14.05 Intermediate Applied Macroeconomics
Problem Set 4 Solutions

Distributed: October 27, 2005 | TA: José Tessada
Due: November 3, 2005 | Frantisek Ricka

Question 1 Diamond Overlapping Generations Model

Consider the Diamond overlapping generations model. L_t individuals are born in period t and live for two periods, working and saving in the first and living off capital in the second period. Assume population is growing at a constant rate, n, and technological progress occurs at exogenous rate g. Markets are competitive and labor and capital are paid their marginal products. There is no capital depreciation. Utility is logarithmic with individual discount rate $0 > \rho > -1$.

$$U = \log(c_t) + \frac{1}{1+\rho}\ \log(c_{t+1}).$$

The production function in per capita terms is

$$y_t = f(k_t) = k_t^{\alpha}.$$

(a) Determine the intertemporal budget constraint for each individual. Set up the consumer's utility maximization problem and derive the equilibrium condition for $\frac{c_{t+1}}{c_t}$ (Euler equation). Use that condition and the budget constraint to solve for first period consumption, c_t, and the savings rate (ie, the fraction of income saved), s_t.

Answer.

The intertemporal budget constraint of each individual is

$$c_t + \frac{c_{t+1}}{1+r_{t+1}} = A_t w_t.$$

The consumer's utility maximization problem is

$$\begin{aligned} \max_{c_t, c_{t+1}} \quad & \log(c_t) + \frac{1}{1+\rho}\ \log(c_{t+1}) \\ \text{subject to} \quad & c_t + \frac{c_{t+1}}{1+r_{t+1}} = A_t w_t. \end{aligned}$$

We can solve this problem using the Langrangean and taking first order conditions with respect to c_t and c_{t+1}. The Lagrangean is given by

$$\mathcal{L} = \log(c_t) + \frac{1}{1+\rho}\log(c_{t+1}) + \lambda\left(w_t A_t - c_t - \frac{c_{t+1}}{1+r_{t+1}}\right).$$

The first order conditions are

$$\begin{aligned} \frac{1}{c_t} &= \lambda \\ \frac{1}{1+\rho}\,\frac{1}{c_{t+1}} &= \lambda\,\frac{1}{1+r_{t+1}}. \end{aligned}$$

Combining the two equations we obtain the Euler equation

$$\frac{c_{t+1}}{c_t} = \frac{1+r_{t+1}}{1+\rho}$$

And using the Euler equation in the budget constraint we obtain

$$\begin{aligned} c_t + \frac{c_t}{1+\rho} &= A_t w_t \\ c_t &= \frac{1+\rho}{2+\rho} w_t A_t \end{aligned}$$

The savings rate, s_t is given by

$$s_t = \frac{w_t A_t - c_t}{w_t A_t} = 1 - \frac{c_t}{w_t A_t} = 1 - \frac{1+\rho}{2+\rho} = \frac{1}{2+\rho}.$$

(b) Using the saving rate derived in part (a) and the production function determine the relationship between k_{t+1} and k_t, and show it in a graph. Write down the expression that implicitly defines the equilibrium capital stock k^*. Is the equilibrium stable?

Answer.

The capital stock is given by

$$K_{t+1} = s_t w_t A_t L_t$$

Therefore, capital per effective unit of labor is

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{s_t w_t}{(1+n)(1+g)} = \frac{1}{2+\rho}\ \frac{w_t}{(1+n)(1+g)}$$

Using the fact that labor earns its marginal product, we have

$$w_t = f(k_t) - k_t f'(k_t) = (1-\alpha)k_t^{\alpha}$$

Substituting above

$$\begin{aligned} k_{t+1} &= \frac{1}{2+\rho}\ \frac{1}{(1+n)(1+g)}\ (1-\alpha)k_t^{\alpha} \qquad (1) \\ k_{t+1} &= m(k_t) \end{aligned}$$

In equilibrium, $k_{t+1} = k_t = k^*$. So, k^* is implicitly defined by

$$k^* = \frac{1}{2+\rho}\ \frac{1}{(1+n)(1+g)}\ (1-\alpha)(k^*)^{\alpha}$$

The equilibrium level is then given by

$$k^* = \left(\frac{1}{2+\rho}\ \frac{1-\alpha}{(1+n)(1+g)} \right)^{1/(1-\alpha)}$$

Figure 1 shows the dynamic system described by equation (1).

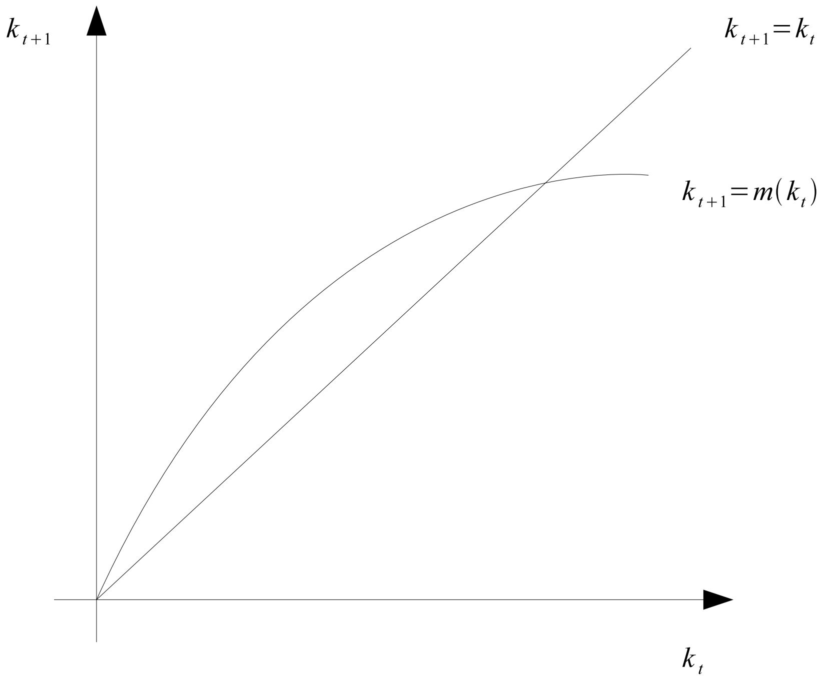

Figure 1: The Dynamic System, Equation (1).

The equilibrium is determined by the intersection of the $k_{t+1} = k_t$ line and the $k_{t+1} = m(k_t)$ curve. This equilibrium is stable because if we start with a capital stock below or above k^* we will always converge to k^*. This result is driven by the diminishing returns to capital.

(c) Describe how each of the following affects the relationship between k_{t+1} and k_t. Show it in a graph and explain intuitively.

(i) A rise in the population growth rate n.

Answer.

A rise in n shifts the $m(k_t)$ curve down (marked as $m_2(k_t)$ in figure 2). Saving in the first period is the same and yields the same amount of capital in the second period as before, but that capital is spread out among more individuals. So, capital per unit of effective labor in the second period (k_{t+1}) is lower for a given k_t. The steady state level of capital per effective unit of labor, k^*, decreases.

(ii) A decrease in the individual discount rate ρ.

Answer.

If ρ decreases the individuals become more patient. They save a higher fraction of their income (s_t increases). Because saving is higher, capital per effective unit of labor in the second period (k_{t+1}) increases for any given k_t. Therefore, the $m(k_t)$ curve shifts up (marked as $m_2(K_T)$ in figure 3) and the steady state level of capital per effective unit of labor increases.

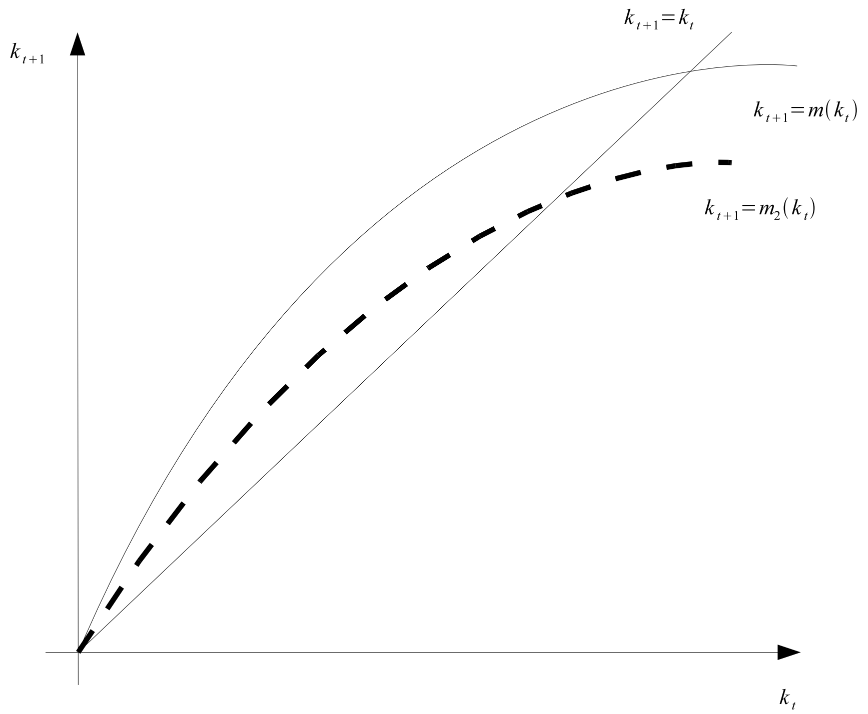

Figure 2: Increase in n.

(iii) A downward shift of the production function. In particular, assume that $f(k_t)$ takes the form Bk_t^α and B falls.

Answer.

Now, the wage is given by

$$w_t = f(k_t) - k_t f'(k_t) = B(1-\alpha)k_t^\alpha$$

And the relationship between k_{t+1} and k_t is given by

$$k_{t+1} = \frac{1}{2+\rho}\ \frac{1}{(1+n)(1+g)}\ B(1-\alpha)k_t^\alpha$$

A decrease in B shifts down the k_{t+1} curve and reduces the steady state level of capital per effective unit of labor (the change in the curve is the same as the one in figure 2). The graph is similar to the one in part (i). A decrease in B does not change the saving rate, but lowers the wage. Individuals save the same fraction of their income, but income is lower. Therefore, the amount of capital per unit of effective labor in the second period is lower for any given k_t.

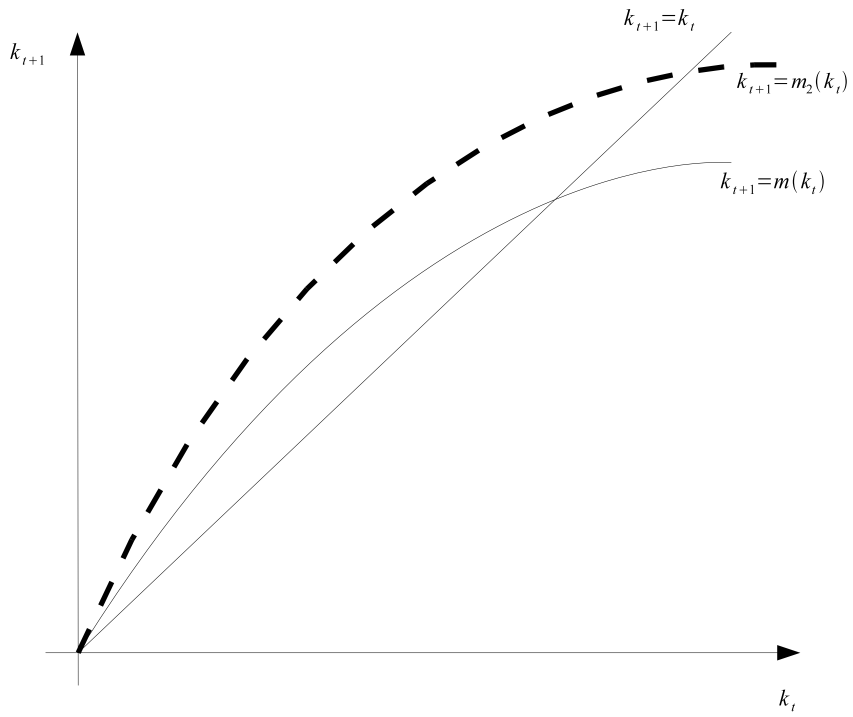

Figure 3: Decrease in ρ.

(d) Now suppose that capital depreciates at rate $\delta > 0$, so that $r_t = f'(k_t) - \delta$. How, if at all, does this affect the savings rate derived in (a)? How does this result depend on the assumption of logarithmic utility?

Answer.

The savings rate is not affected. With logarithmic utility, the savings rate does not depend on the interest rate. In general, with a different utility function this would not be the case and the depreciation rate would affect the savings rate.

Question 2 Social Security using Taxes

Consider a Diamond economy where g is zero (and $A = 1$), production is Cobb-Douglas, population grows at a rate n (from one generation to the next), and utility is logarithmic (see sections 2.9 and 2.10 in the textbook). Suppose the government taxes each young individual an amount T and uses the profits to pay benefits to old individuals; thus each old person receives $(1+n)T$.

(a) Write down the budget constraint faced by an individual born at time t.

Answer.

Note that, with $g = 0$, technology is constant. We can therefore normalize $A = 1$ for this problem. With this done, we can write down the budget constraint as follows

$$\begin{aligned} c_t + s_t &= w_t - T \\ c_{t+1} &= (1 + r_{t+1})\, s_t + (1+n)T \end{aligned}$$

The first equation represents the individual's budget constraint in the first period, and the second equation represents their budget constraint in the second period. Note that, for each individual, the return on their tax, T, is $(1+n)T$, reflecting the fact that there are more individuals in the younger generation whose taxes are being transferred to the older generation. Of course, this rate of return will not in general be equal to the rate of return on private saving, $(1 + r_{t+1})$. Now, since $s_t = w_t - T - c_t$, we can substitute to solve for the intertemporal budget constraint and find

$$c_t + \frac{c_{t+1}}{(1+r_{t+1})} = w_t - T\left(\frac{r_{t+1} - n}{1 + r_{t+1}}\right).$$

(b) Use the budget constraint to determine the saving function of this individual. [*Hint: You should get a function that depends in the wages and the social security tax in the following way:* $s_t = Bw_t - Z_t T$ *where both* B *and* Z_t *are functions of the parameters of the model* (n, ρ) *and the interest rate at* $t+1$ (r_{t+1}) .]

Answer.

With log utility with discounting, the individual will consume $\frac{1+\rho}{2+\rho}$ of their wealth in the first period of their life, see question 1 in the problem set. Thus,

$$c_t = \left(\frac{1+\rho}{2+\rho}\right)\left(w_t - T\left(\frac{r_{t+1} - n}{1 + r_{t+1}}\right)\right)$$

We can then substitute for savings,

$$\begin{aligned} s_t &= w_t - T - c_t \\ &= w_t - T - \left(\frac{1+\rho}{2+\rho}\right)\left(w_t - T\left(\frac{r_{t+1} - n}{1 + r_{t+1}}\right)\right) \\ &= \frac{w_t}{2+\rho} - Z_t T, \end{aligned}$$

where

$$Z_t = 1 - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{r_{t+1} - n}{1 + r_{t+1}}\right)$$

(c) How, if at all, does the introduction of the pay as you go social security affect the relation between k_t and k_{t+1} as described by equation 2.59 in the textbook? [*Hint: Use the individual savings function obtained in part (b) and proceed as usual to determine the relationship between* k_t *and* k_{t+1}.]

Answer.

As usual in this model,

$$\begin{aligned} K_{t+1} &= s_t L_t, \quad \text{so} \\ k_{t+1} &= \frac{s_t}{1+n} \\ &= \frac{1}{1+n}\left(\frac{w_t}{2+\rho} - Z_t T\right) \end{aligned}$$

Now, with Cobb-Douglas production, the wage is $w_t = (1-\alpha)k_t^\alpha$, so we have

$$k_{t+1} = \frac{1}{1+n}\left(\frac{(1-\alpha)k_t^\alpha}{2+\rho} - Z_t T\right).$$

(d) How, if at all, does the introduction of this social security system affect the balanced growth path value of k? [*Hint: The key is to determine the sign of* Z_t.]

Answer.

Without any taxation or social security, the relation between k_t and k_{t+1} would be just as it is above but without the $Z_t T$ term. If Z_t is positive, this means we have shifted down the $k_{t+1} = m(k_t)$ curve relative to the no-tax case. In turn, this reduces the steady state value of k. It follows that all we need to do is find the sign of Z_t.

$$\begin{aligned} Z_t &= 1 - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right) \\ &= \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} \\ &= \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0. \end{aligned}$$

It follows that k^* is reduced relative to the case with no social security.

(e) If the economy is initially on a balanced growth path (BGP) that is dynamically efficient, how does a marginal increase in T affect the welfare of current and future generations? What happens if the initial BGP is dynamically inefficient? [*Hint: Remember that dynamic efficiency means that* $k^* = k(goldenrule)$.]

Answer.

A dynamically efficient economy is one in which saving lies below the golden rule rate of savings. It is "dynamically efficient" in a Pareto sense; although in the long run consumption would increase if people saved more, this would require a short run fall in consumption. But there is no way for future generations to compensate the present generations for this fall in consumption, and therefore the economy is Pareto efficient despite having saving less than the golden rule rate.

This discussion suggests the answer to how a marginal increase in T affects current and future generations. A marginal increase in T reduces savings; since savings were below the golden rule already, now future generations have even lower consumption. The current old generation, however, obviously benefits, since it gets the benefit of the increased contributions to social security without facing a lower capital stock. What happens to the current young is not clear at first sight. They must pay higher taxes but they will also receive a greater benefit when they are old. However, we know that current young are worse off because, in a dynamically inefficient situation, $r_{t+1} = f'(k_{t+1}) > n$.

Note that if the economy were originally dynamically inefficient, the marginal increase in T would increase the welfare of both the current and future generations.[1]

[1] You may want to read again section 2.11 from Romer's textbook.

Question 3 Social Security and Personal Savings Accounts

Consider a Diamond economy where g is zero (and $A = 1$), production is Cobb-Douglas, population grows at a rate n (from one generation to the next), and utility is logarithmic (see sections 2.9 and 2.10 in the textbook). Suppose the government taxes each young individual an amount T and uses the proceeds to purchase capital. Individuals born at T therefore receive $(1 + r_{t+1})T$ when they are old.

(a) Write down the budget constraint faced by an individual born at time t.

Answer.

In this case, the period 2 budget constraint is

$$c_{t+1} = (1 + r_{t+1})s_t + (1 + r_{t+1})T$$

The individual's return on their social security contribution is therefore the same as their return on private saving. Solving for s_t and substituting as in 2(a) gives us

$$c_t + \frac{c_{t+1}}{(1 + r_{t+1})} = w_t$$

which is just the standard intertemporal budget constraint in the OLG model.

(b) Use the budget constraint to determine the saving function of this individual. [*Hint: You should again get a function that depends in the wages and the social security taxin the following way:* $s_t = Bw_t - Z_tT$ *where both* B *and* Z_t *are functions of the parameters of the model* (n, ρ) *and the interest rate at* $t + 1$ (r_{t+1}).]

Answer.

Set up the utility maximization problem exactly as in question 1 of this problem set and derive the first order conditions. From them we obtain the Euler equation

$$\frac{c_{t+1}}{c_t} = \frac{1 + r_{t+1}}{1 + \rho}$$

Substituting in the budget constraint gives the first period consumption and savings

$$\begin{aligned} c_t &= \frac{1+\rho}{2+\rho} w_t \\ s_t &= w_t - c_t - T = \frac{w_t}{2+\rho} - T \end{aligned}$$

The social security contribution reduces *private* savings one-for-one. Note that we assume here (and throughout) that private savings is still positive even after the introduction of the fully funded social security scheme.

(c) How, if at all, does the introduction of the pay as you go social security affect the relation between k_t and k_{t+1} as described by equation 2.59 in the textbook? [*Hint: Use the individual savings function obtained in part (b) and proceed as usual to determine the relationship between* k_t *and* k_{t+1}.]

Answer.

The key element here is to note that the government is saving on behalf of the young, and it does it through the social security system.

In this case, the total capital stock is equal to the sum of private saving and government saving

$$K_{t+1} = s_t \ L_t + T \ L_t$$

It follows that

$$\begin{aligned} k_{t+1} &= \frac{1}{1+n}\left(\frac{w_t}{2+\rho} - T\right) + \frac{1}{1+n}T \\ &= \frac{1}{1+n}\frac{w_t}{2+\rho} \\ &= \frac{1}{1+n}\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha} \end{aligned}$$

The fully-funded social security scheme therefore has no effect on the relation between k_t and k_{t+1}.

(d) How, if at all, does the introduction of this social security system affect the balanced growth path value of k?

Answer.

It immediately follows from the answer above that the fully funded social security scheme also has no effect on the balanced growth path value of k. This is because the government scheme has no real effect in this model. Although the government is doing some of the saving for the young, it is earning precisely the same returns as the young could have earned on their private saving.

(e) Compare the results obtained in questions 2.d and 3.d. Explain *intuitively* why the results are different.

Answer.

Clearly, the equilibrium level of capital per worker is lower in the pay as you go system than in the fully funded system. Intuitively, this is because in the pay as you go system, the tax levied on the young is immediately transferred to the old, whereas in the fully funded system, it is invested first. This intuition is clearest if we let $r_{t+1} = n$ in the pay as you go scheme. Then the pay as you go scheme reduces private saving one for one with tax contributions, as in the fully funded system. But the pay as you go scheme consumes the tax proceeds immediately, while the fully funded scheme places them in capital for a period first. This timing and investment issues translates into differences in the rates of return in both schemes.