14.05: SECTION HANDOUT #4
CONSUMPTION (AND SAVINGS)

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1. Motivation

In our study of economic growth we assumed that consumers saved a fixed (and exogenous) fraction of their income. Then we were not considering the fact that agents can probably do better than that.

Why is this restrictive? Imagine you are told you have an income of \$1 today and that this will grow each month at a 1% rate. What would you do? What if I tell you that your income today is \$1 but it will grow at a rate of 20%? Is your decision the same?

The main idea behind consumption/savings is that the agent maximizes utility deciding how much to consume each period. Consumption/savings theory can be used to analyze many real world issues, including some "hot topics" such as retirement/pensions.

In this handout I present the basic single agent problem. This stylized model will help us understand what variables affect the consumption and savings of people. We will see that this problem looks relatively similar to an standard utility maximization problem, but we need to introduce some extra elements to incorporate the intertemporal dimension in the model.

In this handout we first lay down a more general model, and then we move to analyze the different pieces of the model, using slightly simplified versions of it. There are two elements here: (a) consumption smoothing, and (b) the role of the interest rate. We will deal with them separately.

2. A General Framework

If we want to understand savings, why are we studying consumption? Savings is the part of your income left after you decide how much to consume today, thus savings and consumption are just two sides of the same coin.

The benefit from saving is that we have more resources to consume in periods where for some reason we do not have enough income to buy as many goods and services as we may want to. So, as the benefit is closely related to consumption in different periods of time, we should be able to understand savings as part of a problem where a person decides how much to consume each period. Then, given a certain income profile, we can compute savings on each period. Let us proceed this way and we will see if we obtain results that actually make sense.

2.1. **A Model.** Consider an agent with the following preferences:

$$U = \sum_{t=1}^{T} \frac{1}{(1+\delta)^t} u(C_t), \quad u'(\cdot) > 0,\ u''(\cdot) < 0,\ \delta > 0, \tag{1}$$

where c_t is consumption in period t and δ is the subjective discount rate.[12]

But we still miss the other "half" of the problem. In the simple consumer optimization problem we use a budget constraint to describe the possible consumption bundles. Here we need the same, we need a budget constraint to describe the feasible combinations of consumption in each period; given the "intertemporal" nature of the problem, we call it the "*intertemporal budget constraint.*"

2.1.1. *Deriving the Intertemporal Budget Constraint.* Imagine that there are only two periods: today (1) and tomorrow (2), and you have no assets (wealth) to begin with. Then, we can write savings in period 1 (S_1) as

$$S_1 = Y_1 - C_1. \tag{2}$$

Now, focus on "tomorrow". At the beginning of the period the agent's total resources available for consumption are income (Y_2) plus savings from the previous period including the interest earned ($S_1(1+r)$). As this is the final period, it is not optimal for the agent to leave any resources[3], thus consumption in period 2 is given by

$$\begin{aligned} C_2 &= Y_2 + (1+r)S_1 \\ &= Y_2 + (1+r)(Y_1 - C_1) \end{aligned} \tag{3}$$

rearranging terms in (3) and using equation (2) we obtain

$$C_1 + \frac{C_2}{(1+r} = Y_1 + \frac{Y_2}{(1+r)} \tag{4}$$

Equation (4) is the intertemporal budget constraint in this two period case, it says that the present value of consumption has to be equal to the present value of income over the agent's lifetime, which equals 2 periods in this case.[4] This last interpretation actually allows us to generalize the formula to the case with T periods and initial wealth different from 0. Then, the agent's intertemporal budget constraint is

$$\sum_{t=1}^{T} \frac{1}{(1+r)^t}\ C_t = A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t}\ Y_t \tag{5}$$

This is just a generalization of equation (4). It is worth mentioning two assumptions made when deriving equation(5). First, if you look at equation (2), we did not impose any condition on S_1, this means that the agent can take as much debt as he wants (this means no debt limits, no liquidity constraints). Second, the interest rate is the same for positive and negative savings ("dissaving"). These are simplifications as we

[1] This way to write the intertemporal utility function is said to display additive separable utility and exponential discounting.

[2] Notice that the book uses ρ in the OLG models.

[3] We assume that the agent cannot take debt this period.

[4] Notice that we rule out the chance of taking debt in period 2 to expand the consumption opportunities; we will maintain this assumption unless explicitly noted.

all know, but we will see that in spite of them we are still able to do a good job analyzing consumption and savings.

2.1.2. *How does the optimum looks like?* We now have all the ingredients we need for the recipe. The agent chooses a consumption path, $C_1, C_2, \ldots, C_T$, to maximize (1) subject to the intertemporal budget constraint (5). This problem can be written in the following way using the Lagrangian

$$\max_{\{C_1,\ldots,C_T\}} \sum_{t=1}^{T} \frac{1}{(1+\delta)^t} u\left(C_t\right) + \lambda \left(A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t}\ Y_t - \sum_{t=1}^{T} \frac{1}{(1+r)^t}\ C_t \right).$$

The first order conditions are

$$\frac{u'(C_t)}{(1+\delta)^t} = \frac{\lambda}{(1+r)^t}, \qquad \forall t. \tag{6}$$

Notice that in this problem the agent cares only about the present value of income and not about the exact profile. Given that, let me define

$$\overline{W} \equiv A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t}\ Y_t \tag{7}$$

as the lifetime wealth of the agent. A quick inspection of equation (5) shows that in fact $\overline{W}$ summarizes all the information for the agent needs to solve the utility maximization problem.[5]

I claimed before that this optimization problem can be interpreted in (almost) the same way as any standard utility maximization problem you have seen before. To see this, let me rewrite equation (5) in the following way

$$\sum_{t=1}^{T} p_t\ C_t = \overline{W} \tag{8}$$

where I made use of equation (7), and $p_t \equiv (1+r)^{-t}$ is the price of consumption in period t (C_t) in terms of unit of consumption in period 0. Equation (8) should look familiar and it is basically the same type of budget constraint you have seen many times before. It is important to understand that the price of future consumption p_t is a decreasing function of the interest rate r.

Going back to the optimum consumption decision. Take equation (6) for t and $t+1$, and use one to substitute for λ in the other to obtain

$$\begin{aligned} \frac{\frac{u'(C_t)}{(1+\delta)^t}}{\frac{u'(C_{t+1})}{(1+\delta)^{t+1}}} &= (1+r) \\ \frac{u'(C_t)}{u'(C_{t+1})} &= \left(\frac{1+r}{1+\delta}\right). \end{aligned}$$

This last equation, the *Euler equation*, tells us that in the optimum, the ratio of marginal utilities equals the ratio of the gross interest rate $(1+r)$ and one plus the subjective discount rate. Notice that under the assumption that $u(\cdot)$ is the same every period, if $\delta = r$, then consumption is the same for all t.

[5]Note that $\overline{W}$ is a function of r, so the exact income profile does matter when we analyze the effects of changes in the interest rate (see section 4).

Remark 1. *If $r = \delta$, then the agents chooses a flat consumption profile, so consumption is the same every period.*

The Euler equation (equation 9) gives a relation between C_t and C_{t+1}. With this in hand we can find the consumption level using the budget constraint, equation (5). In particular, if $\delta = r$, then the consumption level is

$$\begin{aligned} A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t} Y_t &= \sum_{t=1}^{T} \frac{1}{(1+r)^t} \overline{C} \\ \overline{C} &= \frac{rW}{1 - \left(\frac{1}{1+r}\right)^T}, \end{aligned} \tag{9}$$

where W is defined in equation (7).

3. Consumption Smoothing

Assumption 1. $\delta = r = 0$.[6]

Using assumption 1 we can write the intertemporal utility function (equation (1)) as

$$U = \sum_{t=1}^{T} u(C_t), \qquad u'(\cdot) > 0,\ u''(\cdot) < 0, \tag{10}$$

where c_t is consumption in period t. In this case the budget constraint of the agent (equation (5)) can be written as

$$\sum_{t=1}^{T} C_t = A_0 + \sum_{t=1}^{T} Y_t \tag{11}$$

The Lagrangian for this problem is

$$\pounds = \sum_{t=1}^{T} u(C_t) + \lambda \left(A_0 + \sum_{t=1}^{T} Y_t - \sum_{t=1}^{T} C_t \right) \tag{12}$$

with FOCs

$$u'(C_t) = \lambda, \qquad \forall t. \tag{13}$$

This is the basic idea of consumption smoothing, individuals will choose a consumption path so as to keep a constant marginal utility of consumption. Under our assumptions, the consumption level uniquely determines the marginal utility, then $C_1 = C_2 = \ldots = C_T = \overline{C}$. Using (11) we obtain

$$\overline{C} = \frac{1}{T} \left(A_0 + \sum_{t=1}^{T} Y_t \right). \tag{14}$$

Equation (14) has a very intuitive interpretation. The right hand side corresponds to the *permanent income*, and this is the basic result from the *permanent income*

[6] Setting both the interest rate and the subjective discount rate equal to 0 will help us simplify the math a bit but the main result will still hold.

hypothesis, consumption is determined by the permanent level of income, not by the current level; savings in this model are equal to the difference between the current and the permanent income level. The *life-cycle hypothesis* relates the basic idea of consumption smoothing to the earnings profile, then an individual borrows when young, pays the debt and saves when adult (working age), and undoes this savings when old (particularly after retirement).

Equation(14) allows to compute the marginal propensity to consume out of current income. For this, notice that an extra dollar of income today leads to a $1/T$ increase in consumption today. More persistent changes lead to higher increases, in fact, if we have a one dollar increase for K periods, then the marginal propensity to consume out of current income is K/T. In fact, the propensity to consume out of a permanent change is exactly 1.

The agent prefers to have a consumption profile that is as smooth as possible. This result is the main idea behind "consumption smoothing". Notice that even if $\delta \neq r$ the agent still tries to maintain a consumption profile without abrupt changes, in the sense that even if it is not constant, there are not jumps in the consumption level.

Other important element is the fact that even with $\delta = r = 0$, there still are incentives to save, thus the interest rate is not the unique reason why people save. Savings are used to move income across periods, not necessarily to earn interest income. Of course, with a positive interest rate the price of consumption in different periods is different, but we still have the same logic. This is a very important implication of "our" model.

4. The 2-period Case

Romer's textbook has a section explaining the income and substitution effects of changes in the interest rate. From your microeconomic classes you must remember that the change in the price of one good has both a substitution and an income effect.[7] We already saw in section 2.1 the interest rate determines the price of future consumption, thus changes in the interest rate can also be interpreted as changes in relative prices.

We do not really need our full model (section 2.1) to try to shed light on the mechanism behind these two effects, we just need two periods to do this. When $T = 2$ and $A_0 = 0$ our model simplifies to

$$\max_{\{C_1, C_2\}} \; u(C_1) + \frac{u(C_2)}{(1+\delta)} + \lambda \left(Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r} \right).$$

The first order conditions are

$$\begin{aligned} u'(C_1) &= \lambda \\ \frac{u'(C_2)}{1+\delta} &= \frac{\lambda}{1+r} \end{aligned}$$

and the Euler equation is

$$\frac{u'(C_2)}{u'(C_1)} = \frac{1+\delta}{1+r}. \tag{15}$$

[7] If you do not remember this, make sure you read Romer's section on this, that is all you need to know about this.

4.1. **The Income Effect.** Let us look first at the mechanism behind the income effect. For this, consider again equation (3):

$$C_2 = Y_2 + (1+r)(Y_1 - C_1).$$

Take C_1 as given, and notice that total resources available for consumption in period 2 will increase with r if and only if $(Y_1 - C_1) > 0$, the agent had positive savings from period 1 (equal to assets given that we assumed $A_0 = 0$); of course, the effect is a reduction in total resources if $(Y_1 - C_1) < 0$. Notice that this is true for a given C_1, and reflects that there is an effect on total resources.

4.2. **The Substitution Effect.** The substitution effect should be clear if we look at equation (15). For simplicity assume that initially $R = \delta$, and thus $C_1 = C_2$, and think of the case of an increase in the interest rate r. If the agent doesn't change the consumption levels the right hand side of the Euler equation will be smaller, and thus the agent would now choose consumption levels such that $C_1 < C_2$, because the rise in the interest rate makes consumption in the second period less expensive in terms of consumption in period 1.[8] However, we just have a condition on the relative levels of both consumptions, but we have not determined the absolute levels of both. For that we need the budget constraint.

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[8]This is true because we assumed that $u(\cdot)$ is the same in every period and is strictly concave.