15.070 Advanced Stochastic Processes

Final Exam

Dec 18, 2005

Problem 1 Let $N(t) = B^3(t) - 3tB(t)$, where $B(t)$ is a standard Brownian motion.

(a) Find Ito representation of $dN(t)$ using Ito formula.

(b) Establish that $N(t)$ is a martingale.

Problem 2 Consider an Ito process

$$X(t,\omega) = X(0,\omega) + \int_0^t u(s)ds + \int_0^t V(s,\omega)dB(s,\omega),$$

where $u \in H^2$ is deterministic function and $V \in H^2$ is a random process. Establish that Ito representation is unique. That is if $X(t,\omega) = X(0,\omega) + \int_0^t u'(s)ds + \int_0^t V'(s,\omega)dB(s,\omega)$ for some deterministic function $u' \in H^2$ and a random process $V' \in H^2$, then $u'(t) = u(t)$ for almost all t and $V'(t,\omega) = V(t,\omega)$ almost surely for all almost all t.

Note: for this problem you may assume that if two integrable functions $f_1, f_2 : [0,\infty) \rightarrow \mathcal{R}$ are such that $\int_0^t f_1(s)ds = \int_0^t f_2(s)ds$ for all t, then $f_1(s) = f_2(s)$ for almost all s.

Problem 3 One of the equivalent definitions of weak convergence (Portmentau Theorem) states that $X_n \Rightarrow X$ if and only if $\lim_{n\rightarrow\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$ for every bounded continuous function f. Let X_n be a random sequence in $C[0,T]$ and let $\mathbf{0}$ stand for zero function $x(t) = 0, t \in [0,T]$.

(a) Use this definition to establish formally the following fact: $X_n \Rightarrow \mathbf{0}$ implies $\|X_n\|_T \rightarrow 0$ in probability. Namely, for every $\epsilon > 0$, $\lim_n \mathbb{P}(\|X_n\|_T > \epsilon) = 0$. HINT: construct a bounded continuous function out of the norm $\|\cdot\|_T$.

(b) Another equivalent definition of weak convergence states that $X_n \Rightarrow X$ if and only if for every open set U, $\mathbb{P}(X \in U) \le \liminf_n \mathbb{P}(X_n \in U)$. Use this definition to establish formally the following fact: $\|X_n\|_T \rightarrow 0$ in probability implies $X_n \Rightarrow \mathbf{0}$.

Problem 4 Let S_n be a symmetric random walk starting at 0. Namely, $S_n = \sum_{1\le k\le n} X_k$, where $X_k = \pm 1$ with probability $1/2$ and is i.i.d. Let m be a positive integer. Define $T = \min\{n : |S_n| \ge m\}$.

(a) Establish that $S_n^2 - n$ is a martingale and use it to prove $\mathbb{E}[T] = m^2$.

(b) Find constants b, c so that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale and use it to compute $\mathbb{E}[T^2]$.

Problem 5 Let $X_n, n \ge 1$ be a sequence of real valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \to X$ almost surely, for some $X : \Omega \to \mathbb{R}$. Establish that there exists $Y : \Omega \to \mathbb{R}$ such that (i) Y is measurable, (ii) $Y = X$ almost surely.

Problem 6 Consider a production facility were finished goods can be produced on average 40 items per minute. The demand for the goods is random and average rate of demand is 20 items per minute. The production facility works only when there is some unfulfilled demand and idles otherwise. Namely, it does not produce goods in advance. The inventory at time zero is 2000 items.

(a) Considering a fluid model approximation for this process, which of the statements is more likely to be true (a) the facility was operating without idling for the first one hour of operation, or (b) the facility idled for at least 10 minutes during the first one hour of operation.

(b) Based on a fluid model approximation estimate time t such that by time t the facility idled for 10 minutes.