14.05: Section Handout #1
Solow Model

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September 16, 2005

Today we will review the basic elements of the Solow model. Be prepared to ask any questions you may have about the derivation of the model, most of the equations we will cover today will be intensively used later in this part of the class.

1 The Economy

We shall first discuss the assumptions used

(a) The economy is closed and there is no government sector, thus the national income identity can be written as:

$$Y_t = C_t + I_t \tag{1}$$

where Y_t is GDP, C_t is consumption, and I_t is investment.
There is some evidence that on average savings and investment tend to move together even in economies that are open, somewhat validating the close economy assumption.[1]

(b) Production is described by a function

$$Y_t = F\left(K_t, A_t L_t\right) \tag{2}$$

which is assumed to be neoclassical. Note the fact that there are three factors: capital (K_t), labor (L_t), and technology (A_t)[2], the last is an intangible factor, but we are certain that it differs from country to country (it is hard to think that Kenya has access to the same technology as the US).
Some properties of the neoclassical production function:

- (i) exhibits constant returns to scale in capital and labor (function is homogeneous of degree one in K and L),
- (ii) marginal product are decreasing (meaning that the second derivatives are negative), and

[1] The most widely cited work on this area is the one by Feldstein and Horioka (1980), even if this apparent puzzle is still unanswered, many authors have found partial explanations for this result. You can find more about this on Romer's textbook.

[2] Strictly speaking $F()$ is also technology, but I will refer to A as technology for simplicity.

(iii) satisfies the *Inada* conditions:

$$\lim_{K\to\infty}\frac{\partial F}{\partial K}=0 \qquad \lim_{K\to 0}\frac{\partial F}{\partial K}=\infty$$
$$\lim_{L\to\infty}\frac{\partial F}{\partial L}=0 \qquad \lim_{L\to 0}\frac{\partial F}{\partial L}=\infty$$

(c) Constant savings rate s; the consumption decision is not explicitly modelled with optimizing consumers (no utility function is explicitly introduced in the model), but as a simplification it is assumed that they follow a simple rule: save s cents out of each dollar on income in every period. Under this assumption we obtain the following equations:

$$\begin{aligned} C_t &= (1-s)\,Y_t && (3)\\ sY_t &= I_t && (4)\end{aligned}$$

where for the last one we also used the fact that the economy is closed. We have just stated that consumption is proportional to GDP (income) and that at the aggregate level, investment is also directly proportional to GDP.

(d) Constant depreciation rate δ.
Firms invest for two reasons:

(i) To increase the stock of capital available for production

(ii) To replace units of capital that deteriorated with the use.[3]

Investment that increases the capital stock is called *"net investment"*, and *"gross investment"* is equal to the sum of net investment and depreciation.
Denoting the net change in capital as $\dot{K} \equiv \frac{dK}{dt}$, we can derive the following expressions:

$$\begin{aligned} I_t &= \dot{K}_t + D_t\\ F(K_t, A_tL_t) &= C_t + I_t\\ &= (1-s)\,F(K_t, A_tL_t) + \dot{K}_t + \delta K_t\\ \text{rearranging terms}&\\ \dot{K}_t &= sF(K_t, A_tL_t) - \delta K_t && (5)\end{aligned}$$

Notice that equation (5) tells us exactly how much capital we will have tomorrow for any combination of K, L, and A today.

(e) Constant population growth and labor equal to population. The population growth rate is denoted by n.

$$\frac{\dot{L}_t}{L_t}=n \qquad (6)$$

[3] An alternative reason is that they become obsolete, mostly because better ones are available now . . . like computers nowadays.

In general, when talking about growth we want to focus on a measure that corrects for the "size" of the economy. In particular, product per capita and/or per worker are the most common measures. The simplest version of the Solow model considers a constant and *exogenous* population growth rate, plus the simplifying assumption that everybody is employed (thus we do not care much about employment and participation rates).
With this assumption, take equation (5) and divide both sides by $A_t L_t$ to obtain

$$\frac{\dot{K}_t}{A_t L_t} = s\frac{F\left(K_t, A_t L_t\right)}{A_t L_t} - \delta\frac{K_t}{A_t L_t} \tag{7}$$

Now, let us denote by $x_t \equiv X_t / L_t$ per equivalent labor unit variables. Then, we can take advantage of the constant returns to scale to re write the production function:

$$\begin{aligned} y &\equiv \frac{Y}{AL} = \frac{1}{AL}F\left(K, AL\right) = F\left(k, 1\right) \\ y &\equiv f\left(k\right) \end{aligned} \tag{8}$$

where equation (8) corresponds to the production function written in "intensive" form.

2 Deriving the Equation for $\dot{k}$

We are now going to obtain the key equation in the Solow model, the equation that describes the dynamic behavior (accumulation/deaccumulation) of capital per equivalent unit of labor (AL).

$$\begin{aligned} \dot{k} &\equiv \frac{\partial k}{\partial t} = \frac{\partial\left(\frac{K}{AL}\right)}{\partial t} \\ &= \frac{\dot{K}LA - K\dot{L}A - KL\dot{A}}{\left(LA\right)^2} \\ &= \frac{\dot{K}}{LA} - \frac{\dot{L}}{L}\frac{K}{LA} - \frac{\dot{A}}{A}\frac{K}{LA} \\ &= \frac{\dot{K}}{AL} - \left(n+g\right)k \\ \dot{k} &= \underbrace{s\ f\left(k\right)}_{\text{Actual investment}} - \underbrace{\left(\delta + n + g\right)}_{\text{Break-even investment}} \end{aligned} \tag{9}$$

The first term in the right hand side of equation (9) corresponds to the total resources that are devoted to "produce" units of capital, and the second term, tells us how much of that is used just to prevent k from falling. Notice that k depreciates not just because K depreciates at a rate δ but also because AL grows at a rate $(n+g)$.

Equation (9) is a first order ordinary differential equation, and we can study its behavior with some very simple tools.

3 Long-run Equilibrium

In our case, equation (9) summarizes all the information we need about the dynamics of the economy. Furthermore, the system is at an equilibrium point when $\dot{k}_t = 0$. Under certain conditions, that we will assume hold, there exists a unique value of k that we will denote by k^* such that $\dot{k}_t(k^*) = 0$.[4]

We can use a phase diagram like with any ordinary differential equation, but we can also plot actual investment and break-even investment separately, in general we use this second approach as it makes easier the comparative statics exercises. (see Romer's textbook and the lecture notes.)

4 An Application: Natural Resources and Land with a Cobb-Douglas Production Function

The basic structure we just put together gives a very simple approach to economic growth. However, this simple framework can easily be extended to richer frameworks. In this part of the handout we will take a quick look at economic growth when other factors of production such as land and natural resources are also used.

4.1 The Model

Take the same model from sections 1 and 2, but with a production function given by

$$Y_t = K_t^{\alpha} \; R_t^{\beta} \; T_t^{\gamma} \; (A_t L_t)^{1-\alpha-\beta-\gamma} \tag{10}$$

where R are natural resources and T is land.

Assume that

$$\begin{aligned} \dot{T}_t &= 0 \\ \dot{R}_t &= -bR_t \qquad b > 0 \end{aligned}$$

The analysis in this case gets a bit more complicated, with land and resources also playing a role in production, the definition of equilibrium used in section 3 is not valid here. We need to find an equilibrium where all variable grow at a constant rate, by assumption this is true for the case of A, L, R, and T, then we need to determine if it is possible for K and Y to grow at a constant rate.

The growth rate of capital is given by

$$\frac{\dot{K}_t}{K_t} = s \; \frac{Y_t}{K_t} - \delta \tag{11}$$

which implies that for K to grow at a constant rate, K/Y must remain constant, meaning that both variables should grow at the same rate.

[4]The Inada conditions guarantee that there exists at least one $k \in (0, \infty)$ such that $\dot{k} = 0$ if $f()$ is continuous $\forall \; k > 0$.

After some manipulations, see Section 1.8 in Romer, we obtain

$$g_Y^{bgp} = g_K^{bgp} = \frac{(1-\alpha-\beta-\gamma)(n+g)-\beta b}{1-\alpha} \tag{12}$$

and using the fact that population grows at the rate n, we can find the product per capita growth rate

$$\begin{aligned} g_{Y/L}^{bgp} &= g_Y^{bgp} - n \\ &= \frac{(1-\alpha-\beta-\gamma)\, g - \beta b - (\beta+\gamma)\, n}{1-\alpha} \end{aligned} \tag{13}$$

Equation (13) tells us that there are two forces that affect growth in different directions. The first terms involves g and shows that technological progress, which spurs economic growth, the mechanics are the same as in the case of the simpler model.[5] But we now have a second term which reflects that we have two other factors of production that do not growth and so become scarcer as the economy keeps cumulating knowledge (A) and labor (L), this affects growth and it may even lead to a negative growth rate if technological progress is not enough.

The superscript *bgp* means "balanced growth path", a concept used to describe an equilibrium where all the variables growth at a constant rate. In this case the variable that remains constant in the steady state (or long-run equilibrium) is the capital-product ratio (K/Y).

[5] Remember that the growth of product per capita equals g in the simpler model.