15.070 Midterm Exam

Date: October 24, 2005

Problem 1 (15 points) *For the following questions/statements just give TRUE or FALSE answers.* ***Do not*** *derive the answers.*

Given two distinct *probability measures* μ_1, μ_2 *on a sample space/*σ*-field pair* $(\Omega, \mathcal{F})$.

- ***A***. *Define* $\nu(A) = \min(\mu_1(A), \mu_2(A))$ *for every* $A \in \mathcal{F}$. *Then* ν *is also a probability measure.*

- ***B***. *Define* $\nu(A) = .5\mu_1(A) + .5\mu_2(A)$ *for every* $A \in \mathcal{F}$. *Then* ν *is also a probability measure.*

- ***C***. *There exist sets* A, B *such that* $\mu_1(A) < \mu_2(A)$ *and* $\mu_2(B) < \mu_1(B)$

Problem 2 (25 points) *On a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *consider a sequence of random variables* $X_1, X_2, \ldots, X_n$ *and* σ*-fields* $\mathcal{F}_1, \ldots, \mathcal{F}_n \subset \mathcal{F}$ *such that* $\mathbb{E}[X_j|\mathcal{F}_{j-1}] = X_{j-1}$ *and* $\mathbb{E}[X_j^2] < \infty$.

- ***A***. *Prove directly (without using Jensen's inequality) that* $\mathbb{E}[X_j^2] \geq \mathbb{E}[X_{j-1}^2]$ *for all* $j = 2, \ldots, n$. *HINT: consider* $(X_j - X_{j-1})^2$.

- ***B***. *Suppose* $X_n = X_1$ *almost surely. Prove that in this case* $X_1 = \ldots = X_n$ *almost surely.*

Problem 3 (25 points) *Given a standard Brownian motion* $B(t)$ *consider the following process on* $t \in [0, 1]$ *which is called* Brownian Bridge*:* $W(t) = B(t) - tB(1), t \in [0, 1]$.

- ***A***. *Compute the covariance* cov$(W(s)W(t))$ *for every* $0 \leq s < t \leq 1$.

- ***B***. *Show that* $W(t)$ *and* $B(1)$ *are independent random variables for every* $t \in [0, 1]$. *You may use the fact that two normal random variables are independent if and only if their covariance is zero.*

Problem 4 (35 points) *Let* B *be a standard Brownian motion. Let* $S = \{t : B(t) = 0\} \subset \mathbb{R}_+$ *- zero set of a Brownian motion. Prove that* S *is almost surely an unbounded infinite set.*

HINT: use the fact that you know the distribution of $M(t) = \sup_{0 \leq s \leq t} B(s)$.