined.

Note: We will supress m,k,n and henceforth write $\overline{SL(2,R)}$ for $\overline{SL(2,R)}_{m,k,n}$

Remark 2.2.1

$\overline{SL(2,R)}$ is an n-fold cover of $SL(2,R)$. That is, there is an exact sequence $1\to T_n\to \overline{SL(2,R)}\to SL(2,R)\to 1$

Also $\alpha(g_1,g_2)$ is a <u>factor set</u> , that is for any $g_1,g_2,g_3 \varepsilon SL(2,R)$

$$\alpha(g_1g_2,g_3)\alpha(g_1,g_2)=\alpha(g_1,g_2g_3)\alpha(g_2,g_3)$$

Since the map $\overline{SL(2,R)} \to SL(2,R)$ given by $(g,t) \mapsto g$ is a homomorphism, we see that $\overline{SL(2,R)}$ acts on the upper half plane H by $\bar{g}z=(g,t)z=gz$, where we have written $\bar{g}=(g,t)$.

We also extend $J(g,z)$ to $J:\overline{SL(2,R)} \times H \to \mathbf{C}$ by $J(\bar{g},z)=t\,J(g,z)$ where again $\bar{g}=(g,t)$. We have then :

<u>Proposition 2.2.1</u>

$J(\bar{g},z)$ is a factor of automorphy for $\overline{SL(2,R)}$.

<u>Proof:</u> Let $\bar{g}_j=(g_j,t_j)\varepsilon\ \overline{SL(2,R)}$ for j=1,2 then

$$J(\bar{g}_1\bar{g}_2,z)=J(\ (g_1,t_1)(g_2,t_2),z)=\ J((g_1g_2,\frac{J(g_1,g_2z)J(g_2z)}{J(g_1g_2,z)}\ t_1t_2),z)$$

$$\frac{J(g_1,g_2z)J(g_2,z)}{J(g_1g_2,z)}\ t_1t_2\ J(g_1g_2,z)=\ t_1J(g_1,g_2z)t_2J(g_2,z)$$

$$=t_1J(g_1,\bar{g}_2z)t_2J(g_2,z)=J(\bar{g}_1,\bar{g}_2z)J(\bar{g}_2,z)$$

//

We will now obtain a decomposition of $\overline{SL(2,R)}$.

<u>Lemma 2.2.1</u>

Let $\bar{N}=\{\ (\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},1)\,|\,x\varepsilon R\}$, $\bar{A}=\{(\ \begin{pmatrix}y^{\frac{1}{2}} & 0\\ 0 & y^{-\frac{1}{2}}\end{pmatrix},1)\ |y>0,\ y\varepsilon R\}$

then $\overline{N}$, $\overline{A}$ are subgroups of $\overline{SL(2,R)}$.

Proof: this follows from the fact that

$$J\left(\left(\begin{pmatrix} y^{\frac{1}{2}} & x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix},1\right),z\right)= 1\cdot\left(\frac{0\cdot z+ y^{-\frac{1}{2}}}{|0\cdot z+y^{-\frac{1}{2}}|}\right)^r =1 , \quad \text{for} \quad y >0$$

//

We now examine how SL(2,R) acts on the point i.

Proposition 2.2.2

a) $\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},1\right)\left(\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix},1\right) i= x+iy , \quad y >0$

b) $\overline{K}= \{\overline{g}\varepsilon\overline{SL(2,R)}\,|\,\overline{g}i=i\}= \{(r(\theta),t)\,|\,0\leqslant \theta<2\pi,\ t\varepsilon T_n\}$

where $r(\theta)= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Proof: Follows from direct computations.

//

We require the following simple result:

Lemma 2.2.2

$J(r(\theta),i)= \exp(ir\theta)$

Proof: we have $J(r(\theta),i)=\left(\frac{i\cdot\sin +\cos}{|i\cdot\sin +\cos|}\right)^r =\left(\frac{\exp(i\theta)}{|\exp(i\theta)|}\right)^r$

$= \frac{\exp(ir\theta)}{1} = \exp(ir\theta)$, as $\theta\varepsilon R$ gives $|\exp(ir\theta)|=1$.

//

We now have:

<u>Proposition 2.2.3</u>

There is an isomorphism $\Psi:\overline{K}\to T \times T_{n/k}$ given by $\Psi(r(\theta),t)=(t\exp(ir\theta),t^k)$.

<u>Proof</u>: We first show that Ψ_1,Ψ_2 given by $\Psi_1(r(\theta),t)=t\exp(ir\theta)$, $\Psi_2(r(\theta),t)=t^k$, are homomorphisms of $\overline{K}\to T$, $\overline{K}\to T_{n/k}$ respectively. We will then find inverses for Ψ_1 and Ψ_2.

Now $\Psi_1[(r(\theta_1),t_1)(r(\theta_2),t_2)]$

$$=\Psi_1[(r(\theta_1)r(\theta_2),t_1t_2\,\frac{J(r(\theta_1),r(\theta_2)z)J(r(\theta_2,z)}{J(r(\theta_1)r(\theta_2),z)})]$$

Letting $z=i$ we appeal to Lemma <u>2.2.2</u>, and noting that $r(\theta_1)r(\theta_2)=r(\theta_1+\theta_2)$ the above becomes:

$$=\frac{\exp(ir(\theta_1+\theta_2))t_1t_2\exp(ir\theta_1)\exp(ir\theta_2)}{\exp(ir(\theta_1+\theta_2))}$$

$$=\Psi_1(r(\theta_1),t_1)\Psi_1(r(\theta_2,t_2))$$

Also we have $\Psi_2((r(\theta_1),t_1)(r(\theta_2),t_2)$

$$=\Psi_2(r(\theta_1+\theta_2),t_1t_2\alpha[(r(\theta_1),t_1),(r(\theta_2),t_2)])$$

$$=t_1^k\ t_2^k\ \ (\alpha[(r(\theta_1),t_1)(r(\theta_2),t_2)])^k=t_1^k\ t_2^k=\Psi_1((r(\theta_1),t_1)\Psi_2((r(\theta_2),t_2))$$

as we have noted that $\alpha(g_1,g_2)^k=1$ for any g_1,g_2.

To show that Ψ is one to one and onto, we find inverse maps to Ψ_1,Ψ_2. We first define: $\overline{r(\theta)}=\exp(i\theta/k)$ for $0\leqslant \theta<2\pi k$. Then let:

$$\Xi_1:\{\overline{r(\theta)}\mid 0\leqslant \theta <2\pi k\} \to \bar{K}, \qquad \Xi_2:T_{n/k}\to \bar{K}$$

$$\Xi_1(\overline{r(\theta)})=(r(2\pi\{\theta/2\pi\},r([\theta/2\pi])); \quad \Xi_2(e(jk/n))=(r(-\frac{2\pi jk}{mn}),e(j/n)),0\leqslant j< \frac{n}{k}$$

where $x=[x]+\{x\}$ denote the integral and fractional parts of x respectively. We then have that $\Psi_1\Xi_1=\text{id}.T$, $\Psi_2\Xi_2=\text{id}.T_{n/k}$. These results follow from a direct computation, and the proposition follows.

//

<u>Corollary 2.2.1</u>

Every element $\bar{u}\varepsilon\bar{K}$ can be uniquely written as $\bar{u}=\overline{r(\theta)}t$, where $0\leqslant \theta<2\pi k$ and $t^k=1$.

<u>Proof</u>: We use Ξ_1,Ξ_2 to identify $\{\overline{r(\theta)}\mid 0\leqslant\theta<2\pi k\}$ and $T_{n/k}$ with their images in $\bar{K}$. The result then follows from Proposition <u>2.2.3</u>.

//

We next prove:

Lemma 2.2.3

Let $\overline{u}\varepsilon\overline{K}'$, and $\overline{u}=\overline{r(\theta)}\cdot t$ as above, then:

$J(\overline{u},i)=\exp(ir\theta)$

Proof: As $J(g,z)$ is a factor of automorphy for $\overline{SL(2,R)}$, we have:

$J(\overline{u},i)=J(\overline{r(\theta)},ti)J(t,i)=J(\overline{r(\theta)},i)J(t,i)$

Now by Propositon 2.2.3 there is a $j\varepsilon Z$ such that

$t=(r(-\frac{2\pi jk}{mn}),e(m/n))$, so by Lemma 2.2.2

$J(t,i)=e(m/n)\exp(-\frac{2\pi ijk}{mn}\frac{m}{k})=e(m/n)e(-m/n)=1$

Also Proposition 2.2.3 gives $\overline{r(\theta)}=(r(2\pi\{\theta/2\pi\},e(r[\theta/2\pi]))$, so:

$J(\overline{r(\theta)},i)=e(r[\theta/2\pi])\exp(2\pi ir\{\theta/2\pi\})=\exp(2\pi ir([\theta/2\pi]+\{\theta/2\pi\}))$

$=\exp(2\pi ir\ \theta/2\pi)=\exp(ir\theta)$

//

If we combine Proposition 2.2.2 and Corollary 2.2.1 we obtain:

Theorem 2.2.1

$\overline{SL(2,R)}$ has the decomposition $\overline{SL(2,R)}=\overline{N}\ \overline{A}\ \overline{K}$ and every $\overline{g}\varepsilon\overline{SL(2,R)}$ can be uniquely written as:

$$(2.2.3) \qquad \bar{g} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \overline{r(\theta)}\cdot t, \quad \text{where} \quad x\in R, y>0, 0\leq\theta<2\pi k, t^{n/k}=1$$

and we have made the obvious identifications for $\bar{N}$, $\bar{A}$.

Remark 2.2.2

(2.2.2) gives us local coordinates for $\overline{SL(2,R)}$, and enables us to carry out analysis on $\overline{SL(2,R)}$.

We now let G be a congruence subgroup of $SL(2,Z)$, $r=m/k\in Q$; $m,k\in Z$, χ a multiplier system of weight r for G, $n(G)$ as in Theorem 2.1 and $n=k\cdot n(G)$ (so $k|n$). We will be considering the Metaplectic group $\overline{SL(2,R)}_{m,k,n}$, which we again denote by $\overline{SL(2,R)}$. We first prove:

Proposition 2.2.4

The map $G \to \overline{SL(2,R)}$ given by $g \mapsto (g,\chi(g))$ is an isomorphism.

Proof:

The map is well defined as we showed in Corollary 2.1.3 that $\chi(g)\in T_n$. By definition we have that

$$\frac{\chi(g_1g_2)}{\chi(g_1)\chi(g_2)} = \frac{J(g_1,g_2z)J(g_2,z)}{J(g_1g_2,z)} \qquad \text{for} \quad g_1,g_2\in G \quad \text{so:}$$

$$(g_1,\chi(g_1))(g_2,\chi(g_2))=(g_1g_2,\chi(g_1)\chi(g_2)\ \frac{\chi(g_1g_2)}{\chi(g_1)\chi(g_2)}\ )=(g_1g_2,\chi(g_1g_2))$$

//

We will use this map to identify G with its image in $\overline{SL(2,R)}$.

Theorem 2.2.2

a) There is an isomorphism $M(r,\chi,G) \to L^2(_G\backslash\overline{SL(2,R)})$ given by $f \mapsto \phi_f$ where $\phi_f(\overline{g})=f(\overline{g}i)J(\overline{g},i)^{-1}$, $\overline{g}\varepsilon\overline{SL(2,R)}$

b) The image of this map is the space $L(r,G) \subseteq L^2(_G\backslash\overline{SL(2,R)})$ of functions ϕ satisfying $\phi(\overline{g}\cdot\overline{r(\theta)}\cdot t)=\phi(\overline{g})\exp(-ir\theta)$, $\overline{g}\varepsilon\overline{SL(2,R)}$ where we have used the representation of $\overline{K}$ given in Corollary 2.2.1.

Proof: We first show that ϕ_f is left G-invariant: letting $\overline{h}=(h,\chi(h))\varepsilon G$, and $\overline{g}=(g,t)\varepsilon\overline{SL(2,R)}$ we have

$\phi_f(\overline{h}\cdot\overline{g})=f(\overline{h}\cdot\overline{g}i)J(\overline{h}\cdot\overline{g},i)^{-1}=f(h\cdot\overline{g}i)\ J(\overline{h}\cdot\overline{g},i)^{-1}$

$=f(\overline{g}i)\chi(h)J(h,\overline{g}i)J(\overline{h}\cdot\overline{g},i)^{-1}=f(\overline{g}i)J(\overline{h},\overline{g}i)J(\overline{h}\cdot\overline{g},i)^{-1}$

$=f(\overline{g}i)J(\overline{h},\overline{g}i)J(\overline{h},\overline{g}i)^{-1}J(\overline{g},i)^{-1}$ as $J(\cdot,\cdot)$ is a factor of automorphy for $\overline{SL(2,R)}$. The above is thus equal to

$=f(\overline{g}i)J(\overline{g},i)^{-1}=\phi_f(\overline{g})$.

We next show that in terms of x,y,θ,t we have:

$\phi_f(x,y,\theta,t)=f(x+iy)\exp(-ir\theta)$.

This will imply the result of b), for if $\phi\varepsilon L(r,G)$, then the above proof shows that $f_\phi(x+iy)=\phi(x,y,0,1)$ satisfies the automorphy condition $f_\phi(gz)=\chi(g)J(g,z)f_\phi(z)$, $g\varepsilon G$.

We thus write $\bar{g}=(\begin{pmatrix} y^{\frac{1}{2}} & x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix},1)\ \overline{r(\theta)}\cdot t$ so $\bar{g}i=x+iy$ and

$$J(g,i)=J[(\begin{pmatrix} y^{\frac{1}{2}} & x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix},1)\ \overline{r(\theta)}\cdot t,i]=J[(\begin{pmatrix} y^{\frac{1}{2}} & x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix},1),\overline{r(\theta)}\cdot ti)J(\overline{r(\theta)}\cdot t,i)$$

$=1\cdot\exp(ir\theta)$ by the proof of Lemma <u>2.2.1</u> and by Lemma <u>2.2.3</u>.

Our result then follows.

Finally we note that the decomposition of $\overline{SL(2,R)}$ given in Theorem <u>2.2.1</u> gives a product measure on $\overline{SL(2,R)}$:

$$d\bar{g}=\frac{dxdy}{y^2}\otimes\frac{d\theta}{2\pi k}\otimes\frac{1}{n/k},$$ so ϕ_f is square integrable iff

$$\int_{G\backslash\overline{SL(2,R)}}|\phi_f(\bar{g})|^2d\bar{g}=\frac{k}{n}\sum_{t\varepsilon T_{\frac{n}{k}}}\int_0^{2\pi k}\iint_{G\backslash H}|f(x+iy)\exp(-ir\theta)|^2\frac{dxdy}{y^2}\,d\theta$$

$$=\frac{k}{n}\sum_{t\varepsilon T_{n/k}}\frac{1}{2\pi k}\int_0^{2\pi k}\iint_{G\backslash H}|f(x+iy)|^2\frac{dxdy}{y^2}\,d\theta=\iint_F|f(x+iy)|^2\frac{dxdy}{y^2}<\infty$$

as $f\varepsilon M(r,\chi,G)$.

The theorem now follows

//

2.3 RELATION TO REPRESENTATION THEORY

In this section we fix a congruence subroup G, $r=m/k\varepsilon Q$ with $m,k\varepsilon Z$, and let $n=n(G)\cdot k$, $\overline{SL(2,R)}=\overline{SL(2,R)}_{m,k,n}$ be as in the last section.

We presently recall that $\overline{SL(2,R)}$ acts on $L^2(G\backslash\overline{SL(2,R)})$ by right multiplication: $R_{\overline{g}}(\phi(\overline{h}))=\phi(\overline{h}\ \overline{g})$, $\phi\varepsilon L^2(G\backslash\overline{SL(2,R)})$, and this is called the <u>Right regular representation</u>.

It has been the approach of Representation Theory to regard the theory of <u>Modular Forms</u> as the analysis of the decomposition of $R_{\overline{g}}$. We shall use this approach to motivate the definition of Modular Forms.

We note that $\overline{K}$ is a <u>Maximal compact subgroup</u> of $\overline{SL(2,R)}$, and that $\overline{u}=\overline{r(\theta)}\cdot t\mapsto\exp(ir\theta)$ is a character of $\overline{K}$. We see that $L(r,G)$, and so,by Theorem 2.2.2, also $M(r,\chi,G)$, corresponds to decomposing $R_{\overline{g}}$ according to characters of $\overline{K}$.

For further decomposition, we must appeal to the Casimir operator Δ for $\overline{SL(2,R)}$, for it is well known that $R_{\overline{g}}$ decomposes under the invariant subspaces of Δ.

We recall that $\overline{SL(2,R)}$ is a covering group of $SL(2,R)$ and thus has the same <u>Lie Algebra</u> as $SL(2,R)$. It follows that $\overline{SL(2,R)}$ has the same Casimir operator as $SL(2,R)$. Using the local coordinates given in Theorem <u>2.2.1</u>,

can be written as:

$$\Delta\phi(x,y,\theta,t)=y^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+y\,\frac{\partial^2\phi}{\partial x\partial\theta}$$

for ϕ a C^∞ function of x,y,θ. Δ is $\overline{SL(2,R)}$ invariant.

We now analyse the restriction of to $L(r,G)$.

<u>Proposition 2.3.1</u>

Let ϕ be a C^∞ function of x,y,θ in $L(r,G)$ and write $\phi=\phi_f$ where $f\varepsilon M(r,\chi,G)$, then $\Delta\phi_f=\phi_{\Delta_r f}$ where

$$\Delta_r f=y^2\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)-iry\,\frac{\partial f}{\partial x}\quad,$$

and this can be extended to

$\Delta_r:M(r,\chi,G)\to M(r,\chi,G)$. Further if $h:H\to \mathbf{C}$ is <u>any</u> C^∞ function of x,y then: $\Delta_r h(z)=\Delta_r(h(gz)J(g,z)^{-1}\overline{\chi(g)}$, $g\varepsilon G$.

<u>Proof</u>: As $\phi_f=f(x+iy)\exp(-ir\theta)$, we see that f is a C^∞ function of x and y. We compute:

$$\Delta\phi_f = \Delta[f(x+iy)\exp(-ir\ )] = y^2\left(\frac{\partial^2(f\ \exp(-ir\theta))}{\partial x^2} + \frac{\partial^2(f\ (\exp(-ir\theta))}{\partial y^2}\right) +$$

$$+ y\ \frac{\partial f}{\partial x}\ \frac{\partial \exp(-ir\theta)}{\partial\theta} = \Delta_r f \cdot \exp(-ir\theta) = \phi_{\Delta_r f}$$

As C^∞ function are dense in $M(r,\chi,G)$, we see that $\Delta\phi_f = \phi_{\Delta_r f}$ is true for any $f \varepsilon M(r,\chi,G)$ and so $\Delta_r : M(r,\chi,G) \to M(r,\chi,G)$ as $\Delta : L^2(G\backslash\overline{SL(2,R)}) \to L^2(G\backslash\overline{SL(2,R)})$.

Finally as Δ is an $\overline{SL(2,R)}$ invariant, we see that the correspondence $\phi_f(\bar{g}) = f(\bar{g}i)J(\bar{g},i)^{-1}$ given in Theorem 2.3.1 gives that Δ_r has the required invariance property.

//

With the decomposition of Δ_r in mind, we define:

<u>Definition 2.3.1</u>

We say that $f \varepsilon M(r,\chi,G)$ is a Modular form if f is C^∞ and

$\Delta_r f + \lambda f = 0$ for some $\lambda \varepsilon R$.

There has been extensive study of the eigenvalues λ.

First of all $\lambda > 0$ unless there are negative eigenvalues :

$(\tfrac{1}{2}r - j)(1 + j - \tfrac{1}{2}r), 0 < j < \tfrac{1}{2}r$.

corresponding to <u>holomorphic functions</u> $y^{-r/2}\, u(z)$, $u \varepsilon M(r,\chi,G)$.

These eigenvalues correspond to the Discrete series representation of SL(2,R).

We say that λ is an Exeptional eigenvalue, if $0<\lambda<\frac{1}{4}$, and denote by Λ the finite set of such eigenvalues.

Now if λ does not correspond to the discrete series representation then we have that $\lambda<\frac{1}{4}$ if λ correspond to the Complementary series representation of SL(2,R), and $\lambda>\frac{1}{4}$ if λ corresponds to the Continuous series representation of SL(2,R).

For $r=0$, the discrete series representation does not occur, and Selberg has conjectured:

Conjecture 2.3.1 (Selberg)

For $r=0$, G a congruence subgroup of SL(2,Z), there are no exceptional eigenvalues.

This can be generalized to:

Problem 2.3.1

Let G be a congruence subgroup, and $r \varepsilon Q$, $0<r<2$, and $R_{\bar{g}}$, $\overline{SL(2,R)}$ as before. For which r does the complementary series representation not occur in $R_{\bar{g}}$?

Selberg showed that Conjecture 2.3.1 is true for $G = SL(2,Z)$. For G a general congruence subgroup, the best that is known is the result of Jacquet-Gelbart, which gives that $\lambda > 3/16$, if $\lambda \neq 0$.

The truth of conjecture 2.3.1 has many application, for example in the work of Iwaniec and Deshouiller [Iwan-Des]. There is a survey article [Vigneras] on the work on this conjecture.

Now let $\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z)$, $z \varepsilon H$, be the famous Jacobi theta function. It is well known that $\theta(z)$ satisfies:

$$\theta(gz) = \chi_\theta(g)(cz+d)^{\frac{1}{2}}\theta(z) \quad , \quad g \varepsilon \Gamma_0(4) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,Z) \mid 4 | c \}.$$

Where

$$\chi_\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \varepsilon_d^{-1} \left(\frac{c}{d}\right) , & \text{for } c \neq 0 \\ 1 & , c = 0 \end{cases}$$

$\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol, and $\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ 1 & \text{if } d \equiv 3 \pmod 4 \end{cases}$

Since it is known that $\theta(z) \neq 0$ for $z \varepsilon H$, Corollary 2.1.1 implies that χ_θ is a multiplier system of weight $\frac{1}{2}$ for $\Gamma_0(4)$-the so called theta multiplier system.

Now it turns out that $3/16$ is an exceptional eigenvalue corresponding to $y^{\frac{1}{4}}\theta(z)$, and $\theta(z)$ is holomorphic.

However Goldfeld and Sarnak have shown that for λ not corresponding to $y^{\frac{1}{4}}\theta(z)$, it must be that $\lambda > 15/64$.

For general real weight r, the best known result is that for $0<r<2$, χ an arbitrary multiplier, G any congruence subgroup: $\lambda > \frac{1}{2}r(1-\frac{1}{2}r)$.

This result is due to [Roelcke], and was actually proved for G any <u>Fuchsian group</u>. That is, a discrete subgroup G of SL(2,R) such that $\iint_F \frac{dxdy}{y^2} < \infty$, where F is a fundamental domain of G.

CHAPTER III

SPECTRAL THEORY OF AUTOMORPHIC FORMS

In this chapter we will be considering a fixed congruence subgroup G, r a real number and χ a multiplier system of weight r for G. We recall from Section 1.1 that we have a complete set of inequivalent cusps of G:

$\infty = \kappa_1, \kappa_2, \ldots, \kappa_h$ with id.$=\sigma_1, \ldots, \sigma_h \varepsilon SL(2,Z)$ satisfying $\sigma_j \infty = \kappa_j$.

Further if $G_\infty = \{\begin{pmatrix} 1 & nq \\ 0 & 1 \end{pmatrix} | n \varepsilon Z\}$, $q > 0$, then we choose $\alpha = \alpha(\chi, G)$, $0 \leq \alpha < 1$, with $\chi\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = e(-\alpha)$. We let $G_{\kappa_j} = \{g \varepsilon G | g\kappa_j = \kappa_j\} = \sigma_j G_\infty \sigma_j^{-1}$.

So every f $M(r,\chi,G)$ has a Fourier expansion at ∞ given by:

$$f(x+iy) = \sum_{n=-\infty}^{n=\infty} a_n(y) e\left(\frac{(n-\alpha)x}{q}\right)$$

we also write $\quad j(g,z) = \chi(g)\left(\frac{cz+d}{|cz+d|}\right)^r$, $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon G$

3.1 EISENSTEIN SERIES

Definition 3.1

For $j=1,\ldots,h$ we define the Eisenstein series

(3.1.1) $$E_j(z,s)=\sum_{g\varepsilon G_{\kappa_j}\backslash G}[\mathrm{Im}(\sigma_j^{-1}gz)]^s j(g,z)^{-1}\ ;\quad z\varepsilon H,\ s\varepsilon C.$$

Proposition 3.1.1

$E_j(z,s)$ converges absolutely for $\mathrm{Re}(s)>1$, and satisfies $E_j(gz,s)=j(g,z)E_j(z,s)$, for $g\varepsilon G$, in this domain.

Proof: We note that $[\mathrm{Im}(gz)]^s=\dfrac{y^s}{|cz+d|^{2s}}$, for $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ while $|j(g,z)|=1$. So for example:

$$|E_1(z,s)|< y^{\mathrm{Re}(s)}\sum_{\begin{pmatrix}* & *\\ c & d\end{pmatrix}\varepsilon G_\infty\backslash G}|cz+d|^{-2\mathrm{Re}(s)}$$

which is well known to converge for $\mathrm{Re}(s)>1$.

Similarily $E_j(z,s)$ converges absolutely for $\mathrm{Re}(s)>1$.

It is then obvious that $E_j(z,s)$ satisfies $E_j(gz,s)=j(g,z)E_j(z,s)$ for $g\varepsilon G,\mathrm{Re}(s)>1$,

//

Remark 3.1.1

If we define $E(z,s)=\sum_{g\varepsilon G_\infty\backslash G}[\mathrm{Im}(gz)]^s$, then it can be shown that for $\mathrm{Re}(s)>1$, s fixed, $|E(z,s)-y^s|$ is bounded for all $z\varepsilon H$. From this it follows that $E(z,s)$ is <u>not</u> square integrable, in other words $E(z,s)\notin M(r,\chi,G)$.

Selberg has proved the following:

<u>Theorem 3.1.1</u>

$E_j(z,s)$ has an analytic continuation in s to the whole complex plane, except for a possible finite set of simple poles on the interval $(\frac{1}{2},1]$. Furthermore the residues $\theta_1,\ldots,\theta_\gamma$ at these poles $\rho_1,\ldots,\rho_\gamma$ are square integrable automorphic forms which are <u>not</u> cusp forms. Also, the poles of $E_j(z,s)$ coincide with the poles of the constant term in the Fourier expansion of $E_j(z,s)$.

<u>Proof</u>: [Kubota]

//

We will compute the Fourier expansion of $E_j(z,s)$, but first we need some special functions.

<u>Definition 3.1.2</u>

Let $a,b\varepsilon\mathbf{C}$, we define the <u>Whittaker function</u> for $a-b-\frac{1}{2}$

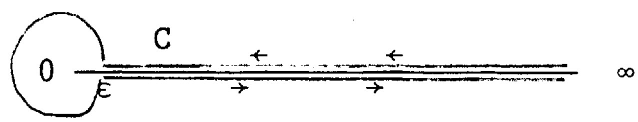
not a negative integer by:

$$W_{a,b}(y) = \frac{1}{2\pi i}\Gamma(a-b+\tfrac12)\exp(-\tfrac12 y)\int_C (-t)^{b-a-\frac12}\,(1+\tfrac{t}{y})^{a+b-\frac12}\exp(-t)\,dt$$

and C is the contour given by:

$$(3.1.2)\qquad C:\ \begin{cases} -t & : \ t< -\varepsilon \\ \varepsilon\cdot e(\frac{t+\varepsilon}{2\varepsilon}) & : \ -\varepsilon < t<\varepsilon \\ t & : \ t>\varepsilon \end{cases} \qquad \text{for any } \varepsilon>0$$

and looks like



It is known that $W_{a,b}(y)$ and $W_{-a,b}(-y)$ are a fundamental set of solutions of Whittaker's equation:

$$(3.1.3)\qquad u''(y) + \left(-\tfrac14 + \frac{a}{y} + \frac{\tfrac14 - b^2}{y^2}\right)u(y) = 0\ , \qquad y>0.$$

As $W_{a,b}(z) \sim \exp(-\tfrac12 z)$ as $z \to \infty$, we see that only $W_{a,b}(y)$, $y>0$, satisfies the regularity condition at ∞.

We can now prove:

Theorem 3.1.2

Let $E_j(z,s) = \sum_{n=-\infty}^{n=\infty} B_{n,j}(y,s)\ e(\frac{(n-\alpha)x}{q})$ then

$$B_{n,j}(y,s) = \delta_0^n\, y^s - \frac{\left(\frac{\pi(n-\alpha)}{4q}\right)^{s-1} D_{n,j}(s)}{i^r\ \Gamma(s+r/2)}\ W_{\frac{r}{2}\mathrm{sgn}(n),\,s-\frac12}\left(\frac{4\pi|n-\alpha|}{q}\,y\right)$$

Here $D_{n,j}(s)$ is the <u>Dirichlet series</u>:

$$D_{n,j}(s)=\sum_{c>0} c^{-2s} \sum_{\substack{0<d<qc \\ g=\begin{pmatrix}* & * \\ c & d\end{pmatrix}\in\sigma_j^{-1}G}} \overline{\chi(g)}\; e\left(\frac{(n-\alpha)d}{qc}\right)$$

<u>Proof</u>: We write $\frac{n-\alpha}{q}=\beta$, and note that for $\mathrm{Re}(s)>1$

$$B_{n,j}(y,s)=\delta_0^n\, y^s+\int_0^q \sum_{G_{\kappa_j}\backslash G} [\mathrm{Im}(\sigma_j^{-1}gz)]^s\, j(g,z)^{-1}\, e(-\beta x)\,dx$$

$$=\delta_0^n\, y^s+\int_0^q \sum_{G_\infty\backslash\sigma_j^{-1}G} [\mathrm{Im}(gz)]^s j(g,z)^{-1} e(-\beta x)\,dx, \quad \text{as} \quad G_{\kappa_j}=\sigma_j G_\infty \sigma_j^{-1}.$$

$$=\delta_0^n\, y^s+\sum_{\substack{c,d \\ g=\begin{pmatrix}* & * \\ c & d\end{pmatrix}\in\sigma_j^{-1}G,\, c>0}} \int_0^q \frac{y^s}{|cz+d|^{2s}}\; \frac{|cz+d|^r}{\chi(g)(cz+d)^r}\, e(-\beta x)\,dx, \text{ since, by}$$

by absolute convergence, we can interchange summation and integration, and we have expanded $\mathrm{Im}(gz)$, $j(g,z)$. We now write $d=qr_1c+d_1$ $(0\leqslant d_1<qc)$ and obtain:

$$=\delta_0^n\, y^s+ y^s\sum_{c>0} \sum_{\substack{0<d<qc \\ g=\begin{pmatrix}* & * \\ c & d\end{pmatrix}\in\sigma_j^{-1}G}} \overline{\chi(g)}\, e\left(\frac{\beta d}{c}\right)\int_{-\infty}^{\infty} \frac{e(-\beta x)}{|z|^{2s-r}\, z^r}\,dx$$

$$=\delta_0^n\, y^s+ D_{n,j}(s)\, y^s\int_{-\infty}^{\infty} \frac{e(-\beta x)}{|z|^{2s-r}\, z^r}\,dx$$

The theorem will therefore follow if we prove:

Lemma 3.1.1

$$y^s\int_{-\infty}^{\infty}\frac{e(-\beta x)}{|z|^{2s-r}z^r}\,dx= -\frac{\pi(\pi\beta/4)^{s-1}\,W_{\frac{r}{2}\mathrm{sgn}(n),\,s-\frac12}(4\pi|\beta|\ y)}{i^r\ \Gamma(s+r/2)}$$

Proof: $$y^s\int_{-\infty}^{\infty}\frac{e(-\beta x)}{|z|^{2s-r}\ z^r}\,dx = y^s\int_{-\infty}^{\infty}\frac{e(-\beta x)}{(x^2+y^2)^{s-\frac12 r}\ (x+iy)^r}\,dx$$

now we have $i^r(y-ix)=(x+iy)^r$, $x^2+y^2=(y+ix)(y-ix)$, so above is:

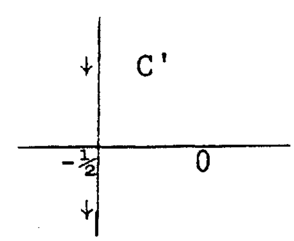
$$=\frac{y^s}{i^r}\int_{-\infty}^{\infty}\frac{e(-\beta x)}{(y-ix)^{s-\frac12 r}(y+ix)^{s-\frac12 r}\ (y-ix)^r}\,dx=\frac{y^s}{i^r}\int_{-\infty}^{\infty}\frac{e(-\beta x)}{(y-ix)^{s+\frac12 r}(y+ix)^{s-\frac12 r}}\,dx$$

We now let $x\to 2yx$ and obtain:

$$=\frac{1}{i^r 2(4y)^{s-1}}\int_{-\infty}^{\infty}\frac{e(2\beta xy)}{(\frac12-ix)^{s+\frac12 r}(\frac12+ix)^{s-\frac12 r}}\,dx,$$ we let $x\to -x$ and expand:

$$=\frac{\exp(-2\pi\beta y)}{i^r 2(4y)^{s-1}}\int_{-\infty}^{\infty}\frac{\exp(4\pi\beta y(\frac12+ix))}{(\frac12+ix)^{s+\frac12 r}(\frac12-ix)^{s-\frac12 r}}\,dx.$$ We put $z=-\frac12-ix$ to get:

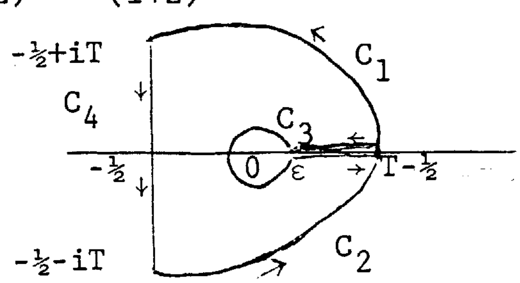
$$=-\frac{\exp(-2\pi\beta y)}{i^r(4y)^{s-1}}\,\frac{1}{2\pi i}\int_{C'}\frac{\exp(4\pi\beta y(-z))}{(-z)^{s+\frac12 r}(1+z)^{s-\frac12 r}}\,dz$$



Where C' is the contour C': $-\frac12-it$, $t=-\infty$ to $t=\infty$.

Now let $$I_j=\int_{C_j}\frac{\exp(4\pi\beta y(-z))}{(-z)^{s+\frac12 r}(1+z)^{s-\frac12 r}}\,dz$$ Where C_j: $j=1,\ldots,4$

are given by



, $T>\frac12$, $0<\varepsilon<\frac12$.

C_2, C_3 are segments of the circle $\{|z+\frac{1}{2}|=T\}$, while C_3, C_4 are segments of C, C' defined above, respectively.

It is now seen that as $T \to \infty$: $I_1, I_2 \to 0$; $I_3 \to I_C$; $I_4 \to I_{C'}$ where $I_C, I_{C'}$ are defined similarily to I_j, $j=1,\ldots,4$.

As the residue theorem gives that $I_{C_1+C_4+C_2+C_3}=0$, we conclude that $I_C = I_{C'}$. The above quantity is thus:

$$= -\frac{\pi\exp(-2\pi\beta y)}{i^r\ (4y)^{s-1}}\frac{1}{2\pi i}\int_C \frac{\exp(-4\pi\beta yz)}{(-z)^{s+\frac{1}{2}r}(1+z)^{s-\frac{1}{2}r}}\,dz \text{ , we let } z\to\frac{z}{4\pi\beta y}:$$

$$=(4\pi\beta y)^{s+\frac{1}{2}r-1}\frac{\pi\exp(-2\pi\beta y)}{i^r(4y)^{s-1}}\int_C \exp(-z)\ (-z)^{-s-\frac{1}{2}r}\left(1+\frac{z}{4\pi\beta y}\right)^{-s+\frac{1}{2}r}\,dz$$

$$= \frac{\pi(\pi\beta)^{s-1}}{i^r 4^{s-1}\Gamma(s+\frac{1}{2}r)}\Big\{\frac{-1}{2\pi i}\,\Gamma(-(s-\tfrac{1}{2})+\tfrac{1}{2}+\tfrac{1}{2}r)\exp(-\tfrac{1}{2}\pi\beta y)(4\pi\beta y)^{\frac{1}{2}r}\ \cdot$$

$$\cdot\int_C (-z)^{-\frac{1}{2}r-\frac{1}{2}+(-s+\frac{1}{2})}\left(1+\frac{z}{4\pi\beta y}\right)^{\frac{1}{2}r-\frac{1}{2}+(-s+\frac{1}{2})}\exp(-z)\,dz\ \Big\}$$

But, by Definition 3.1.2, the expression in brackets is

$W_{\frac{1}{2}r\,\mathrm{sgn}(n),-s+\frac{1}{2}}(4\pi|\beta|y)=W_{\frac{1}{2}r\,\mathrm{sgn}(n),s-\frac{1}{2}}(4\pi|\beta|\ y)$, as $0\leq\alpha<1$ implies $\mathrm{sgn}(\beta)=\mathrm{sgn}(n)$, and also $W_{a,b}=W_{a,-b}$.

The result follows directly.

//

Corollary 3.1.1

Let $\theta(z)$ be the residue of $E_j(z,s)$ at ρ, as in Theorem 3.1.1,

then $\theta(z)$ has the Fourier expansion at ∞ given by:

$$\theta(x+iy)=\sum_{n=-\infty}^{n=\infty} a_n(y)\ e\left(\frac{(n-\alpha)}{q}x\right), \quad \text{where}$$

$$a_n(y)=\delta_0^n\, y^\rho - \frac{\pi\left(\frac{(n-\alpha)}{4q}\right)^{\rho-1}}{i^r\Gamma(\rho+\frac{1}{2}r)} \operatorname*{Res}_{s=\rho} D_{n,j}(s)\, W_{\frac{1}{2}r\,\mathrm{sgn}(n),\rho-\frac{1}{2}}\left(\frac{4\pi|n-\alpha|y}{q}\right)$$

Proof: Follows by taking the residue of $B_{n,j}(s)$ at $s=\rho$ and then applying Theorem 3.1.1

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We conclude this section by proving that θ_j: $j=1,\ldots,\gamma$ are modular forms.

Proposition 3.1.1

Let $\theta(z)$ correspond to $\operatorname*{Res}_{s=\rho} E_j(z,s)$ then

$\Delta_r\theta+\rho(1-\rho)\theta=0$, thus corresponds to the eigenvalue $\rho(1-\rho)$.

Proof: We show that for $\mathrm{Re}(s)>1$

$\Delta_r E_j(z,s)+s(1-s)E_j(z,s)=0$. The result will then follow by analytic continuation.

$$\Delta_r[(\mathrm{Im}z)^s] = \Delta_r y^s = y^2\left(\frac{\partial^2 y^s}{\partial x^2} + \frac{\partial^2 y^s}{\partial x^2}\right) - iry\frac{\partial y^s}{\partial x}$$

$=s(s-1)y^{s-2}y^2=s(s-1)y^s$.

We now use the invariance property of Δ_r given in Proposition 2.3.1 to get $\Delta_r[(\mathrm{Im}(gz)^s j(g,z)^{-1}] = s(s-1)(\mathrm{Im}z)^s\, j(g,z)^{-1}$.

Summing over $g\varepsilon G_{\kappa_j}\backslash G$ gives the result. //

<u>Remark 3.1.2</u>

For $\rho_j \neq 1$, θ_j corresponds to an exceptional eigenvalue, as $0<\rho_j<1$ implies that $0<\rho_j(1-\rho_j)<\frac{1}{4}$.

We will adopt the notation ν_j to denote $E_{\nu_j}(z,s)$ where

$\theta_j(z) = \operatorname*{Res}_{s=\nu_j} E_{\nu_j}(z,s)$.

3.2 SPECTRAL DECOMPOSITION OF $M(r,\chi,G)$

Definition 3.2.1

Let $C(r,\chi,G)$ be the space of functions:

$$u(z)=\frac{1}{4\pi}\sum_{j=1}^{h}\int_{-\infty}^{\infty}<f,E_j(\cdot,\tfrac{1}{2}+it)>E_j(z,\tfrac{1}{2}+it)dt, \quad \text{for } f\in M(r,\chi,G)$$

$C(r,\chi,G)$ is the Continuous spectrum.

Definition 3.2.2

Let $\theta_1,\ldots,\theta_\gamma$ be the residues of all $E_j(z,s)$ at $\rho_1,\ldots,\rho_\gamma$, as in Theorem 3.1.1. We denote by $R(r,\chi,G)$ the subspace of $M(r,\chi,G)$ generated by $\theta_1,\ldots,\theta_\gamma$.

We also recall that $S(r,\chi,G)$ is the subspace of cusp forms, that is, function satisfying $a_{0,j}(y)\equiv 0$ for $j=1,\ldots,h$ where

$$\sum_{n=-\infty}^{n=\infty} a_{n,j}(y)\ e\left(\frac{(n-\alpha)}{q}x\right) \quad \text{is the Fourier expansion}$$

of f at κ_j.

Selberg's spectral decomposition is then:

Theorem 3.2.1 (Selberg)

$$M(r,\chi,G)=S(r,\chi,G)\oplus C(r,\chi,G)\oplus R(r,\chi,G)$$

We further have:

Theorem 3.2.2 (Selberg)

$S(r,\chi,G)$ has a complete orthonormal set given by eigenfunctions of Δ_r. $\{u_j\}_{j=-\nu}^{\kappa}$, $\Delta_r u_j+\lambda_j u_j=0$, where, if $\kappa=\infty$, then $\lambda_0=0<\lambda_1<\lambda_2 \ldots \quad \lambda_j\to\infty$ as $j\to\infty$. And λ_j, $j<0$ correspond to holomorphic functions $y^{-\frac{1}{2}r}\, u_j(z)$.

Putting these results together yealds:

Theorem 3.2.3

Let $f\in M(r,\chi,G)$ then

$$f(z)=\sum_{j=-\nu}^{-1} \langle f,u_j\rangle u_j(z)+\sum_{j>0}\langle f,u_j\rangle u_j(z)+\sum_{j=1}^{\gamma}\langle f,\theta_j\rangle\theta_j(z) \tag{3.2.1}$$

$$+\frac{1}{4\pi}\sum_{j=0}^{h}\int_{-\infty}^{\infty}\langle f,E_j(\cdot,\tfrac{1}{2}+it)\rangle E_j(z,\tfrac{1}{2}+it)\,dt$$

We now compute the Fourier expansion at of the eigenfunctions u_j, $j>0$, given in Theorem 3.2.2

Proposition 3.2.1

For $j>0$ let $u_j(x+iy)=\sum_{n\neq 0} a_{n,j}(y)\; e\left(\frac{(n-\alpha)}{q}\,x\right)$ then:

$a_{n,j}(y)=a_{n,j}W_{\frac{1}{2}r\,\mathrm{sgn}(n),\sqrt{\frac{1}{4}-\lambda_j}}\left(4\pi\frac{|n-\alpha|}{q}y\right)$

<u>Proof</u>: Equating Fourier coefficients in $\Delta_r u_j + \lambda_j u_j = 0$ gives:

$$a''_{n,j}(y) + \left(-4\pi^2\beta^2 + \frac{\pi r\beta}{y} + \frac{\lambda_j}{y^2}\right) a_{n,j}(y) = 0 \tag{3.2.1}$$

where again we have written $\beta = \frac{n-\alpha}{q}$.

We see that (3.2.1) is just Whittaker's equation (3.1.3).

So after a change of variables we get the solution

$$a'_{n,j}\, W_{\frac{1}{2}r,\sqrt{\frac{1}{4}-\lambda_j}}(4\pi\beta y) + a''_{n,j}\, W_{-\frac{1}{2}r,\sqrt{\frac{1}{4}-\lambda_j}}(-4\pi\beta y)$$

But we note that the square integrability of u_j implies

$a_{n,j}(y) \to 0$ as $y \to \infty$. So as in Section <u>3.1.2</u> we have

$a''_{n,j}=0$ if $\beta>0$, that is $n>0$, as $\alpha<1$. And $a'_{n,j}=0$ if $n<0$.

The result then follows.

//

3.3 POINCARE SERIES

Definition 3.3.1

Let $m > 0$ be an integer, then for $z \varepsilon H$, $s \varepsilon \mathbb{C}$

$$P_m(z,s) = \sum_{g \varepsilon G_\infty \backslash G} \operatorname{Im}(gz)^s \, j(g,z)^{-1} \qquad (3.3.1)$$

is the Non-holomorphic Poincare series.

Proposition 3.3.1

For $\operatorname{Re}(s) > 1$, (3.3.1) converges absolutely and $P_m(z,s) \varepsilon M(r,\chi,G)$.

Proof: It is clear that if (3.3.1) converges absolutely then $P_m(z,s)$ satisfies the automorphy condition.

We note that $\left| e\left(\frac{(m-\alpha)}{q} gz \right) \right| < 1$ so $|P_m(z,s)| < \sum_{g \varepsilon G_\infty \backslash G} (\operatorname{Im} gz)^{\operatorname{Re}(s)}$

which converges abolutely for $\operatorname{Re}(s) > 1$ as in the proof of Proposition 3.1.1. Further, we have that:

$(\operatorname{Im} z)^s \, e\left(\frac{(m-\alpha)}{q} z \right) = y^{\operatorname{Re}(s)} \exp\left(-2\pi \frac{(m-\alpha)}{q} y\right)$ which has a

maximum $\mu = \left(\frac{(\operatorname{Re}(s)-1)q}{2\,(m-\alpha)}\right) \exp(1-\operatorname{Re}(s))$ at $y = \frac{(\operatorname{Re}(s)-1)q}{2\,(m-\alpha)}$.

So in fact:

$$|P_m(z,s)| < \left| (\operatorname{Im}(z)^s \, e\left(\frac{(m-\alpha)}{q} z \right) \right| + |E(z,s) - y^s| < \mu + |E(z,s) - y^s| < Cst.$$

Where, as before, $E(z,s)=\sum_{g\epsilon G_\infty \backslash G} (Im(gz))^s$ and $|E(z,s)-y^s|$ is bounded for $z\epsilon H$, s fixed.

We therefore have that $P_m(z,s)$ is bounded for $z\epsilon H$, s fixed, $Re(s)>1$. Hence $\iint_F \frac{dxdy}{y^2} < \infty$ implies that $\iint_F |P_m(z,s)|^2 \frac{dxdy}{y^2} < \infty$

and the proposition follows

//

The important property of $P_m(z,s)$ is that it "picks" the n^{th} Fourier coefficient of automorphic forms:

<u>Proposition 3.3.2</u>

Let $f\epsilon M(r,\chi,G)$ have the Fourier expansion at ∞

$f(x+iy)=\sum_{n=-\infty}^{n=\infty} a_n(y)\; e\left(\frac{(n-\alpha)}{q}\, x\right)$ then

$$<f,P_m(\cdot,s)>= \int_0^\infty \exp\left(-2\pi\frac{(m-\alpha)}{q}\, y\right)\; a_m(y)\; y^{\bar{s}-2}\, dy$$

<u>Proof</u>: $<f,P_m(\cdot,s)>= \iint_F f(z)\overline{P_m(z,s)}\, \frac{dxdy}{y^2}$

$= \sum_{g\epsilon G_\infty \backslash G} \iint_F f(z)\; Im(gz)^{\bar{s}}\; j(g,z)\; e\left(-\frac{(m-\alpha)}{q}\, gz\right) \frac{dxdy}{y^2}$, by absolute convergence we have interchanged integration and summation.

$= \sum_{g\epsilon G_\infty \backslash G} \iint_{gF} (Im(z))^{\bar{s}}\; f(z)\; e\left(-\frac{(m-\alpha)}{q}\, z\right) \frac{dxdy}{y^2}$

$$= \iint_{G_\infty \backslash H} y^{\bar{s}} f(z)\, e\!\left(-\frac{(m-\alpha)}{q} z\right) \frac{dx\,dy}{y^2} = \int_0^\infty \int_0^q y^{\bar{s}-2} f(z)\, e\!\left(-\frac{(m-\alpha)}{q} z\right) dx\,dy$$

$$= \int_0^\infty y^{\bar{s}-2} \exp\!\left(-2\pi \frac{(m-\alpha)}{q} y\right) \int_0^q f(z)\, e\!\left(-\frac{(m-\alpha)}{q} x\right) dx\, dy$$

$$= \int_0^\infty a_m(y) \exp\!\left(-2\pi \frac{(m-\alpha)}{q}\right) y^{\bar{s}-2}\, dy$$

//

For further investigation of $P_m(z,s)$ we will require its Fourier expansion at ∞.

<u>Proposition 3.3.3</u>

Let $P_m(z,s) = \sum_{n=-\infty}^{n=\infty} Q_{m,n}(y,s)\, e\!\left(\frac{(n-\alpha)}{q} x\right)$, then

$$Q_{m,n}(y,s) = \delta_m^n\, y^s + y^s \sum_{c>0} c^{-2s}\, S(m,n,c,\chi,G) \int_{-\infty}^{\infty} \frac{e\left[\left(-(n-\alpha)x - \frac{(m-\alpha)}{c^2 z}\right)/q\right]}{|z|^{2s-r}\, z^r}\, dx$$

Where
$$S(m,n,c,\chi,G) = \sum_{\substack{g \varepsilon G_\infty \backslash G / G_\infty \\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \overline{\chi(g)}\, e\!\left[\frac{(m-\alpha)a + (n-\alpha)d}{qc}\right]$$

is the generalized Kloosterman sum of Section <u>1.4</u>.

<u>Proof</u>: We have $Q_{m,n}(y,s) = \int_0^q P_m(z,s)\, e\!\left(-\frac{(n-\alpha)}{q} x\right) dx$

$$= \delta_m^n y^s + \sum_{\substack{g \varepsilon G_\infty \backslash G \\ g \neq id.}} \int_0^q e\!\left(\frac{(m-\alpha)}{q} gz\right) (\mathrm{Im}(gz))^s \frac{|cz+d|^r}{\chi(g)(cz+d)^r}\, e\!\left(-\frac{(n-\alpha)}{q} x\right) dx$$

by absolute convergence for $\mathrm{Re}(s) > 1$.

Now writing $\operatorname{Im}(gz) = \frac{y}{|cz+d|^2}$ and $gz = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c^2(z+d/c)}$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, And the above is:

$$= \delta_m^n y^s + \sum_{\substack{g \in G_\infty \backslash G \\ g \neq id.}} y^s c^{-2s} \int_0^q \frac{e\left[\frac{(m-\alpha)}{q}\left(\frac{a}{c} - \frac{1}{c^2(z+d/c)}\right)\right] \left|z+\frac{d}{c}\right|^r e\left(-\frac{(n-\alpha)}{q}x\right) dx}{\left|z+\frac{d}{c}\right|^{2s} \chi(g) \left(z+\frac{d}{c}\right)^r}$$

As in the proof of Theorem 3.1.2 we write: $d = \xi qc + d_1$, where $\xi \in Z$ and $0 \leq d_1 < qc$. The result then follows directly.

//

Remark 3.3.1

We see from the above proposition that the generalized Kloosterman sum occurs naturally in the Fourier expansion of $P_m(z,s)$, In fact it occurs as a coefficient in the Dirichlet series: $Z(s,m,n,\chi,G) = \sum_{c>0} \frac{S(m,n,c,\chi,G)}{c^{2s}}$

which is the Kloosterman-Selberg Zeta function

Using Perron's formula: $\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s} ds = \begin{cases} 1 & \text{for } x > 1 \\ 0 & \text{for } x < 1 \end{cases}$

we have: $\sum_{0<c<x} \frac{S(m,n,\chi,G)}{c} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z\left(\frac{1+s}{2}, m, n, \chi, G\right) \frac{x^s}{s} ds$

and this is the key to the Goldfeld-Sarnak proof of Theorem 1.4.

We will give the spectral decomposition of $P_m(z,s)$ for which we will need:

Proposition 3.3.4

(3.3.2) $$\langle P_m(\cdot,s),E_j(\cdot,t)\rangle = i^r\left(4\pi\frac{(m-\alpha)}{q}\right)^{1-s} \pi \left(\frac{(m-\alpha)\pi}{4q}\right)^{\bar{t}-1} D_{m,j}(\bar{t}) \cdot \frac{\Gamma(s-t)\Gamma(s+t-1)}{\Gamma(s-\frac{1}{2}r)\Gamma(t+\frac{1}{2}r)}$$

Proposition 3.3.5

For $j>0$, u_j as in Theorem 3.2.2, $a_{n,j}$ as in Proposition 3.2.1:

$$\langle P_m(\cdot,s),u_j\rangle = \bar{a}_{m,j}\left(4\pi\frac{(m-\alpha)}{q}\right)^{1-s} \frac{\Gamma(s-\sqrt{\frac{1}{4}-\lambda_j}-\frac{1}{2})\Gamma(s+\sqrt{\frac{1}{4}-\lambda_j}-\frac{1}{2})}{\Gamma(s-\frac{1}{2}r)}$$

Before proving these propositions, we note that Proposition 3.3.4 gives:

Corollary 3.3.1

Let $\theta(z) = \operatorname{Res}_{s=\rho} E_j(z,s)$ be as in Theorem 3.1.1 then:

$$\langle P_m(\cdot,s),\theta\rangle = \left(4\pi\frac{(m-\alpha)}{q}\right)^{1-s}\left(\frac{(m-\alpha)\pi}{4q}\right)^{\rho-1} \operatorname{Res}_{s=\rho} D_{m,n}(s)\ \frac{\Gamma(s-\rho)\Gamma(s+\rho-1)}{\Gamma(s-\frac{1}{2}r)\Gamma(\rho+\frac{1}{2}r)}$$

Proof: Follows by taking the residue at $s=\rho$ in (3.3.2), and by noting that $\rho\varepsilon(\frac{1}{2},1]$ and so is a real number. //

Proof of Proposition 3.3.4: Writing $E_j(z,s)=\sum_{n=-\infty}^{n=\infty} B_{n,j}(y,s)e(\frac{n-\alpha}{q}x)$

as before, we have, as in Proposition 3.3.2, with $\beta=\frac{m-\alpha}{q}$

$$\langle P_m(\cdot,s),E(\cdot,t)\rangle=\overline{\langle E(\cdot,t),P_m(\cdot,s)\rangle}=\int_0^\infty \overline{B_{m,j}(y,s)}\exp(-2\pi\beta y)y^{s-2}dy$$

$$=\int_0^\infty \frac{\overline{-\pi(\frac{\pi(m-\alpha)}{4q})^{t-1}}}{i^r}\ \frac{\overline{D_{m,j}(t)}}{\Gamma(s+\frac{1}{2}r)}\ \exp(-2\pi\beta y)\ y^{s-2}\ dy$$

$$= -\pi(\frac{\pi(m-\alpha)}{4q})^{\overline{t}-1}\ \frac{i^r\ D_{m,j}(\overline{t})}{\Gamma(t+\frac{1}{2}r)}\ \int_0^\infty W_{\frac{1}{2}r,\overline{t}-\frac{1}{2}}(4\pi\frac{|m-\alpha|}{q}y)\ \exp(-2\pi\frac{(m-\alpha)}{q}y)y^{s-2}dy$$

as all function are Real analytic.

We now require:

Theorem 3.3.1 (Barnes)

$$W_{a,b}(y)=\frac{\exp(-\frac{1}{2}y)}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(w-a)\Gamma(-w-b+\frac{1}{2})\Gamma(-w+b+\frac{1}{2})}{\Gamma(-a-b-\frac{1}{2})\Gamma(-a+b+\frac{1}{2})}\ y^w\ dw$$

Proof: [Whittaker and Watson §16.4] //

We have: $W_{\frac{1}{2}r,\overline{t}-\frac{1}{2}}(4\pi\beta y)=$

$$\frac{\exp(-2\pi\beta y)}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(w-\frac{1}{2}r)\Gamma(-w-\overline{t}+1)\Gamma(-w+\overline{t})}{\Gamma(-\frac{1}{2}r-\overline{t}+1)\Gamma(-\frac{1}{2}r+\overline{t})}\ (4\pi\beta y)^w\ dw$$

Substituting in the above gives: $\frac{-\pi(\frac{\pi\beta}{4})^{\overline{t}-1}\ i^r\ D_{m,j}(\overline{t})}{\Gamma(\overline{t}+\frac{1}{2}r)\Gamma(\overline{t}-\frac{1}{2}r)\Gamma(-\overline{t}-\frac{1}{2}r+1)}$ ^

$$\cdot\ \frac{1}{2\pi i}\int_0^\infty \exp(-4\pi\beta y)\ y^{s-2}\int_{-\infty i}^{\infty i}\Gamma(w-\frac{1}{2}r)\Gamma(-w-\overline{t}+1)\Gamma(-w+\overline{t})(4\pi\beta y)^w\ dw\ dy$$

We interchange the order of integration in the above and note that:

$$\int_0^\infty \exp(-4\pi\beta y)(4\pi\beta y)^w \, y^{s-2} \, dy = \frac{\Gamma(w+s-1)}{(4\pi\beta)^{s-1}}$$

. So we obtain the integral:

$$\frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \Gamma(w-\tfrac{1}{2}r)\Gamma(-w-\bar{t}+1)\Gamma(-w+\bar{t})\Gamma(w+s-1) \, dw$$

$$= \frac{\Gamma(-\tfrac{1}{2}r-\bar{t}+1)\Gamma(\bar{t}-\tfrac{1}{2}r)\Gamma(s-\bar{t})\Gamma(s+\bar{t}-1)}{\Gamma(s-\tfrac{1}{2}r)}$$

by <u>Barne's Lemma</u> [Whittaker and Watson, 14.52].

The result now follows directly //

<u>Proof of Proposition 3.3.5</u>: This is similar to the previous proof, as we have:

$$\langle P_m(\cdot,s),u_j\rangle = \bar{a}_{m,j} \int_0^\infty W_{\frac{1}{2}r,\sqrt{\frac{1}{4}-\lambda_j}}\left(4\pi\frac{(m-\alpha)}{q}y\right)\exp\left(-2\pi\frac{m-\alpha}{q}\,y\right) y^{s-2} \, dy$$

$$= \bar{a}_{m,j} \left(4\pi\,\frac{m-\alpha}{q}\right)^{1-s} \frac{\Gamma(s-\sqrt{\frac{1}{4}-\lambda_j}-\frac{1}{2})\Gamma(s+\sqrt{\frac{1}{4}-\lambda_j}-\frac{1}{2})}{\Gamma(s-\frac{1}{2}r)}$$

By the same computation as above. //

Putting the above three results together yealds:

<u>Theorem 3.3.2</u>

$P_m(z,s)$ has the spectral decomposition:

$$P_m(z,s)=\left(4\pi\frac{m-\alpha}{q}\right)^{1-s}\frac{1}{\Gamma(s-\frac{1}{2}r)}\cdot\Big\{\sum_{j>0}\bar{a}_{m,j}\,\Gamma(s-\sqrt{\tfrac{1}{4}-\lambda_j}-\tfrac{1}{2})\Gamma(s+\sqrt{\tfrac{1}{4}-\lambda_j}-\tfrac{1}{2})\;u_j(z)$$

$$+\frac{\pi i^r}{4}\sum_{j=1}^{h}\int_{-\infty}^{\infty}\left(\frac{m-\alpha}{4q}\pi\right)^{-\frac{1}{2}-it}\frac{D_{m,j}(\frac{1}{2}-it)\Gamma(s-\frac{1}{2}+it)\Gamma(s-\frac{1}{2}-it)}{\Gamma(\frac{1}{2}-it+\frac{1}{2}r)}E(z,\tfrac{1}{2}+it)\,dt$$

$$+\;\pi i^r\sum_{j=1}^{\gamma}\left(\frac{m-\alpha}{4q}\pi\right)^{\rho_j-1}\frac{\Gamma(s-\rho_j)\Gamma(s+\rho_j-1)}{\Gamma(\rho_j+\frac{1}{2}r)}\operatorname*{Res}_{s=\rho_j}D_{m,\nu_j}(s)\;\theta_j(z)\Big\}$$

If Δ_r has no negative eigenvalues in $M(r,\chi,G)$.

CHAPTER IV

APPLICATIONS TO EXPONENTIAL SUMS

4.1 THE DEDEKIND η FUNCTION AND MULTIPLIER SYSTEMS

The classical Dedekind η function is given by:

$$\eta(z)=e(z/24)\prod_{n=1}^{\infty}(1-e(nz)), \quad \text{for} \quad z\in H. \tag{4.1.1}$$

If we recall that the Dedekind sum is given by:

$$s(d,c)=\sum_{j=1}^{c}((j/c))\cdot((jd/c)), \quad \text{where} \quad c,d\in Z \quad \text{and} \quad ((x))=x-[x]-\tfrac{1}{2},\ x\in R.$$

Then we have the following transformation law:

Theorem 4.1.1 (Dedekind)

Let $\log z$ denote the principal branch of log, then for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,Z)$ we have:

$$\log(\eta(gz))=\begin{cases} \log \eta(z)+\tfrac{1}{2}[\log(\frac{cz+d}{i})+2\pi i(\frac{a+d}{12c}-s(d,c))], & \text{for} \quad c\neq 0 \\ \log \eta(z)+\tfrac{1}{2}(\frac{2\pi ib}{12}), & \text{for} \quad c=0 \end{cases} \tag{4.1.2}$$

Proof: There are many proofs of this, an elegant one given in [Siegel].

//

We will show that $\eta^{2r}(z)=\exp(2r\log\eta(z))$ satisfies an automorphy condition: $\eta^{2r}(gz)=\chi_r(g)\, j_r(g,z)\, \eta^{2r}(z)$, Where $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,Z)$, and $j_r(g,z)=(cz+d)^r$.

For this we require:

Theorem 4.1.2:

Let $g_1, g_2, g_3 \varepsilon SL(2,R)$, where $g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, $j=1,2,3$.

and $j_1(g_j, z) = c_j z + d_j$. Then for $g_3 = g_2 g_1$ we have:

(4.1.3) $\log(j_1(g_1,z)) + \log(j_1(g_2, g_1 z)) = \log(j_1(g_3,z)) + 2\pi i W(g_2,g_1)$, where

$$W(g_2,g_1) = \begin{cases} \frac{1}{4}[\mathrm{sgn}(c_1)+\mathrm{sgn}(c_2)-\mathrm{sgn}(c_3)-\mathrm{sgn}(c_1c_2c_3); & c_1c_2c_3 \neq 0 \\ -\frac{1}{4}(1-\mathrm{sgn}(c_1))(1-\mathrm{sgn}(c_2)); & c_1c_2 \neq 0,\ c_3=0 \\ \frac{1}{4}(1+\mathrm{sgn}(c_1))(1-\mathrm{sgn}(d_2)), & c_1c_3 \neq 0, c_2=0 \\ \frac{1}{4}(1-\mathrm{sgn}(a_1))(1+\mathrm{sgn}(c_2)); & c_2c_3 \neq 0, c_1=0 \\ \frac{1}{4}(1-\mathrm{sgn}(a_1))(1-\mathrm{sgn}(d_2)); & c_1=c_2=c_3=0 \end{cases}$$

Proof: [Maass] Theorem 16, p.115

//

Corollary 4.1.1

Let the notation be as in Theorem 4.1.2, and define

$\sigma(g_2,g_1) = e(r \cdot W(g_2,g_1))$, then we have:

(4.1.4) $j_r(g_1,z) j_r(g_2,g_1) = j_r(g_3,z)\ \sigma(g_2,g_1)$

Proof: It is seen that (4.1.4) is obtained by $e(r\cdot)$ of equation (4.1.3).

//

Proposition 4.1.1.

Let $f(z)=\exp(2r\log \eta(z))$, then for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)$

$f(gz)=\chi_r(g)\, j_r(g,z)\, f(z)$, where

$$(4.1.5) \qquad \chi_r(g)=\begin{cases} e[\, r(\,-1/4 + \frac{a+d}{12c} - s(d,c))] \;, & c>0 \\ e(rb/12), & \text{if } c=0 \end{cases}$$

Proof: This follows by applying $e(2r\cdot)$ to equation (4.1.2) and appealing to Corollary 4.1.1 to get that: $(\frac{cz+d}{i})^r=(cz+d)^r e(-r/4)$, where $i^{-1}=e(-1/4)$ and $c>0$. As we have identified g and $-g$, we can always choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \geq 0$. //

Corollary 4.1.2

Let G be a congruence subgroup of SL(2,Z), r a real number then χ_r given by (4.1.5) is a multiplier system of weight r for G.

Proof: By restriction, we see that (4.1.5) holds for any $g \in G$, a congruence subgroup of SL(2,Z). Now we see from (4.1.1) that $\eta(z)\neq 0$ for $z \in H$, therefore $\eta^{2r}(z)=f(z)\neq 0$, for $z \in H$. Therefore Corollary 2.1.1 implies that χ_r is a multiplier system of weight r for G.

//

4.2 APPLICATIONS TO DEDEKIND AND KLOOSTERMAN SUMS

We recall that for G a congruence subgroup we have $G_\infty = \{\begin{pmatrix} 1 & nq \\ 0 & 1 \end{pmatrix} | n \varepsilon Z\}$, where $q=q(G)>0$ is uniquely defined. Furthermore it is clear that $q(\Gamma(N))=N$, where

$$\Gamma(N)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL(2,Z) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\}, \quad N \varepsilon \mathbb{N}.$$

Let us now fix $r=h/k \varepsilon Q$, with $h,k \varepsilon N$. We let χ_r denote the multiplier system of weight r for $\Gamma(12k)$ given in Proposition 4.1.1:

$$\chi_r(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{cases} e(r(-\tfrac{1}{4} + \frac{a+d}{12c} - s(d,c))), & \text{for } c > 0 \\ e(rb/12), & \text{if } c=0 \end{cases}$$

We will be studying the Generalized Kloosterman sum:

$$S(m,n,c,\chi_r,\Gamma(12k)) = \sum_{\substack{0<d<12kc \\ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon \Gamma(12k)}} \overline{\chi_r(g)}\, e\left(\frac{am+dn}{12kc}\right)$$

where we note that $\chi_r(\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}) = e(qr/12) = e(12kh/12k) = e(h) = 1 = e(0)$ as $q=q(\Gamma(12k))=12k$, and therefore we have $\alpha(\chi_r, \Gamma(12k))=0$.

We immediately have:

Theorem 4.2.1

Let m,n,c be positive integers, then:

$$(4.2.1)\quad S(m,n,c,\chi_r,\Gamma(12k))=e(\tfrac{1}{4}r)\sum_{\substack{0<d<12kc\\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma(12k)}} e\left(\frac{a(m-h)+d(n-h)}{12kc}+\frac{hs(d,c)}{k}\right)$$

$$(4.2.2)\quad =e(\tfrac{1}{4}r)\sum_{\substack{0<d<12kc\\ (d,c)=1\\ d\equiv 1 (\mathrm{mod}\ 12k)\\ c\equiv 0\ (\mathrm{mod}\ 12k)\\ a\cdot d\equiv 1\ (\mathrm{mod}\ c)}} e\left(\frac{a(m-h)+d(n-h)}{12kc}+\frac{hs(d,c)}{k}\right)$$

<u>Proof</u>: Substituting the explicit formula for χ_r in the Kloosterman sum, and noting that $c>0$ gives $\tfrac{1}{4}[(1-\mathrm{sgn}(c))(\mathrm{sgn}(d)-1)]=0$, and we have $S(m,n,c,\chi_r,\Gamma(12k))$

$$=\sum_{\substack{0<d<12kc\\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma(12k)}} e\left(-\frac{h}{k}\left[(-\tfrac{1}{4})+\frac{(a+d)}{12c}-s(d,c)\right]+\frac{am+dn}{12kc}\right)$$

and the first equation (4.2.1) follows.

We obtain (4.2.2) from (4.2.2) by noting that $\begin{pmatrix}*&*\\c&d\end{pmatrix}\in\Gamma(12k)$ iff $c\equiv 0$ (mod 12k), $d\equiv 1$ (mod 12k) and $(c,d)=1$; also $ad-bc=1$ implies that $ad\equiv 1$ (mod c).

//

<u>Corollary 4.2.1</u>

$$S(h,h,c,\chi_{h/k},\Gamma(12k))= e(\tfrac{1}{4}\,\frac{h}{k})\sum_{\substack{0<d<12kc\\ c\equiv 0\ (\mathrm{mod}\ 12k)\\ d\equiv 1\ (\mathrm{mod}\ 12k)\\ (c,d)=1}} e\left(\frac{h}{k}\,s(d,c)\right)$$

We now apply Theorem 1.4,p.10, to $S(m,n,c,\chi_r,\Gamma(12k))$ and get:

Theorem 4.2.2

$$\sum_{0<c<x} \frac{S(m,n,c,\chi_r,\Gamma(12k))}{c} = \sum_{\lambda_j \varepsilon \Lambda} A_j x^{\tau_j} + \mathcal{O}(x^{\beta/3+\varepsilon}) \tag{4.2.3}$$

for any $\varepsilon>0$. Where Λ is the set of exceptional eigenvalues $0<\lambda_j<\frac{1}{4}$ associated to $M(r,\chi_r,\Gamma(12k))$, $\tau_j=2\sqrt{\frac{1}{4}-\lambda_j}$ and A_j's are constants. Also $= \operatorname{Inf}_{\delta>0} \{ \sum_{c>0} \frac{|S(m,n,c,\chi_r,\Gamma(12k))|}{c^{1+\delta}} <\infty\}$, so $\beta\leqslant 1$.

Corollary 4.2.2

For $0<r<2$, and for every $\varepsilon>0$, we have:

$$\sum_{0<c<x} S(m,n,c,\chi_r,\Gamma(12k)) = \mathcal{O}(x^{\max\{ 2\sqrt{\frac{1}{4}-(\frac{1}{2}r(1-\frac{1}{2}r)}, 1/3\} +\varepsilon})$$

in particular, for $2/3<r<4/3$ this gives:

$$\sum_{0<c<x} \frac{S(m,n,c,\chi_r,\Gamma(12k))}{c} << x^{1/3+\varepsilon} \quad \text{for any} \quad \varepsilon>0.$$

Proof: This follows from Roelcke's result [Roelcke], which gives that $\lambda> \frac{1}{2}r(1-\frac{1}{2}r)$ for any $\lambda\varepsilon\Lambda$, and from the trivial estimate $\beta \leq 1$.

//

We now let $r=\frac{h}{k}$ be an arbitrary rational number, $h,k\in N$. We examine in detail the case $m=n=h$ in Theorem 4.2.2.

Proposition 4.2.1

There is a $\delta>0$ such that:

$$\sum_{0<c<x} \frac{S(h,h,c,\chi_r,\Gamma(12k))}{c} << x^{1-\delta} \tag{4.2.4}$$

Proof: We note that $0<\lambda<\frac{1}{4}$ implies that $\tau= 2\sqrt{\frac{1}{4}-\lambda} <1$, so as Λ is a finite set, by Theorem 3.2.2, we have $\mu= \underset{\lambda_j\varepsilon\Lambda}{\mathrm{Max}}\{\tau_j\}< 1$. Let $\mu'= \mathrm{Max}\{\mu,1/3\}<1$, then we see from Theorem 4.2.2 that $\delta= \frac{1-\mu'}{2}$ will give us the required estimate, if we choose $\varepsilon<\delta$. //

To prove our next result we require the following well known technique.

Lemma 4.2.1 (Partial Summation)

Let $f(t)$ be a continuously differentiable function for $1\leq t<x$, and $A(x)=\sum_{0<n<x} a(n)$ then:

$$\sum_{1\leq n<x} a(n)f(n)= A(x)f(x)-A(1)f(1)- \int_1^x A(t)f'(t)dt$$

Proof: A proof can be found in [Apostol] p.77. //

Proposition 4.2.2

There is a $\delta'>0$ such that:

$$\sum_{0<c<x} S(h,h,c,\chi_r,\Gamma(12k)) \ll x^{2-\delta'} \tag{4.2.5}$$

Proof: We apply partial summation with $a(n)= S(h,h,c,\chi_r,\Gamma(12k))$, $f(t)= t$. So there is a constant K with $A(x)<K\cdot x^{1-\delta}$, by Proposition 2.4.1. We therefore have:

$$\sum_{0<c<x} S(h,h,c,\chi_r,\Gamma(12k))= \sum_{0<c<x} a(c)f(c)= A(x)\cdot x - 0\cdot 0+ \int_1^x A(t)\cdot 1dt$$

$$\ll K\cdot x^{1-\delta}\cdot x+ \int_1^x K\cdot t^{1-\delta}\cdot dt = K(1+ \frac{1}{2-\delta})x^{2-\delta} \ll x^{2-\delta}.$$ And we can take $\delta'=\delta <2$.

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To prove our concluding Theorem, we will require:

Theorem 4.2.3 (Weyl)

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of complex numbers of absolute value one, then the sequence $\{\xi_j\}_{j=1}^{\infty}$ is equidistributed iff for every $m\in N$, we have:

$$\sum_{j<x} \xi_j^m = o(x)$$

Proof: A proof of this is given in [Polya-Szego] Part II,N°164.

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Theorem 4.2.4

Let $\{x\} = x-[x]$ denote the fractional part of x, $x \in R$.

The sequence of fractional parts

$$\left\{\left\{\frac{h\, s(d,c)}{k}\right\}\right\}_{\substack{c>0 \\ c\equiv 0 \pmod{12k} \\ 0<d<12kc \\ (d,c)=1,\ d\equiv 1 \pmod{12k}}}$$

is equidistributed on $[0,1)$.

Proof: We first note that this is equivalent to the statement :

that the sequence $$\left\{e\left(\frac{h\, s(d,c)}{k}\right)\right\}_{\substack{c>0 \\ c\equiv 0 \pmod{12k} \\ 0<d<12kc \\ (d,c)=1,\ d\equiv 1 \pmod{12k}}}$$

is equidistributed on the unit circle T.

We next note that: $$\sum_{\substack{0<c<x,\, c\equiv 0 \pmod{12k} \\ 0<d<12kc,\ (d,c)=1 \\ d\equiv 1 \pmod{12kc}}} 1 \geqslant \sum_{\substack{0<c<x \\ c\equiv 0 \pmod{12k}}} \phi(c)$$

where $\phi(n) = \sum_{(j,n)=1} 1$ is Euler's ϕ function.

Now it is well known that $$x^2 \ll \sum_{\substack{0<c<x \\ c\equiv 0 \pmod{12k}}} \phi(c)$$

for example in [Apostol] Theorem 3.7.

If we now let m be a positive integer, and we combine Corollary 4.2.1,

Proposition 4.2.2

$$e(\tfrac{1}{4}\tfrac{mh}{k}) \sum_{\substack{0<c<x \\ c\equiv 0 \pmod{12k} \\ 0<d<12kc,\,(d,c)=1 \\ d\equiv 0 \pmod{12k}}} e\left(\frac{mh\ s(d,c)}{k}\right) = \sum_{0<c<x} S(hm,hm,\chi_r,\Gamma(12k)) \ll x^{2-\delta}$$

for a $\delta>0$.

Combining this with:

$$x^2 \ll \sum_{\substack{0<c<x \\ c\equiv 0 \pmod{12k} \\ 0<d<12k,\,(d,c)=1 \\ d\equiv 0 \pmod{12k}}} 1$$

by above, we see that the sequence

$$\left\{ e\left(\frac{h\ s(d,c)}{k}\right) \right\}_{\substack{c>0 \\ c\equiv 0 \pmod{12k} \\ 0<d<12kc,\,(d,c)=1 \\ d\equiv 1 \pmod{12k}}}$$

satisfies the conditions of Weyl's criterion Theorem 4.2.3, and is therefore equidistributed.

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