ON UNITARY REPRESENTATIONS WITH REGULAR INFINITESIMAL CHARACTER

by

SUSANA A. SALAMANCA RIBA

Sc.B. Universidad Autonoma Metropolitana
Mexico D.F. Mexico
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Signature of Author ....
Department of Mathematics
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Certified by ....
David A. Vogan
Supervisor

Accepted by ....
Nesmith C. Ankeny
Chairman, Departmental Graduate Committee

Dedication

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ABSTRACT

For $G = SL(n,\mathbb{R})$, $SU(p,q)$, and $SP(n,\mathbb{R})$ we prove that every irreducible unitary representation of G with the same infinitesimal character as that of a finite dimensional, arises as cohomological parabolic induction from a one-dimensional unitary character. The techniques used involve case-by-case arguments that do not use any special features of these groups. So it seems reasonable to hope that these arguments could be extended in order to solve the problem for other groups.

For G as above let K be a maximal compact subgroup of G, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, the Cartan decomposition of $\mathfrak{g}_0 = \mathrm{Lie}(G)$, $(\pi,\mathcal{H})$ an irreducible Hermitian representation of G on which $Z(\mathfrak{g})$ acts as on a finite dimensional module, $\mathcal{H}_K$ the Harish-Chandra module of π and (μ,V_μ) a lowest K-type of $\mathcal{H}_K$. We prove that either $\mathcal{H}_K$ is isomorphic to a Zuckerman module $A_q(\lambda)$, for some θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ and one-dimensional unitary character λ of $L = N_G(\mathfrak{q})$, or else the Hermitian form restricted to $V_\mu \oplus (\mathfrak{p}\otimes V_\mu)$ is indefinite.

Thesis Supervisor: Dr. David A. Vogan, Jr.
Title: Professor of Mathematics

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Chapter 1. Introduction

The problem of classifying those irreducible unitary representations of a Lie group whose infinitesimal character is the same as that of an irreducible finite dimensional, has interested many people for many years. These representations are important because they appear in interesting applications like the theory of automorphic forms (see for example, Borel-Wallach [1980], ch. VII.5-VII.6). Here is an example:

If G is a semisimple Lie group, Γ a discrete subgroup such that $\Gamma\backslash G$ is compact, and F an irreducible finite dimensional G-module, then the right action of G on $L^2(\Gamma\backslash G)$ gives a Hilbert space decomposition, with finite multiplicities $m(\pi,\Gamma)$,

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\hat{G}_u} m(\pi,\Gamma)\pi,$$

with $\hat{G}_u$ the set of irreducible, unitary representations $(\pi,\mathcal{H})$ of G. Matsushima's formula (see Borel-Wallach [1980], p. 223) is

$$H^*(\Gamma,F) = \bigoplus_{\pi\in\hat{G}_u} m(\pi,\Gamma)H^*(\mathfrak{g},K;\mathcal{H}_K\otimes F^*).$$

Here $\mathfrak{g}$ is $\mathrm{Lie}(G)\otimes\mathbb{C}$, and $\mathcal{H}_K$ is the Harish-Chandra module of $(\pi, \mathcal{H})$. The relative Lie algebra cohomology groups on the right are non-zero only when the infinitesimal character of π is the same as that of F.

In Vogan-Zuckerman [1984], an algebraic construction of the modules $\mathcal{H}_K$ with non-vanishing cohomology groups in terms of cohomological parabolic induction is given. The derived functor modules constructed this way are conjectured to exhaust a larger family of unitary representations described below.

We refer to chapter 2 for precise definitions of the terms used below.

Let G be a reductive Lie group, K a maximal compact subgroup, θ the corresponding Cartan involution, and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the Cartan decomposition of $\mathfrak{g}_0$.

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ-stable parabolic subalgebra (cf. 2.3), that is

$$\theta\mathfrak{l} = \mathfrak{l}, \qquad \theta\mathfrak{u} = \mathfrak{u},$$

$\bar{\mathfrak{l}} = \mathfrak{l}$, with $^-$ denoting complex conjugation

and $\bar{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}$.

Let L be the normalizer of $\mathfrak{q}$ in G, and λ a one-dimensional unitary character of L. A $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is a Harish-Chandra module constructed as in Vogan [1981] Chapter 6 by cohomological parabolic induction from the one-dimensional unitary character λ. (See Definitions 2.4.14, 2.5.2).

The conditions on the infinitesimal character mentioned above can be weakened to include a larger family of representations.

If X is a Harish-Chandra module with infinitesimal character χ, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, then up to Weyl-group orbit considerations χ corresponds to a weight $\gamma \in \mathfrak{h}^*$. Choose a positive root system $\Delta^+(\mathfrak{g},\mathfrak{h})$ such that γ is dominant.

Conjecture 1.1. Suppose X is an irreducible unitary Harish-Chandra module such that $\gamma - \rho$ is dominant for $\Delta^+(\mathfrak{g},\mathfrak{h})$. Then, there are a θ-stable parabolic subalgebra $\mathfrak{q}$ and a unitary one-dimensional character λ of L such that

$$X \cong A_{\mathfrak{q}}(\lambda).$$

Some progress has been made when we assume that γ is regular and integral. Namely, for G a complex group, T. J. Enright [1979] proved that if γ is regular integral then there exists a $(\mathfrak{g},K)$ module $A_{\mathfrak{q}}(\lambda)$ isomorphic to X. Also, in Speh [1981], the same result for $G = SL(n,\mathbb{R})$ is proved.

The following result is proved in this thesis.

Theorem 1.2. If G is $SL(n,\mathbb{R})$, $SU(p,q)$, or $SP(n,\mathbb{R})$ and γ regular and integral, then, for some $\mathfrak{q}$ and λ

$$X \cong A_{\mathfrak{q}}(\lambda).$$

The proof for $SL(n,\mathbb{R})$ is new and quite different from Speh's original one.

The proof is by induction on the dimension of G. It involves choosing an appropriate proper subgroup $L \subset G$ and embedding $\mathcal{H}_K$ as the Langlands submodule of a derived functor module induced from a representation of L, in such a way that the information about unitarity or non-unitarity of the representation of L can be carried up to G and our representation $\mathcal{H}_K$.

The thesis is organized as follows. In Chapter 2 we set up the notation and results needed to restate and prove the result in the following form:

Theorem 1.3. Suppose X is an irreducible Harish-Chandra module with regular integral infinitesimal character, equipped with a non-zero Hermitian form $\langle\ ,\ \rangle$. Then, either

a) $X \cong A_{\mathfrak{q}}(\lambda)$, for some $\mathfrak{q}$, λ as above, or

b) There are a lowest-K-type V_{δ_1} and a K-type $V_{\delta_2} \subseteq V_{\delta_1} \otimes \mathfrak{p}$, such that

$$\mathrm{Hom}_K(V_{\delta_i}, X) \neq 0 \qquad i = 1,2,$$

and the restriction of $\langle\ ,\ \rangle$ to the sum $V_{\delta_1} \oplus V_{\delta_2}$ is indefinite.

Sections 2.1 through 2.4 are devoted to notation and the results that will be needed for the proof. The two main issues are the definition of the Zuckerman derived functor modules $\mathcal{R}_{\mathfrak{q}}(Y)$ and Vogan's embedding of any irreducible Harish-Chandra module into some Zuckerman derived functor module. We also give some useful properties of these modules.

In Section 2.5 we define the modules $A_{\mathfrak{q}}(\lambda)$ and prove some nice features that we will use in later chapters.

Sections 2.6 and 2.7, and Chapters 3 and 5 are the actual proof of Theorem 1.3. The main results are Theorems 2.6.7 and 2.6.8, which say that we can exhibit X as a submodule of a derived functor module $\mathcal{R}_{\mathfrak{q}}(X_L)$ in such a way that we can reduce the problem to the representation X_L of the group L. Chapters 3, 4 and 5 are the proof of Theorem 2.6.7.

We argue by contradiction: With the help of Vogan's embedding result we find another θ-stable parabolic subalgebra and another Zuckerman module containing X. We have to check several conditions that will ensure the reduction, but mainly 2.6.7 b) and c). Then assuming that the representation X_L is not isomorphic to a module $A_{\mathfrak{q}^0}(\lambda^0)$ we prove non-unitarity on X_L. For this we use the properties of the $A_{\mathfrak{q}}(\lambda)$ modules discussed in 2.5 and some techniques discussed in 2.7, primarily Lemma 2.7.1.

Chapter 2

In this chapter we set up notation, state the basic results we will need and our main result, and provide a scheme for the proof.

For undefined terms in this section see, for example, Vogan [1981] Chapter 0.

2.1. Structure Theory

We will denote Lie groups by upper case roman letters such as G, H, L and complex Lie algebras by script letters such as $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{l}$. We will make the distinction between the real Lie algebra of a Lie group and its complexification as follows:

$$\mathfrak{g}_0 = \text{Lie }(G) \qquad \mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}, \quad \text{etc.}$$

Let $U(\mathfrak{g})$ = universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ = center of $U(\mathfrak{g})$.

Although we will eventually study connected real simple linear Lie groups, we will consider connected real reductive linear Lie groups. These are Lie groups satisfying:

a) G is connected

b) $\mathfrak{g}_0$ is a real reductive Lie algebra

c) G has a faithful finite dimensional representation

Let θ be a Cartan involution of $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the Cartan decomposition of $\mathfrak{g}_0$ into the $+1$ and -1 eigenspaces of θ.

Fix once and for all a nondegenerate, invariant symmetric bilinear form on $\mathfrak{g}_0$. We will denote this and its various complexifications, restrictions and dualizations by $(\, , \,)$. We may choose it so that the Cartan decomposition of $\mathfrak{g}_0$ is orthogonal and

$$(\, , \,)\big|_{\mathfrak{p}_0} > 0$$

$$(\, , \,)\big|_{\mathfrak{k}_0} < 0.$$

Let H be a Cartan subgroup of G. Denote by $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$ the roots of $\mathfrak{h}$ in $\mathfrak{g}$.

In general if $\mathfrak{s}$ is an abelian reductive Lie subalgebra of $\mathfrak{g}$ and V is an $\mathrm{ad}(\mathfrak{s})$-stable subspace of $\mathfrak{g}$ then $\Delta(V,\mathfrak{s})$ is the set of weights of $\mathfrak{s}$ in V (with multiplicities). For any $B \subset \Delta(V,\mathfrak{s})$ let $\rho(B) = \frac{1}{2} \sum_{\alpha \in B} \alpha$. When there is no confusion we will use $\Delta(V)$ for $\Delta(V,\mathfrak{s})$.

If H is a θ-stable Cartan subgroup, then

$$H = TA; \quad \text{with} \quad T = H \cap K, \quad A = H \cap (\exp \mathfrak{p}_0) = \exp(\mathfrak{h}_0 \cap \mathfrak{p}_0)$$

and $\Delta(\mathfrak{g},\mathfrak{h})$ is θ-stable.

Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ and

$$W(G,H) = N_G(H)/H \cong N_K(H)/H \cap K.$$

Let $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ be a set of positive roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, the corresponding Borel subalgebra and $\rho = \rho_{\mathfrak{g}} = \rho(\mathfrak{n})$.

Let $\mathfrak{t}_0^c \subseteq \mathfrak{k}_0$ be a Cartan subalgebra. Define $\mathfrak{h}^c$ (resp. H^c) to be the centralizer in $\mathfrak{g}$ (resp. G) of $\mathfrak{t}_0^c$. H^c is θ-stable, so we can write

$$H^c = T^c A^c, \quad \text{with} \quad T^c = H^c \cap K$$

a Cartan subgroup of K.

H^c is called the fundamental or maximally compact Cartan subgroup of G.

On the other extreme, if $\mathfrak{a}_0^s \subseteq \mathfrak{p}_0$ is a maximal abelian subalgebra and $\mathfrak{h}_0^s = \mathfrak{t}_0^s + \mathfrak{a}_0^s$ is maximal abelian then $\mathfrak{h}_0^s$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Its centralizer H^s in G is a Cartan subgroup of G, the maximally split one.

2.2. Harish-Chandra Modules

Let $(\pi, \mathcal{H})$ be a continuous complex Hilbert space representation and $\mathcal{H}_K$ the subset of $\mathcal{H}$ of K-finite vectors. If $(\pi, \mathcal{H})$ is *admissible*, that is, if all the K-isotypic components of $\mathcal{H}_K$ are finite dimensional, then the limit

$$\pi(x)v = \lim_{t \to 0} \frac{1}{2}(\pi(\exp tx)v - v),$$

exists for all $x_0 \in \mathfrak{g}_0$ and $v \in \mathcal{H}_K$ and defines a representation of $\mathfrak{g}_0$ in $\mathcal{H}_K$.

$\mathcal{H}_K$ is a $(\mathfrak{g}, K)$ module since we can complexify the representation $\mathcal{H}_K$ to a representation of $\mathfrak{g}$ ($\mathcal{H}$ being complex). Also $\mathcal{H}_K$ is a representation of K. We call $\mathcal{H}_K$ the *Harish-Chandra module of* $(\pi, \mathcal{H})$ (cf. Harish-Chandra [1953].

Any irreducible unitary representation of G is admissible.

Denote by $\mathcal{M}(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$ modules and by $\mathcal{A}(\mathfrak{g}, K)$ the category of admissible $(\mathfrak{g}, K)$ modules.

An irreducible $(\mathfrak{g}, K)$ module is automatically admissible.

A $\mathfrak{g}$ module X is *quasisimple* if $Z(\mathfrak{g})$ acts by scalars on X. Then we have a homomorphism

$$\chi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$$

$$\chi(z)x = \pi(z)x$$

called the infinitesimal character of X.

Any irreducible $(\mathfrak{g},K)$ module is quasisimple.

If (π,X), $(\pi'.X') \in \mathcal{M}(\mathfrak{g},K)$ we say that X and X' are equivalent if there is an invertible map which is an element of the set of $(\mathfrak{g},K)$-module maps defined by

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi,\pi') = \mathrm{Hom}_{\mathfrak{g},K}(X,X') = \{L : X \longrightarrow X \mid$$

$$L \text{ is complex linear and } \pi'L = L\pi\}.$$

Write $\hat{G}$ for the set of equivalence classes of irreducible $(\mathfrak{g},K)$ modules. If $(\pi,\mathcal{H})$, $(\pi',\mathcal{H}')$ are representations of G we say they are infinitesimally equivalent if $(\pi,\mathcal{H}_K)$, $(\pi',\mathcal{H}'_K)$ are equivalent.

2.3 Parabolic subalgebras

Let $t_0^c \subseteq k_0$. Fix

$$x \in i(t_0^c)^*.$$

We define a θ-stable parabolic subalgebra $\mathfrak{q}$ as follows.

Let

$$\Delta(\mathfrak{l}) = \Delta(\mathfrak{l}, \mathfrak{t}^c) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c) \mid \langle\alpha, x\rangle = 0\}$$

$$\Delta(\mathfrak{u}) = \Delta(\mathfrak{u}, \mathfrak{t}^c) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}^c) \mid \langle\alpha, x\rangle > 0\}$$

$$\mathfrak{l} = \bigoplus_{\alpha\in\Delta(\mathfrak{l})} \mathbb{C}X_\alpha + \mathfrak{t}^c, \qquad \mathfrak{u} = \bigoplus_{\alpha\in\Delta(\mathfrak{u})} \mathbb{C}X_\alpha$$

then $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is θ-stable.

2.4. Derived Functor Modules

In this section we consider that part of the classification of Harish-Chandra modules that consists of attaching a certain set of parameters to an irreducible Harish-Chandra module. We are going to exhibit each irreducible $(\mathfrak{g}, K)$ module as a submodule of a derived functor module.

We will first consider a particular set of irreducible $(\mathfrak{g}, K)$ modules when G is quasisplit. To define these groups we need some notation.

Let $\mathfrak{a}_0^s \subseteq \mathfrak{p}_0$ be a maximal abelian subalgebra and A^s the corresponding connected subgroup of G.

Let $M = K^A$ = centralizer of A^s in K.

Define $\Delta^s = \Delta(\mathfrak{g}/(\mathfrak{m}+\mathfrak{a}^s), \mathfrak{a}^s)$ = the nonzero roots of $\mathfrak{a}^s$ in $\mathfrak{g}$.

Fix a positive system $\Delta^+ \subset \Delta^s$ and let

$$\mathfrak{n}_0 = \bigoplus_{\alpha \in \Delta^+} \mathbb{R}X_\alpha$$

and N the corresponding connected subgroup of G.

Define $P^s = MA^sN$.

Definition 2.4.1. For a fixed representation (δ, V) of M and $\nu \in \hat{A}^s$, define the Hilbert space

$$\mathcal{H}_{\delta,\nu} = \{f : G \longrightarrow V \mid f \text{ measurable};$$

$$f(gman) = a^{-(\nu+\rho)}\delta(m)^{-1}f(g);$$

$$m \in M; \quad a \in A; \quad n \in N \text{ and } f|_K \in L^2(K)\}.$$

The action of G on $\mathcal{H}_{\delta,\nu}$ given by

$$(\pi_{\delta,\nu}(g))f(g_0) = f(g^{-1}g_0)$$

defines a representation $I(\delta \otimes \nu) = \mathrm{Ind}_{P^s}^G(\delta \otimes \nu)$, the induced representation of G.

Definition 2.4.2. G is quasisplit if $\mathfrak{m}_0 + \mathfrak{a}_0^s$ is abelian.

Hence, if G is quasisplit $\mathfrak{h}_0^s = \mathfrak{m}_0 + \mathfrak{a}_0^s$ is a Cartan subalgebra of $\mathfrak{g}_0$ and $H^s = MA^s = T^sA^s$ a Cartan subgroup of G.

Therefore

$$\hat{H}^s = \text{homomorphisms} : H \longrightarrow \mathbb{C}^{\times} = \hat{M} \times \hat{A}^s \cong \hat{M} \times (\mathfrak{a}^s)^*$$

Definition 2.4.3. a) A representation $\delta \in \hat{M}$ is fine if $d\delta$ is trivial on $\mathfrak{m}_0 \cap [\mathfrak{g},\mathfrak{g}]$.

b) Consider the set

$$\bar{\Delta} = \{\alpha \in \Delta^s \mid \tfrac{1}{2}\alpha \notin \Delta^s\}.$$

A root $\alpha \in \bar{\Delta}$ is real if $\alpha = \beta\big|_{\mathfrak{a}^s}$ for some $\beta \in \Delta^s$ real.

Let $\mathfrak{a}_0^\alpha = \{X \in \mathfrak{a}_0^s \mid \alpha(X) = 0\}$ and $G^{A^\alpha} = M^\alpha A^\alpha$; $K^\alpha = M^\alpha \cap K$. Choose an injection

$$\phi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{m}_0^\alpha$$

such that

$$\phi_\alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathfrak{a}_0$$

$$\phi_\alpha \circ \theta = \theta \circ \phi_\alpha$$

$$\phi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \alpha\text{-root space.}$$

Put $$Z_\alpha = \Phi_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in k_0^\alpha .$$

c) A representation $\mu \in \hat{K}$ is fine (for G) if

i) For $\alpha \in \bar{\Delta}$ <u>real</u> $\mu(iZ_\alpha)$ has eigenvalues included in $\{0, \pm 1\}$.

ii) For each complex root $\alpha \in \bar{\Delta}$

$$\mu\Big|_{(k_0^\alpha \cap [\mathfrak{g}_0, \mathfrak{g}_0])} \quad \text{is trivial.}$$

d) If $\delta \in \hat{M}$ is a fine representation set $A(\delta) = \{\mu \in \hat{K} \mid \mu$ is fine, and δ occurs in $\mu|_M\}$.

Proposition 2.4.5. See Vogan [1981] 4.4.8.

Suppose G is quasisplit and $\mu \in \hat{K}$ is fine. Then the restriction of μ to M is a sum of fine representations of M, each occurring with multiplicity one.

Say that $\mu \in A(\delta)$ for some δ fine. Let X be an irreducible $(\mathfrak{g}, K)$ module containing the K-type μ.

Then there is a character $\nu \in \hat{A}^s$ such that

$$\mathrm{Hom}_{g,K}(X, I(\delta \otimes \nu)) \neq 0.$$

For an arbitrary linear reductive Lie group we will define the notion of minimal (or lowest) K-type of a representation and attach to it certain parameters. In the next section we will then construct a $(\mathfrak{g},K)$ module with these parameters using a reduction to a quasisplit group. The irreducible representation with that lowest K-type is a subquotient of this module. For proofs of these results see Vogan [1981] Chapters 5 and 6.

Fix a Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$, a positive root system

$$\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{k},\mathfrak{t}^c),$$

and a $\Delta^+(\mathfrak{k})$-dominant weight $\mu \in \hat{T}^c$; write $\mu \in (\mathfrak{t}^c)^*$ for its differential. Define

$$2\rho_c = 2\rho(\Delta^+(\mathfrak{k})) \in (\mathfrak{t}^c)^*.$$

Let $\mathfrak{h}^c$ as in 2.1. Then there exists a θ-stable positive root system $\Delta^+(\mathfrak{g},\mathfrak{h}^c)$ which makes $\mu + 2\rho_c$ dominant. See for example, Vogan [1981], p. 239.

Fix a noncompact imaginary root $\beta \in \Delta(\mathfrak{g},\mathfrak{h}^c)$. Write X_β for a root vector for the root β. Put

$$X_{-\beta} = \bar{X}_\beta$$

$$Z_\beta = X_\beta + X_{-\beta} \in \mathfrak{p}_0$$

$$\mathfrak{g}_0^\beta = \text{centralizer of } Z_\beta \text{ in } \mathfrak{g}_0$$

$$(\mathfrak{h}_0^c)^\perp = \{x \in \mathfrak{h}_0^c \mid \beta(x) = 0\} \qquad (2.4.6)$$

$$= (\mathfrak{t}_0^c)^\perp \oplus \mathfrak{a}_0^c$$

$$\mathfrak{h}_0^\beta = (\mathfrak{h}_0^c)^\perp \oplus \langle Z_\beta \rangle .$$

Then $\mathfrak{g}_0^\beta$ is reductive, the subgroup G_0^β of G with Lie algebra $\mathfrak{g}_0^\beta$ is real reductive linear and $\mathfrak{h}_0^\beta$ is a Cartan subalgebra of $\mathfrak{g}_0^\beta$ and of $\mathfrak{g}_0$. See Vogan [1981], p. 235.

Proposition 2.4.7 (Vogan [1981], 5.3.3). For each $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$-dominant weight $\mu \in \hat{T}^c$, there is a unique element $\lambda_V(\mu) = \lambda_V^G(\mu) \in (\mathfrak{t}^c)^*$ having the following properties: fix a θ-stable positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ making $\mu + 2\rho_c$ dominant; and write $\rho = \rho(\Delta^+(\mathfrak{g}, \mathfrak{h}^c))$. Then $\lambda_V(\mu)$ is dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$, and there is a set $\{\beta_1, \ldots, \beta_r\} \subseteq \Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ of imaginary roots, satisfying

a) If we put

$$c_i = \frac{-2\langle \beta_i, \mu + 2\rho_c \rangle}{\langle \beta_i, \beta_i \rangle} ,$$

$$v = \sum c_i \beta_i ,$$

then

$$0 \leq c_i \leq 1,$$

and

$$\lambda_V^G(\mu) = \lambda_V(\mu) = \mu + 2\rho_c - \rho + \tfrac{1}{2}v.$$

b) If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^c)$ is imaginary, and $\langle \alpha, \lambda_V(\mu) \rangle = 0$, then $\langle \alpha, \beta_i \rangle \neq 0$ for some i.

c) The root β_1 is noncompact; and either it is simple, or there is a complex simple root α of $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ such that $\beta_1 = \alpha + \theta\alpha$.

d) Either all $c_i = 0$, or $c_1 \neq 0$.

e) Write

$$\mathfrak{g}^1 = \mathfrak{g}^{\beta_1}, \quad \mathfrak{h}^1 = \mathfrak{h}^{\beta_1}$$

as in equations (2.4.6). Then the positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}^c) \cap \beta_1^\perp \cong \Delta^+(\mathfrak{g}^1, \mathfrak{h}^1)$ and its subset $\{\beta_2, \ldots, \beta_r\}$ satisfy these same conditions for $\mathfrak{g}^1$ and the weight $\mu|_{\mathfrak{g}^1 \cap \mathfrak{t}}$.

Definition 2.4.8. For a $\Delta^+(k)$-dominant weight $\mu \in \hat{T}^c$ let $q_V^G(\mu)$ be the parabolic defined by $\lambda_V^G(\mu)$ as described in 2.3. If π is a K-type of highest weight μ we also define $q_V^G(\pi) = q_V(\mu)$.

Proposition 2.4.9 (Vogan [1981] §5.3). Suppose $\mu \in \hat{T}^c$ is $\Delta^+(k)$-dominant and $q_V(\mu) = \ell + u$. Then

(a) ℓ_0 is quasisplit

(b) $q_V^L(\mu - 2\rho(u \cap p)) = \ell$

(c) If $\pi \in \hat{K}$ has highest weight μ, then π is fine $\Leftrightarrow q_V^G(\mu) = g$.

We will now define a preordering on $\hat{K}$.

Definition 2.4.10.

a) If $\pi \in \hat{K}$ has highest weight $\mu \in \hat{T}^c$, put $\lambda = \lambda_V^G(\mu)$ as in proposition 2.4.6. Define

$$\|\pi\|_{\text{lambda}} = \|\mu\|_{\text{lambda}} = \langle\lambda,\lambda\rangle.$$

b) If X is a nonzero (g,K)-module define $X(\pi)$ to be the π-isotypic component of X. Then the set

$$\{\pi \in \hat{K} \mid X(\pi) \neq 0 \text{ and } \|\pi\|_{\text{lambda}} \text{ is minimal}\}$$

is nonempty. We define it to be the set of lowest K-types. We will refer to such a π as an LKT.

c) Define $\lambda_V(X) = \lambda_V(\mu)$ for μ a highest weight of a LKT of X and let $\mathfrak{q}_V^G(X)$ be the parabolic associated to μ.

When there is no confusion we will refer to these parameters as $\mathfrak{q}_V$ and λ_V.

Let $\mathfrak{l}_V$ be the Levi factor of $\mathfrak{q}_V$, and L_V the normalizer of $\mathfrak{q}_V$ in G.

$L_V \supset T^c$, so let π_0 be the irreducible representation of $L_V \cap K$ generated by the μ-weight space, inside $\pi\big|_{L_V \cap K}$.

Let

(2.4.11)
$$\pi^{L_V} = \pi_0 \otimes [\Lambda^R(\mathfrak{u}\cap\mathfrak{p})]^* \in (L_V\cap K)^\wedge$$
$$R = \dim \mathfrak{u} \cap \mathfrak{p}.$$

Notice that $\mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})$ is a highest weight of π^{L_V}. By proposition 2.4.9 b) and c)

$$\pi^{L_V} \text{ is fine for } L_V.$$

Fix $H_V = TA$ a maximally split Cartan subgroup of L_V and choose $\delta^{L_V} \in \hat{T}$, fine occurring in $\pi^{L_V}\big|_T$.

This triple $(\mathfrak{q}_V, H_V, \delta^{L_V})$ is, by definition a set of discrete θ-stable data attached to X.

We have attached all these parameters to a representation X with LKT π.

Now with these parameters we will exhibit a $(\mathfrak{g}, K)$ module that contains X as a subquotient. We need more definitions.

Definition 2.4.12. A set of θ-stable data for G is a quadruple $(\mathfrak{q}, H, \delta, \nu)$ with the conditions:

a) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$.

b) L is quasisplit, and $H = TA \subseteq L$ is a θ-stable maximally split Cartan subgroup of L.

c) $\delta \in \hat{T}$ is fine for L and $\nu \in \hat{A}$.

d) If $\lambda^L \in \mathfrak{t}^*$ is the differential of δ and

$$\lambda^G = \lambda^L + \rho(\Delta(\mathfrak{u}, \mathfrak{t}^c)) \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$$

then i) $\langle \lambda^G, \alpha \rangle > 0$ for all $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$

ii) $\langle \lambda^G, \beta \rangle = 0$ for all $\beta \in \Delta(\mathfrak{l}, \mathfrak{h})$.

Notice that the discrete θ-stable data attached to some $X \in \mathcal{M}(\mathfrak{g}, K)$ together with any character $\nu \in \hat{A}$ are a set $(\mathfrak{q}_V, H_V, \delta^{L_V}, \nu)$ of θ-stable data for G.

In order to give a generalization of Proposition 2.4.5 for non-quasisplit groups we need to define the objects in which we are going to realize the Harish-Chandra modules of G.

Definition 2.4.13 Zuckerman Functors.

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ-stable parabolic subalgebra as defined in 2.3 and $L \subseteq G$ the reductive Lie subgroup corresponding to $\mathfrak{l}_0$.

Since G is connected, then so are K, T, L and $L \cap K$.

Let Z be any $(\mathfrak{q}, L\cap K)$ module. Define

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Z) = \mathrm{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}(Z) = \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Z)_{L\cap K\text{-finite}}.$$

This is a $(\mathfrak{g}, L\cap K)$ module.

Now, if W is any $(\mathfrak{g}, L\cap K)$ module define

$$\Gamma W = \{v \in W \mid \dim U(\mathfrak{k})\cdot v < \infty\}.$$

ΓW is a $(\mathfrak{g}, K)$ module and $\Gamma : \mathcal{M}(\mathfrak{g}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is a left exact functor.

The Zuckerman functors $\{(\Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K})^i\}_{i\geq 0}$ are the right derived functors of Γ (written Γ^i). That is, if W is a $(\mathfrak{g}, L\cap K)$ module then W admits an injective resolution

$$0 \longrightarrow W \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

then we have a cochain complex

$$0 \longrightarrow \Gamma W \xrightarrow{\phi_{-1}} \Gamma I_0 \xrightarrow{\phi_0} \Gamma I_1 \xrightarrow{\phi_1} \cdots .$$

Define $\Gamma^i W = \dfrac{\ker \phi_i}{\mathrm{Im}\ \phi_{i-1}}$. So $\Gamma^0 W = \Gamma W$.

Definition 2.4.14. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and $L \subseteq G$ as in 2.3. Then L is a reductive linear (connected) Lie group and $L \cap K$ is a maximal compact subgroup of G.

We will define the i-th cohomological parabolic induction functor $(\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}})^i : \mathcal{M}(\mathfrak{l}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$, as follows.

Let $V \in \mathcal{M}(\mathfrak{l}, L\cap K)$. We make $V \in \mathcal{M}(\mathfrak{q}, L\cap K)$ by letting $\mathfrak{u}$ act trivially.

Let

$$W = \mathrm{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}(V \otimes_{\mathbb{C}} \Lambda^{\dim \mathfrak{u}} \mathfrak{u})$$

then

$$(\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}})^i(V) = \mathcal{R}_{\mathfrak{q}}^i(V) = \mathcal{R}^i(V)$$

$$= \Gamma^i(W).$$

We are now in a position to state the generalization of Proposition 2.4.5. However, it is convenient at this moment to mention some properties of these derived functor modules that we will need later.

Suppose $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i$ are two parabolic subalgebras such that

$$\mathfrak{l}_1 \subseteq \mathfrak{l}_2, \quad \mathfrak{u}_2 \subseteq \mathfrak{u}_1, \quad L_1 \cap K \subseteq L_2 \cap K;$$

set

$$\mathfrak{u} = \mathfrak{l}_2 \cap \mathfrak{u}_1$$

then $\mathfrak{q} = \mathfrak{l}_1 + \mathfrak{u} \subseteq \mathfrak{l}_2$ is a parabolic subalgebra of $\mathfrak{l}_2$.

Proposition 2.4.15 (Zuckerman [1977]; Vogan [1981] 6.3.10). With notation as above if W is an $(\mathfrak{l}_1, L_1 \cap K)$ module such that for $q \neq q_0$

$$\left[\mathcal{R}_{\mathfrak{q}}^{\mathfrak{l}_2}\right]^q W = 0$$

then

$$\left[\mathcal{R}_{\mathfrak{q}_2}^{\mathfrak{g}}\right]^p \left[\left[\mathcal{R}_{\mathfrak{q}}^{\mathfrak{l}_2}\right]^{q_0}(W)\right] \cong \left[\mathcal{R}_{\mathfrak{q}_1}^{\mathfrak{g}}\right]^{p+q_0}(W).$$

Proposition 2.4.16 (Zuckerman [1977]; Vogan [1981] 6.3.11). Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra; $\mathfrak{h} \subseteq \mathfrak{l}$, a Cartan subalgebra. Let Y be an $(\mathfrak{l}, L \cap K)$ module with infinitesimal character $\lambda \in \mathfrak{h}^*$. Then

$$\mathcal{R}^i_{\mathfrak{q}}(W) \text{ has infinitesimal character } \lambda + \rho(\mathfrak{u})$$

Definition 2.4.17. Standard Representations.

Let $(\mathfrak{q}, H, \delta, \nu)$ be a set of θ-stable data for G (definition 2.4.12). Let $H = TA \subset L$ and choose $N \subseteq L$ such that $P = TAN$ is a minimal parabolic subgroup of L and for all α in the corresponding positive system $\Delta(\mathfrak{n}, \mathfrak{a})$

$$\langle \mathrm{Re}\ \nu, \alpha \rangle \leq 0.$$

Let $I_L(\delta \otimes \nu)$ be the principal series representation $\mathrm{Ind}_P^L(\delta \otimes \nu \otimes 1)$. We define the standard $(\mathfrak{g}, K)$ module with θ-stable data $(\mathfrak{q}, H, \delta, \nu)$ by

$$X_G(\mathfrak{q}, \delta \otimes \nu) = \mathcal{R}^s_{\mathfrak{q}}(I_L(\delta \otimes \nu))$$

as in 2.4.14 where $s = \dim \mathfrak{u} \cap \mathfrak{k}$.

We will now state some properties of these standard modules. Fix $\mathfrak{t}_0^c \subseteq \mathfrak{l}_0 \cap \mathfrak{k}_0$ a Cartan subalgebra containing $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ and a positive root system $\Delta^+(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t}^c)$. Set $\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t}^c) \cup \Delta(\mathfrak{u} \cap \mathfrak{k})$.

Proposition 2.4.18 (Vogan [1981] 6.5.9). Let $(\mathfrak{q}, H, \delta, \nu)$ be a set of θ-stable data for G, and $\lambda^G = d\delta + \rho(\mathfrak{u})$ then

a) $\mathcal{R}_{\mathfrak{q}}^{s}(I_L(\delta\otimes\nu))$ has infinitesimal character (λ^G, ν).

b) If π is a K-type occurring in $\mathcal{R}_{\mathfrak{q}}^{s}(I_L(\delta\otimes\nu))$ with highest weight η then there exists an $L \cap K$-type π^L of highest weight η^L such that

$$\delta \subseteq \pi^L\big|_T \quad \text{and} \quad \eta = \eta^L + 2\rho(\mathfrak{u}\cap\mathfrak{p}) + \sum_{\substack{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{p}) \\ n_\alpha\in\mathbb{N}}} n_\alpha \alpha .$$

c) Moreover if π is a LKT, then the last summation term is zero and η^L is fine for L.

Proposition 2.4.19 (Vogan [1981] 6.5.9 (g) and the proof of 6.5.12 (b)). Suppose X is an irreducible $(\mathfrak{g}, K)$ module and $(\mathfrak{q}_V, H, \delta_V^L)$ a set of discrete θ-stable data attached to X. Then there is a character $\nu_V \in \hat{A}$ such that if $\mathcal{R}_{\mathfrak{q}_V}^{s}(I_V^L(\delta_V^L\otimes\nu))$ is the standard $(\mathfrak{g}, K)$ module with parameters $(\mathfrak{q}_V, H, \delta_V^L, \nu_V)$, then $\mathrm{Hom}_{\mathfrak{g},K}(X, \mathcal{R}_{\mathfrak{q}_V}^{s}(I_V^L(\delta_V^L\otimes\nu_V)))$ is one dimensional.

Lemma 2.4.20. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ-stable parabolic subalgebra and Y an $(\mathfrak{l}, L\cap K)$ module. Write

$s = \dim \mathfrak{u} \cap \mathfrak{k}$, $\lambda_V^L = \lambda_V(Y)$ and $X = \mathcal{R}_{\mathfrak{q}}^s(Y)$. Assume $\langle \lambda_V^L + \rho(\mathfrak{u}), \alpha \rangle > 0$; $\alpha \in \Delta(\mathfrak{u})$. Choose $\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l} \cap \mathfrak{k}) \cup \Delta(\mathfrak{u} \cap \mathfrak{k})$.

Suppose μ^L is the highest weight of a LKT (for $L \cap K$) of Y with respect to the positive system $\Delta^+(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t}^c)$ and that we choose $\Delta^+(\mathfrak{l})$ so that $\mu^L + 2\rho_{\mathfrak{l} \cap \mathfrak{k}}$ is dominant. Then

a) $\mu = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is dominant for $\Delta^+(\mathfrak{k})$.

b) $\mu + 2\rho_c$ is dominant for $\Delta^+(\mathfrak{g})$, where $\Delta^+(\mathfrak{g})$ is compatible with $\Delta^+(\mathfrak{k})$ and

$$\Delta^+(\mathfrak{g}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u}).$$

Proof. $\mu + 2\rho_c = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p}) + 2\rho(\mathfrak{u} \cap \mathfrak{k}) + 2\rho_{\mathfrak{l} \cap \mathfrak{k}}$

$$= \mu^L + 2\rho_{\mathfrak{l} \cap \mathfrak{k}} + 2\rho(\mathfrak{u}).$$

Suppose $\alpha \in \Delta^+(\mathfrak{g})$ is simple, then

i) If $\alpha \in \Delta^+(\mathfrak{l})$,

$$\langle \mu + 2\rho_c, \alpha \rangle = \langle \mu^L + 2\rho_{\mathfrak{l} \cap \mathfrak{k}}, \alpha \rangle + \langle 2\rho(\mathfrak{u}), \alpha \rangle$$

$$= \langle \mu^L + 2\rho_{\mathfrak{l} \cap \mathfrak{h}}, \alpha \rangle + 0$$

ii) If $\alpha \in \Delta(\mathfrak{u})$ then, for any simple root γ, $\langle \alpha, \gamma \rangle \leq 0$; hence $\langle \alpha, \beta \rangle \leq 0$ for $\beta \in \Delta^+(\mathfrak{l})$.

Choose $\{\beta_i\} \subseteq \Delta(\ell \cap p)$ as in Proposition 2.4.7, such that

$$\lambda_V^L = \mu^L + 2\rho_{\ell\cap k} - \rho_\ell + \frac{1}{2} \sum_{0 \le c_i \le 1} c_i \beta_i$$

then

$$(2.4.21) \qquad \mu + 2\rho_c = \lambda_V^L + \rho_\ell - \frac{1}{2} \sum c_i \beta_i + 2\rho(u)$$

then

$$\langle \check{\alpha}, \mu + 2\rho_c \rangle = \langle \check{\alpha}, \lambda_V^L + \rho(u) \rangle + \langle \check{\alpha}, \rho - \sum c_i \beta_i \rangle$$

$$> 0 + 1.$$

This proves b) of the lemma.

For (a), it is enough to prove that if $\gamma \in \Delta^+(k)$ is simple, then

$$\langle \check{\gamma}, \mu + 2\rho_c \rangle \ge 2.$$

i) If $\gamma \in \Delta^+(\ell \cap k)$

$$\langle \check{\gamma}, \mu + 2\rho_c \rangle = \langle \check{\gamma}, \mu^L + 2\rho(\mathfrak{l}\cap\mathfrak{k}) \rangle + \langle \check{\gamma}, 2\rho(\mathfrak{u}) \rangle$$

$$= \langle \check{\gamma}, \mu^L + 2\rho(\mathfrak{l}\cap\mathfrak{k}) \rangle + 0$$

$$\geq 2,$$

since μ^L is dominant for $\Delta^+(\mathfrak{l}\cap\mathfrak{k})$.

ii) If $\gamma \in \Delta(\mathfrak{u}\cap\mathfrak{k})$ then, as for b): $\langle \check{\gamma}, \mu + 2\rho_c \rangle > 1$, and it is an integer since $\mu + 2\rho_c$ exponentiates.

q.e.d.

Lemma 2.4.22. In the setting of Lemma 2.4.20, if μ is dominant for $\Delta^+(\mathfrak{g})$, then $\lambda_V^G(\mu) = \lambda_V^L + \rho(\mathfrak{u})$.

Proof. By 2.4.21, $\mu + 2\rho_c - \rho = \lambda_V^L + \rho(\mathfrak{u}) - \frac{1}{2}\sum c_i\beta_i$.

We claim that $\lambda_V^L + \rho(\mathfrak{u})$ satisfies the conditions for $\lambda_V(\mu)$ in Proposition 2.4.7:

Condition (a) holds by the definition of $\lambda_V^L + \rho(\mathfrak{u})$.

Since $\Delta(\bar{\mathfrak{u}}) = -\Delta(\mathfrak{u})$ and $\langle \check{\alpha}, \lambda_V^L + \rho(\mathfrak{u}) \rangle > 0$ for $\alpha \in \Delta(\mathfrak{u})$, 2.4.7 (b) holds because $\langle \check{\alpha}, \lambda_V^L + \rho(\mathfrak{u}) \rangle = 0$ implies $\alpha \in \Delta(\mathfrak{l})$.

But then, $\langle \alpha, \rho(\mathfrak{u}) \rangle = 0$; hence $\langle \check{\alpha}, \beta_i \rangle \neq 0$ for some β_i.

Since simple roots for $\Delta(\mathfrak{l}, \mathfrak{h}^c)$ are simple for $\Delta(\mathfrak{g}, \mathfrak{h}^c)$, 2.4.7 (c) (d) (e) hold.

q.e.d.

In the setting of Lemma 2.4.20, let (π, Z) be a K-type occurring in X with highest weight η.

We want to estimate the lambda-norm of η.

Let $\mathfrak{q}_V^L = \mathfrak{q}_V^L(Y) = \mathfrak{u}_1 + \mathfrak{l}_1 \subseteq \mathfrak{l}$.

$$\mathfrak{q}_V = \mathfrak{l}_1 + \mathfrak{u}_1 + \mathfrak{u} \subseteq \mathfrak{q}$$

and $(\mathfrak{q}_V^L, H_V, \delta_V^L, \nu_V^L)$ a set of θ-stable data for L attached to Y (Definition 2.4.12).

Let $\lambda^G = d\delta_V^L + \rho(\mathfrak{u}_V)$ as in Definition 2.4.12.

Lemma 2.4.23. With notation as in Lemma 2.4.20

a) $(\mathfrak{q}_V, H_V, \delta_V^L, \nu_V^L)$ is a set of θ-stable data for G.

b) If V_η is a K-type in X, then $\langle \lambda_V(\eta), \lambda_V(\eta) \rangle \geq \langle \lambda^G, \lambda^G \rangle$.

c) If equality holds in b) then $\eta = \eta^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ for η^L a highest weight of a L(L∩K)T of Y and V_η is a LKT of X.

d) Conversely if $\eta = \eta^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$, then V_η is a LKT of X and equality holds.

Proof. For (a) we only need to see that $\langle \lambda^G, \check{\alpha} \rangle > 0$ for all roots in $\Delta(\mathfrak{u})$.

By hypothesis on $(\mathfrak{q}_V^L, H_V, \delta_V^L, \nu_V^L)$,

$$\lambda_V^L = d\delta_V^L + \rho(\mathfrak{u}_1).$$

$$\langle \lambda_V^L, \check{\alpha} \rangle > 0 \quad \text{for} \quad \alpha \in \Delta(\mathfrak{u}_1),$$

$H = TA$ is maximally split Cartan of L_1 and δ_V^L is fine.

Now

$$\lambda^G = d\delta_V^L + \rho(\mathfrak{u}_V) = d\delta_V^L + \rho(\mathfrak{u}_1) + \rho(\mathfrak{u}) = \lambda_V^L + \rho(\mathfrak{u}).$$

If $\alpha \in \Delta(\mathfrak{u}_1) \subseteq \Delta(\mathfrak{l})$ then

$$\langle \lambda_V^L + \rho(\mathfrak{u}), \check{\alpha} \rangle = \langle \lambda_V^L, \check{\alpha} \rangle > 0.$$

If $\alpha \in \Delta(\mathfrak{u})$, by hypothesis in Lemma 2.4.20,

$$\langle \lambda^G, \check{\alpha} \rangle > 0.$$

The proof for b) and c) is exactly the proof of Lemma 6.5.6 in Vogan [1981].

q.e.d.

2.5. The Modules $A_{\mathfrak{q}}(\lambda)$.

Let G be a connected real reductive linear Lie group, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ a θ-stable parabolic subalgebra and L the normalizer of $\mathfrak{q}$ in G. Then $\mathfrak{l}_0 = \text{Lie}(L)$.

Let $\lambda : \mathfrak{l} \longrightarrow \mathbb{C}$ be a one-dimensional representation of $\mathfrak{l}$. Assume that

$$(2.5.1)\quad \begin{cases} \text{a)}\ \lambda \text{ is the differential of a unitary} \\ \qquad \text{character of } L \text{ (call it } \lambda \text{ also).} \\ \text{b)}\ \langle \lambda|_{t^c}, \alpha\rangle \geq 0 \text{ for all } \alpha \in \Delta(u, t^c). \end{cases}$$

We say that λ is an admissible representation of ℓ.

Definition 2.5.2. With notation as above, we define the Harish-Chandra module $A_q(\lambda)$ by

$$A_q(\lambda) = \mathcal{R}^s_q(\mathbb{C}_\lambda) \qquad \text{(Definition 2.4.14)}$$

with

$$s = \dim u \cap k.$$

Fix positive root systems

$$\Delta^+(\ell \cap k) \quad \text{and}$$

$$\Delta^+(\ell) = \Delta^+(\ell, t), \quad \text{compatible with } \Delta^+(\ell \cap k).$$

Then

$$\Delta^+(k) = \Delta^+(\ell \cap k) \cup \Delta(u \cap k)$$

and

$$\Delta^+(g) = \Delta^+(q) = \Delta^+(\ell) \cup \Delta(u)$$

are positive t-root systems for k and g, respectively. Choose a fundamental Cartan subalgebra $h^c = t^c + a^c$ and a positive root system $\Delta^{+}(\mathfrak{g},\mathfrak{h}^{c})$ so that

$$\Delta^{+}(\mathfrak{g},\mathfrak{h}^{c})\Big|_{\mathfrak{t}^{c}} = \Delta^{+}(\mathfrak{g}).$$

Then
$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^{+}(\mathfrak{g},\mathfrak{h}^{c})}\alpha = \frac{1}{2}\sum_{\beta\in\Delta^{+}(\mathfrak{g})}\beta.$$

Proposition 2.5.3 (Vogan-Zuckerman [1984]. See also Speh-Vogan [1980] and Vogan [1981]). Regard $\lambda\Big|_{\mathfrak{t}^{c}}$ as a weight in $(\mathfrak{t}^{c})^{*}$. Let

$$\mu = \lambda\Big|_{\mathfrak{t}^{c}} + 2\rho(\mathfrak{u}\cap\mathfrak{p}) \in (\mathfrak{t}^{c})^{*}.$$

a) The $(\mathfrak{g},K)$ module $A_{\mathfrak{q}}(\lambda)$ is the unique irreducible module satisfying:

i) As a K-representation, $A_{\mathfrak{q}}(\lambda)$ contains the K-type with highest weight μ.

ii) $Z(\mathfrak{g})$ acts on $A_{\mathfrak{q}}(\lambda)$ by the character $\chi_{\lambda+\rho} : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$; where $\chi_{\lambda+\rho}(z) = (\lambda+\rho)(\xi(z))$ and ξ is the Harish-Chandra homomorphism.

iii) Any K-type occurring in $A_{\mathfrak{q}}(\lambda)$ has a highest weight of the form

$$\eta = \lambda\Big|_{\mathfrak{t}^{c}} + 2\rho(\mathfrak{u}\cap\mathfrak{p}) + \sum_{\substack{\beta\in\Delta(\mathfrak{u}\cap\mathfrak{p})\\ n_{\beta}\in\mathbb{N}}} n_{\beta}\beta.$$

b) Moreover μ is the unique LKT of $A_{\mathfrak{q}}(\lambda)$.

Proof. The infinitesimal character of the representation $\lambda : L \longrightarrow \mathbb{C}$ is $\lambda + \rho_{\mathfrak{l}}$. Then by Proposition 2.4.16 $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho_{\mathfrak{l}} + \rho(\mathfrak{u}) = \lambda + \rho$. So ii) holds.

If $\mu^L = \lambda|_{\mathfrak{t}^c}$ then μ^L is the highest weight of the (lowest) $L \cap K$-type of $\mathbb{C}_\lambda$.

Choose $\Delta^+(\mathfrak{l})$ making $\mu^L + 2\rho_{\mathfrak{l}\cap\mathfrak{k}}$ dominant. Then

$$\lambda_V^L = \lambda_V^L(\mu^L) = \lambda\Big|_{\mathfrak{t}^c} + 2\rho_{\mathfrak{l}\cap\mathfrak{k}} - \rho_{\mathfrak{l}} + \sum c_i\beta_i$$

$$= \lambda\Big|_{\mathfrak{t}^c} + Q$$

with Q a sum of roots in $\mathfrak{l}$.

But $\mu^L\big|_{\Delta(\mathfrak{l})} \equiv 0$ so $\Delta^+(\mathfrak{l})$ can be chosen so that Q is dominant.

Then if $\alpha \in \Delta(\mathfrak{u})$ is simple, $\langle\lambda_V^L + \rho(\mathfrak{u}),\alpha\rangle > 0$. In fact

$$\langle\lambda_V^L + \rho(\mathfrak{u}),\alpha\rangle = \langle\lambda\Big|_{\mathfrak{t}^c} + \rho(\mathfrak{u}),\alpha\rangle \geq \langle\rho(\mathfrak{u}),\alpha\rangle$$

$$> 0.$$

By Lemmas 2.4.20, 2.4.22, and 2.4.23, i) and b) hold.

The irreducibility and uniqueness of $A_{\mathfrak{q}}(\lambda)$ take more work, and since we won't be using these facts we refer to Speh-Vogan [1980]. See also Vogan [1981].

By 2.5.4 and Theorem 1.3 in Vogan [1984], we have the following.

Proposition 2.5.5. In the above setting, the modules $A_{\mathfrak{q}}(\lambda)$ are unitarizable.

Proposition 2.5.6. Fix $\Delta^{+}(\mathfrak{k})$. Let $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i \subseteq \mathfrak{g}$; $i = 1,2$, be θ-stable parabolic subalgebras such that $\Delta(\mathfrak{q}_i) \supseteq \Delta^{+}(\mathfrak{k})$ and $\lambda_i \in \mathfrak{t}^{*}$ admissible one-dimensional representations of $\mathfrak{l}_i$ (Definition 2.5.1). Then,

$$A_{\mathfrak{q}_1}(\lambda_1) \cong A_{\mathfrak{q}_2}(\lambda_2)$$

$$\Leftrightarrow \qquad \lambda_1 = \lambda_2 \quad \text{and} \quad \mathfrak{u}_1 \cap \mathfrak{p} = \mathfrak{u}_2 \cap \mathfrak{p}.$$

Proof. We need a few lemmas:

Lemma 2.5.7. Suppose $\tilde{\mathfrak{q}} = \tilde{\mathfrak{l}} + \tilde{\mathfrak{u}}$, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$, are θ-stable parabolic subalgebras, and $\lambda : \mathfrak{l} \longrightarrow \mathbb{C}$ admissible representations such that

$$(2.5.8)\qquad \begin{array}{ll} 1) & \tilde{\mathfrak{q}} \supseteq \mathfrak{q}, \quad \text{that is,} \quad \tilde{\mathfrak{l}} \supseteq \mathfrak{l} \quad \text{and} \quad \mathfrak{u} \supseteq \tilde{\mathfrak{u}}. \\ 2) & \lambda \perp \Delta(\tilde{\mathfrak{l}}). \\ 3) & \mathfrak{u} \cap \mathfrak{p} = \tilde{\mathfrak{u}} \cap \mathfrak{p}. \end{array}$$

Then $A_{\tilde{\mathfrak{q}}}(\lambda) \cong A_{\mathfrak{q}}(\lambda)$.

Proof. By induction by stages (Proposition 2.4.15),

$$\mathcal{R}^{s}_{\mathfrak{q}}(\mathbb{C}_{\lambda}) \cong \mathcal{R}^{\tilde{s}}_{\tilde{\mathfrak{q}}}(\mathcal{R}^{\dim\, \tilde{\mathfrak{l}}\cap(\mathfrak{u}\cap\mathfrak{k})}_{\mathfrak{q}\cap\tilde{\mathfrak{l}}}(\mathbb{C}_{\lambda}))$$

but $\mathfrak{q} \cap \tilde{\mathfrak{l}} = \mathfrak{l} + \mathfrak{u} \cap \tilde{\mathfrak{l}}$ and, by 3), $\mathfrak{u} \cap \tilde{\mathfrak{l}} \subseteq \mathfrak{k}$ so

$$\mathcal{R}^{\dim\, \mathfrak{u}\cap\tilde{\mathfrak{l}}}_{\mathfrak{q}\cap\tilde{\mathfrak{l}}}(\mathbb{C}_{\lambda}) \cong \mathbb{C}_{\lambda}.$$

Hence $\mathcal{R}^{s}_{\mathfrak{q}}(\mathbb{C}_{\lambda}) \cong \mathcal{R}^{\tilde{s}}_{\tilde{\mathfrak{q}}}(\mathbb{C}_{\lambda}) = A_{\tilde{\mathfrak{q}}}(\lambda)$ this proves the lemma.

q.e.d.

By this lemma, we may assume that both $\mathfrak{q}_i$'s in the proposition are maximal with respect to conditions 1) - 3).

Lemma 2.5.9. In the above setting

$$\Delta(\mathfrak{l}_i\cap\mathfrak{k}) = \{\alpha \in \Delta(\mathfrak{g}) \mid \langle\alpha,\lambda_i + 2\rho(\mathfrak{u}_i\cap\mathfrak{p})\rangle = 0\}.$$

Proof. Suppose $\alpha \in \Delta^+(k, t^c)$ is a simple root so that

a) $\alpha \notin \Delta(l_i \cap k)$.

b) $\langle \alpha, \mu_i \rangle = 0, \quad \mu_i = \lambda_i + 2\rho(u_i \cap p)$.

Let $$\Delta(\bar{l}) = \mathrm{Span}(\Delta(l_i), \alpha) \cap \Delta(g)$$

$$\Delta(\bar{u}) = \Delta(u_i) \setminus \Delta(\bar{l})$$

$$\bar{q} = \bar{l} + \bar{u}.$$

We want to contradict the maximality of q_i.

Breaking up $\Delta(u_i \cap p)$ in maximal α strings

$$\{\gamma_0; \gamma_0 + \alpha; \ldots \gamma_0 + r\alpha\},$$

$$(\text{i.e. } \gamma_0 - \alpha,\ \gamma_0 + (r+1)\alpha \notin \Delta(u_i \cap p))$$

and using representation theory of $sl(2)$ we can conclude that

$$\langle \alpha, 2\rho(u_i \cap p) \rangle \geq 0$$

and we have equality if and only if $u_i \cap p$ is invariant under the three dimensional subalgebra g^α that contains the α-root vector X_α.

But, by definition of λ_i, $\langle \alpha, \lambda_i \rangle \geq 0$.

So, (a) and (b) imply that $\mathfrak{u}_i \cap \mathfrak{p}$ is invariant under $\mathfrak{g}^\alpha$ and

$$\langle \alpha, \lambda_i \rangle = 0 = \langle \alpha, 2\rho(\mathfrak{u}_i \cap \mathfrak{p}) \rangle .$$

Now we want to prove that

$$(2.5.10) \qquad \bar{\mathfrak{u}} \cap \mathfrak{p} = \mathfrak{u}_i \cap \mathfrak{p}.$$

If $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ and $\beta\Big|_{\mathfrak{t}^c} = \alpha$ then

$$s_\alpha \left[\beta \Big|_{\mathfrak{t}^c} \right] = -\beta \Big|_{\mathfrak{t}^c} .$$

If β is complex, then the non-compact root of $-\beta\Big|_{\mathfrak{t}^c}$ is not in $\Delta(\mathfrak{u}_i \cap \mathfrak{p})$ so it contradicts invariance under $\mathfrak{g}^\alpha$.

Hence α is an imaginary root of $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$. α is also simple for $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$. In fact, since α is simple for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$, and $\alpha \notin \Delta(\mathfrak{l} \cap \mathfrak{k})$ we can assume that if γ, $\delta \in \Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ and $\alpha = \gamma + \delta$ then

$$\gamma \in \Delta(\mathfrak{u}_i \cap \mathfrak{p}),$$

say, and $\gamma - \alpha = -\beta \notin \Delta(\mathfrak{u}_i \cap \mathfrak{p})$; contradicting invariance again.

Consider a simple factor $\mathfrak{l}_0 \subseteq \bar{\mathfrak{l}}$, not contained in $\mathfrak{l}$. Then $\mathfrak{l}_0$ is not orthogonal to α. Let $\{\beta_1, \beta_2, \ldots, \beta_\ell\}$ be a set of simple roots for $\mathfrak{l}_0$ containing α.

Say $\alpha = \beta_{i_0}$ and β_{i_0+1} is adjacent to α.

Suppose $\mathfrak{l}_0 \cap \mathfrak{p} \neq 0$. Then there is a non-compact root $\beta = \sum n_i \beta_i$ with some $n_{i_0+1} > 0$ and such that

$$\langle \alpha, \beta \rangle = \sum n_i \langle \alpha, \beta_i \rangle < 0.$$

$\alpha + \beta = \delta$ is a non-compact root, and $\delta \in \Delta(\mathfrak{u}_i \cap \mathfrak{p})$. So the string through δ is not complete.

Hence $\mathfrak{l}_0$ is compact and $\mathfrak{q}$ $(\subseteq \bar{\mathfrak{q}})$ is not maximal satisfying (2.5.8).

This proves Lemma 2.5.9.

q.e.d.

We are now able to prove Proposition 2.5.6.

By Lemma 2.5.9,

$$\mathfrak{l}_1 \cap \mathfrak{k} = \mathfrak{l}_2 \cap \mathfrak{k}$$

$$\mathfrak{u}_1 \cap \mathfrak{k} = \mathfrak{u}_2 \cap \mathfrak{k}$$

hence $\lambda_1 + 2\rho(\mathfrak{u}_1) = \lambda_2 + 2\rho(\mathfrak{u}_2)$. But $\langle \lambda_i \beta \rangle = \langle 2\rho(\mathfrak{u}_i), \beta \rangle = 0$ for all $\beta \in \Delta(\mathfrak{l}_i)$ and $\langle 2\rho(\mathfrak{u}_i), \alpha \rangle > 0$, $\langle \lambda_i, \alpha \rangle \geq 0$, $\alpha \in \Delta(\mathfrak{u}_i)$. So

$$\Delta(\mathfrak{l}_i) = \{\beta \in \Delta(\mathfrak{g},\mathfrak{t}^c) \mid \langle \lambda_i + 2\rho(\mathfrak{u}_i),\beta\rangle = 0\}$$

$$\Delta(\mathfrak{u}_i) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{t}^c) \mid \langle \lambda_i + 2\rho(\mathfrak{u}_i),\alpha\rangle > 0\}.$$

Hence
$$\mathfrak{u}_1 \cap \mathfrak{p} = \mathfrak{u}_2 \cap \mathfrak{p}$$

and
$$\lambda_1 = \lambda_2.$$

This proves Proposition 2.5.6.

q.e.d.

2.6. Reduction step for the proof of Theorem 1.3.

We are now in a position to prove the main result stated in Chapter 1. We will argue by contradiction and reduction to a proper subgroup $L \subsetneq G$.

Suppose $X \in \mathcal{A}(\mathfrak{g},K)$ is irreducible and has a Hermitian form $\langle\ ,\ \rangle$. We will assume X cannot be realized as an $A_{\mathfrak{q}}(\lambda)$ module, but will exhibit X as a Langlands submodule of some derived functor module induced from an $(\mathfrak{l},L\cap K)$ module X_L, making sure that this information can be carried over to G and X.

We need to keep track of the existence of Hermitian forms at different steps of induction as well as of their signatures on some finite sets of K-types.

Recall from Vogan [1984] (Definition 2.10) the Hermitian dual of a $(\mathfrak{g},K)$ module Y

$$Y^h = \{f : Y \longrightarrow \mathbb{C} \mid \dim U(\mathfrak{k})\cdot f < \infty;$$

$$f(\lambda x) = \bar{\lambda} f(x), \quad \lambda \in \mathbb{C} \quad x \in Y\}.$$

Y^h is a $(\mathfrak{g},K)$ module.

Definition 2.6.1. An invariant, symmetric Hermitian form on a $(\mathfrak{g},K)$ module Y is a pairing

$$\langle\ ,\ \rangle : Y \times Y \longrightarrow \mathbb{C}$$

such that

a) $\langle x, ay+bz\rangle = \bar{a}\langle x,y\rangle + \bar{b}\langle x,z)$

$$\langle ax+bw, y\rangle = a\langle x,y\rangle + b\langle w,y\rangle$$

for $a, b \in \mathbb{C}$, $x, y, z, w \in Y$.

b) $\langle (U+iV)x, y\rangle = -\langle x, (U-iV)y\rangle$

$$U, V \in \mathfrak{g}_0 \quad y, x \in Y.$$

c) $\langle k\cdot x, y\rangle = \langle x, k^{-1}\cdot y\rangle \qquad k \in K.$

d) $\langle x,y\rangle = \overline{\langle y,x\rangle}.$

The radical of $\langle\ ,\ \rangle$ is

$$\mathrm{Rad}(\langle\ ,\ \rangle) = \{x \in Y \mid \langle x, Y\rangle = 0\}.$$

It is clear that invariant symmetric Hermitian forms on Y are given by $(\mathfrak{g}K)$ maps $f : Y \longrightarrow Y^h$ such that $f = f^h : Y^h \longrightarrow Y$. Moreover we have

Proposition 2.6.2. Suppose $X \in \mathcal{A}(\mathfrak{g}, K)$ is irreducible. Then X admits a non-zero invariant Hermitian form if and only if

$$X \cong X^h.$$

In this case the Hermitian form is non-degenerate and any two such forms differ by multiplication by a real constant.

Proposition 2.6.3. Let $X \in \mathcal{A}(\mathfrak{g}, K)$ be irreducible and $(\mathfrak{q}_V, H_V, \delta_V, \nu_V)$ a set of θ-stable data attached to X, so that

$$\dim\left[\mathrm{Hom}_{\mathfrak{g},K}(X, \mathcal{R}^s_{\mathfrak{q}_V}(I^{L_V}(\delta_V \otimes \nu_V))\right] = 1$$

(see Proposition 2.4.19). Let $H_V = TA$. Then $X \cong X^h$ if and only if there is an element

$$\omega \in W(L,A) \quad \text{such that}$$

$$\omega\delta = \delta \quad \text{and} \quad \omega\nu = -\bar{\nu}.$$

In this case we get a Hermitian form on X from a form on $\mathcal{R}^s_{\mathfrak{q}_V}(I^{L_V}(\delta_V \oplus \nu_V))$.

This result is essentially due to Knapp and Zuckerman [1976].

A formulation close to this one is in Vogan [1984], Corollary 2.15.

Corollary 2.6.4. Let $X \in \mathcal{A}(\mathfrak{g},K)$, irreducible, endowed with a non-zero Hermitian form $\langle\ ,\ \rangle$. Write $\mathfrak{q}_V = \mathfrak{q}_V(X)$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra such that $\mathfrak{l} \supset \mathfrak{l}_V$, $\mathfrak{u} \subset \mathfrak{u}_V$ and $(\mathfrak{q}_V, H_V, \delta_V, \nu_V)$, a θ-stable data attached to X. Write

$$X_L = \mathcal{R}^{\mathfrak{l}}_{\mathfrak{l}\cap\mathfrak{q}_V}(I_{L_V}(\delta_V \oplus \nu_V)).$$

Then

$$X^h_L \text{ has a Hermitian form } \langle\ ,\ \rangle^L.$$

Proof. This is a formal consequence of Proposition 2.6.3.

q.e.d.

Proposition 2.6.5. Fix $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$, a θ-stable parabolic subalgebra. Suppose $Y \in \mathcal{M}(\mathfrak{l}, L\cap K)$ is equipped with a (possibly degenerate) invariant Hermitian form $\langle\ ,\ \rangle^L$.

Then there is a natural invariant Hermitian form $\langle\ ,\ \rangle^G$ on $[\mathcal{R}^s_{\mathfrak{q}}(Y^h)]^h$.

Proof. Recall from Vogan [1981] Chapter 6, Definition 6.1.5 the functors

$$\mathrm{ind}^{\mathfrak{g}}_{\bar{\mathfrak{q}}} : \mathcal{M}(\mathfrak{l}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, L\cap K).$$

$$\mathrm{ind}^{\mathfrak{g}}_{\bar{\mathfrak{q}}} Y = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} Y.$$

Write

$$\mathcal{L}^j : \mathcal{M}(\mathfrak{l}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$$

$$\mathcal{L}^j_{\bar{\mathfrak{q}}} Y = \mathcal{L}^j Y = \Gamma^j\ \mathrm{ind}^{\mathfrak{g}}_{\bar{\mathfrak{q}}}(Y \otimes \Lambda^{top}\mathfrak{u})$$

where $\Gamma^j : \mathcal{M}(\mathfrak{g}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ are the Zuckerman's functors (cf. Definition 2.4.14). Set $\tilde{Y} = Y \otimes \Lambda^{top}\ \mathfrak{u}$.

By hypothesis, we have a map

$$\phi^L : Y \longrightarrow Y^h.$$

This induces a map

$$\phi^{\mathfrak{q}} : \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\tilde{Y}) \longrightarrow \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(\tilde{Y}^h)$$

and

$$\phi^G : \mathcal{L}_{\bar{\mathfrak{q}}}^s Y \longrightarrow \mathcal{R}_{\mathfrak{q}}^s Y^h.$$

By Theorem 5.3 (Enright-Wallach) in Vogan [1984]

$$\mathcal{R}_{\mathfrak{q}}^{2s-i}(Y^h) \cong (\mathcal{L}_{\bar{\mathfrak{q}}}^i Y)^h.$$

Let

$$\langle \ , \ \rangle : \mathcal{L}_{\bar{\mathfrak{q}}}^s Y \longrightarrow (\mathcal{L}_{\bar{\mathfrak{q}}}^s Y)^h$$

be the natural pairing given by Definition 2.10 in Vogan [1984].

Define

$$\langle u, v\rangle^G = \langle u, \phi^G v\rangle.$$

This gives an invariant Hermitian form on $\mathcal{L}^s(Y)$ (cfr. the proof of Corollary 5.5, Vogan [1984]).

q.e.d.

Definition 2.6.6. If $Z \in \mathcal{M}(\mathfrak{g},K)$ and $\delta \in \hat{K}$, write

$$Z(\delta) = \mathrm{Hom}_K(V_\delta, Z).$$

Then,

$$(2.6.7) \qquad Z \cong \bigoplus_{\delta \in \hat{K}} Z(\delta) \otimes V_\delta.$$

If we fix a positive definite form on V_δ, $Z(\delta)$ inherits a Hermitian form. Suppose Z is equipped with a non-zero Hermitian form $\langle\ ,\ \rangle$. Write $p(\delta)$ (resp. $q(\delta)$, $z(\delta)$), for the multiplicity of V_δ in the subspace of $Z(\delta)$ where $\langle\ ,\ \rangle$ is positive (resp. negative or zero).

Write the signature of $\langle\ ,\ \rangle$ on $Z(\delta)$ as $\mathrm{sgn}(\langle\ ,\ \rangle|_{Z(\delta)}) = (p(\delta), q(\delta), z(\delta))$.

Then write, formally

$$\mathrm{sgn}(\langle\ ,\ \rangle) = \sum_{\delta \in \hat{K}} (p(\delta), q(\delta), z(\delta)).$$

We will prove in the next chapters the following result.

Theorem 2.6.7. Let $G = SL(n,\mathbb{R})$, $SU(p,q)$ or $SP(n,\mathbb{R})$ and $X \in \mathcal{A}(\mathfrak{g},K)$ irreducible, endowed with a non-zero invariant Hermitian form $\langle\ ,\ \rangle$ and regular integral infinitesimal character.

If $X \cong A_{\mathfrak{q}'}(\lambda')$, for any q' and λ'. Then there are a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, an $(\mathfrak{l}, L\cap K)$-module X_L and $(L\cap K)$-types δ_i^L $i = 1,2$ such that

a) X is the unique irreducible submodule of $\mathcal{R}_{\mathfrak{q}}(X_L)$, and X occurs only once as a composition factor of $\mathcal{R}_{\mathfrak{q}}(X_L)$.

b) X_L^h is endowed with a Hermitian form $\langle\ ,\ \rangle^L \neq 0$. Write (p_L, q_L, z_L) for its signature. Then

$$p_L(\delta_1^L) \neq 0 \qquad \text{and} \qquad q_L(\delta_2^L) \neq 0.$$

c) Choose $\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l}\cap\mathfrak{k}) \cup \Delta(\mathfrak{u}\cap\mathfrak{k})$. Then, if δ_i^L has highest weight μ_i^L, $\mu_i = \mu_i^L + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant.

Chapters 3, 4, 5 will be devoted to the proof of this result. Assume this for the moment.

Using this result, we want to prove non-unitarity of X. We need to check that the Hermitian form $\langle\ ,\ \rangle^G$ induced on $\mathcal{R}_{\mathfrak{q}}(X_L)^h$ by Proposition 2.6.5 is a multiple of $\langle\ ,\ \rangle$ on X; that for the $L \cap K$ types satisfying c) of Theorem 2.6.7, the corresponding K types occur in X and that the signature of the form on these K-types is the same as that of $\langle\ ,\ \rangle^L$ on the δ_i^L.

Theorem 2.6.8. Suppose $X \in \mathcal{A}(\mathfrak{g}, K)$ is irreducible and has a non-zero Hermitian form $\langle\ ,\ \rangle$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be θ-stable and X_L an $(\mathfrak{l}, L\cap K)$ module such that X is the unique irreducible submodule of $\mathcal{R}_{\mathfrak{q}}(X_L)$, X occurs only once as composition factor in $\mathcal{R}^s_{\mathfrak{q}}(X_L)$ and X^h_L has a non-zero Hermitian form $\langle\ ,\ \rangle^L$. If $\delta^L \in (L\cap K)^{\wedge}$ is an $(L\cap K)$-type of X_L with highest weight μ^L such that $\mu = \mu^L + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ is dominant for $\Delta(\mathfrak{u}\cap\mathfrak{k})$ then if $\delta \in \hat{K}$ has highest weight μ, $X(\delta) \neq 0$ and

$$\mathrm{Sign}\left[\langle\ ,\ \rangle\big|_{X(\delta)}\right] = \mathrm{Sgn}\left[\langle\ ,\ \rangle^L\big|_{X_L(\delta^L)}\right]$$

Proof. Applying the appropriate definitions and results to K and $\mathfrak{q} \cap \mathfrak{k}$ we have maps

$$\mathcal{R}^i_{\mathfrak{q}\cap\mathfrak{k}} : \mathcal{M}(\mathfrak{l}\cap\mathfrak{k}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{k}, K)$$

$$\mathcal{L}^j_{\bar{\mathfrak{q}}\cap\mathfrak{k}} : \mathcal{M}(\mathfrak{l}\cap\mathfrak{k}, L\cap K) \longrightarrow \mathcal{M}(\mathfrak{k}, K).$$

If $Y \in \mathcal{M}(\mathfrak{l}, L\cap K)$ there are natural maps

$$\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}\tilde{Y} \longrightarrow\!\!\!\!\rightarrow \mathrm{pro}^{\mathfrak{k}}_{\mathfrak{q}\cap\mathfrak{k}}\tilde{Y}$$

$$\mathrm{ind}^{\mathfrak{k}}_{\bar{\mathfrak{q}}\cap\mathfrak{k}}\tilde{Y} \longrightarrow \mathrm{ind}^{\mathfrak{g}}_{\bar{\mathfrak{q}}}\tilde{Y}.$$

These induce (k,K)-maps

$$\mathcal{R}^{i}_{\mathfrak{q}}Y \xrightarrow{\ r\ } \mathcal{R}^{i}_{\mathfrak{q}\cap k}Y$$

$$\mathcal{L}^{j}_{\bar{\mathfrak{q}}}Y \xrightarrow{\ \iota\ } \mathcal{L}^{j}_{\bar{\mathfrak{q}}}Y \ .$$

Then, the following diagram is commutative

$$\begin{array}{ccc} [\mathcal{L}^{i}_{\bar{\mathfrak{q}}}Y]^{h} & \xrightarrow{\ \cong\ } & \mathcal{R}^{2s-i}_{\mathfrak{q}}(Y^{h}) \\ \downarrow{\scriptstyle \iota^{h}} & & \downarrow{\scriptstyle r} \\ [\mathcal{L}^{i}_{\bar{\mathfrak{q}}\cap k}Y]^{h} & \xrightarrow{\ \cong\ } & \mathcal{R}^{2s-i}_{\mathfrak{q}}(Y^{h}) . \end{array}$$

The isomorphisms across are Theorem 5.3 in Vogan [1984] for $(G,\mathfrak{q})$ and $(K,\mathfrak{q}\cap k)$, respectively.

Arguing as in the Proof of 2.6.5 (for K) we have maps

$$\phi^{\mathfrak{q}\cap k} : \mathrm{ind}^{k}_{\bar{\mathfrak{q}}\cap k} Y \longrightarrow \mathrm{pro}^{k}_{\mathfrak{q}\cap k}Y^{h}$$

$$\phi^{k} : \mathcal{L}^{s}_{\bar{\mathfrak{q}}\cap k} Y \longrightarrow \mathcal{R}^{s}_{\mathfrak{q}\cap k}Y^{h}.$$

and we have the following commutative diagram

(2.6.10)

$$\begin{array}{ccccc} \mathcal{L}^{s}_{\bar{q}} Y & \xrightarrow{\phi^{G}} & \mathcal{R}^{s}_{q}(Y^{h}) & \cong & (\mathcal{L}^{s}_{\bar{q}} Y)^{h} \\ \downarrow \iota & & \downarrow r & & \downarrow \iota^{h} \\ \mathcal{L}^{s}_{\bar{q}\cap k} Y & \xrightarrow{\phi^{K}} & (\mathcal{R}^{s}_{q\cap k} Y^{h}) & \cong & (\mathcal{L}^{s}_{\bar{q}\cap k} Y)^{h} \end{array}$$

And we have a Hermitian form on $\mathcal{L}^{s}_{\bar{q}\cap k}(Y)$

$$\langle x,y\rangle^{K} = \langle x,\phi^{K}y\rangle.$$

Since $\phi^{K} = r\circ\phi^{G}\circ\iota$, and by Proposition 6.10 in Vogan [1984], ι is a unitary map,

(2.6.11) $$\langle x,y\rangle^{K} = \langle \iota x,\iota y\rangle^{G}.$$

(2.6.12) Write

$$\operatorname{sign}(\langle\ ,\ \rangle^{K}) = (p_K,q_K,z_K); \quad p_K,\ q_K,\ z_K : \hat{K} \longrightarrow \mathbb{N}$$

$$\operatorname{sign}(\langle\ ,\ \rangle^{G}) = (p_G,q_G,z_G); \quad p_G,\ q_G,\ z_G : \hat{K} \longrightarrow \mathbb{N}$$

and again

$$\operatorname{sign}(\langle\ ,\ \rangle^{L}) = (p_L,q_L,z_L); \quad p_L,\ q_L,\ z_L : (L\cap K)^{\wedge} \longrightarrow \mathbb{N}$$

By 2.6.11,

$$p_G(\delta) \geq p_K(\delta)$$

$$q_G(\delta) \geq q_K(\delta)$$

$$z_G(\delta) \geq z_K(\delta).$$

The main ingredient in the proof of Proposition 2.6.8 is the following result due to T. Enright.

Proposition 2.6.13 (Enright [1984]). Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ θ-stable parabolic.

Let $\delta^L \in (L\cap K)^\wedge$ with highest weight μ^L. Set $\mu = \mu^L + 2\rho(\mathfrak{u}\cap\mathfrak{p})$.

a) If μ is not $\Delta(\mathfrak{u}\cap\mathfrak{k})$-dominant, then

$$\mathcal{L}^s_{\mathfrak{q}\cap\mathfrak{k}} Y(\delta^L) = 0.$$

b) If μ is $\Delta(\mathfrak{u}\cap\mathfrak{k})$ dominant, write $\delta \in \hat{K}$ for the representation of K with highest weight μ. Then

$$p_K(\delta) = p_L(\delta^L)$$

$$q_K(\delta) = q_L(\delta^L)$$

$$z_K(\delta) = z_L(\delta^L).$$

For a proof of this result see Vogan [1984] 6.5–6.8.

Lemma 2.6.14. Suppose V is a module of finite length and S is irreducible.

Assume

a) $S \subseteq V$ occurs exactly once as a composition factor of V.

b) Any non-zero $W \subseteq V$ contains S.

c) S is equipped with a Hermitian form.

Then, up to scalars, V^h has a unique Hermitian form $\langle\ ,\ \rangle_1$ and

$$S \cong V^h / \mathrm{rad}(\langle\ ,\ \rangle_1).$$

The proof of this lemma is standard. We can now prove Theorem 2.6.8. By Proposition 2.6.13 and 2.6.11

$$p_G(\delta) \geq p_L(\delta^L) \tag{2.6.15}$$

$$q_G(\delta) \geq q_L(\delta^L)$$

and

$$z_G(\delta) \geq z_L(\delta^L).$$

Apply Lemma 2.6.14 to

$$V = \mathcal{R}_{\mathfrak{q}}^{s}(X_L) \qquad \text{and} \qquad S = X.$$

We know that a) - c) hold in this Lemma since they are part of our assumptions on X. We also know that $\langle\ ,\ \rangle^G \neq 0$ by 2.6.15.

Hence, we have the following result:

Proposition 2.6.16. In the setting of Theorem 2.6.8

$$\langle\ ,\ \rangle^G\big|_X = c\langle\ ,\ \rangle$$

$$X \cong [\mathfrak{R}_{\mathfrak{q}}^s(X_L)]^h/\mathrm{rad}(\langle\ ,\ \rangle^G).$$

So $\langle\ ,\ \rangle^G\big|_X$ is nondegenerate and has signature

$$\mathrm{sgn}(\langle\ ,\ \rangle) = (p_G, q_G).$$

q.e.d.

It is now straightforward to prove Theorem 1.3. Using Theorem 2.6.7, proved in chapters 3-5 for our groups in question, we have that the hypotheses in Theorem 2.6.8 are true and by 2.6.15

$$p_G(\delta^1) > 0$$

and $$q_G(\delta^2) > 0$$

and the form $\langle\ ,\ \rangle$ on X is indefinite too.

q.e.d.

2.7. Methods to detect non-unitarity.

To prove Theorem 2.6.7, we will need a few techniques that we will discuss here. Fix a positive root system $\Delta^+(k)$.

Lemma 2.7.1 (Parthasarathy's Dirac operator inequality. See Borel-Wallach [1980] II.6.1.1.) Let $(\pi, \mathcal{H})$ be a unitary representation of G and $\mathcal{H}_K$ its Harish-Chandra module.

Fix a positive t-root system $\Delta^+(g)$ compatible with $\Delta^+(k)$ and a k-type δ occurring in $\mathcal{H}_K$ with highest weight $\mu \in t^{c*}$. Write

$$\rho = \rho(\Delta^+(g)) \in (t^c)^*$$

$$\rho_c = \rho(\Delta^+(k)) \in (t^c)^*$$

$$\rho_n = \rho(\Delta^+(p)) = \rho - \rho_c \in (t^c)^*.$$

Let c_0 be the eigenvalue of the Casimir operator of g acting on $\mathcal{H}_K$, and $\omega \in W(k,t)$ making $\omega(\mu-\rho_n)$ dominant for $\Delta^+(k)$. Then

$$\langle \omega(\mu-\rho_n) + \rho_c,\ \omega(\mu-\rho_n) + \rho_c \rangle \geq c_0 + \langle \rho, \rho \rangle.$$

Lemma 2.7.2. Let $X \in \mathcal{M}(\mathfrak{g}, K)$ with a non-zero, invariant Hermitian form $\langle\ ,\ \rangle$. Suppose the Dirac inequality fails on a K-type δ, for some choice of $\Delta^+(\mathfrak{p})$. Then

1) There is a $\mathfrak{k}$-type η occurring in $V_\delta \otimes \mathfrak{p}$ such that

$$\langle\ ,\ \rangle\big|_{V_\delta \oplus V_\eta}$$

is indefinite.

2) Suppose G/K is Hermitian symmetric with a one-dimensional compact center, so that we can choose $z \in i\mathfrak{k}_0$ with the property that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is the decomposition of $\mathfrak{g}$ into the eigenspaces 0, +1, −1 of z, respectively.

Set $\rho_n^{\pm} = \rho(\Delta(\mathfrak{p}^{\pm}))$. Then, if the Dirac inequality fails on δ for $\rho_n^{\pm}$, there is a $\mathfrak{k}$-type $\eta^{\mp}$ occurring in $V_\delta \otimes \mathfrak{p}^{\mp}$ such that

$$\langle\ ,\ \rangle\big|_{(V_\delta \oplus V_{\eta^{\mp}})}$$

is indefinite.

Proof. Recall from Borel-Wallach [1980], II §6, the definition of $(\gamma, S(V))$, the space of spinors of a finite dimensional vector space V defined over $\mathbb{R}$, with a positive definite inner product $\langle\ ,\ \rangle$. Write $\langle\ ,\ \rangle_S$ for the unitary structure on $S(V)$ such that

$$\langle \gamma(v)x, y\rangle_S = -\langle x, \gamma(v)y\rangle_S$$

$$v \in V \qquad x,\ y \in S(V).$$

Recall also, the definition of the Dirac operator

$$D : H \otimes S \longrightarrow H \otimes S$$

for (π, H) a unitary $(\mathfrak{g}_0, \mathfrak{k}_0)$-module and $S = S(\mathfrak{p}_0)$.

(2.7.3)
$$D(v \otimes s) = \sum_{\alpha \in \Delta(\mathfrak{p})} \pi(X_\alpha)v \otimes \gamma(X-\alpha)s.$$

Since

$$S = \bigoplus_{\Delta^+(\mathfrak{g}) \supseteq \Delta^+(\mathfrak{k})} m \cdot V_{\rho(\Delta^+(\mathfrak{g}) - \rho_c)}$$

(where $m = 2^{[\dim \mathfrak{a}^c/2]}$) (cfr. Borel-Wallach [1984] II §6) then $\omega(\mu - \rho_n)$ is the highest weight of a $\mathfrak{k}$-representation occurring in $V_\delta \otimes V_{\rho_n} \subseteq H \otimes S$.

Let $\xi = v \otimes s$ be a weight vector for $\omega(\mu-\rho_n)$.

Write also $\langle\ ,\ \rangle_D$ for the tensor product inner product on $H \otimes S$; then the proof of Lemma 2.7.1 shows that

$$0 > \langle D\xi, D\xi\rangle_D =$$

$$(\langle\omega(\mu-\rho_n) + \rho_c, \omega(\mu-\rho_n)+\rho_c\rangle - c_0 - \langle\rho,\rho\rangle)\langle\xi,\xi\rangle_D.$$

So $D\xi \neq 0$ and

$$D\xi = \sum_{\alpha\in\Delta(\mathfrak{p})} \pi(X_\alpha)v \otimes \gamma(X_{-\alpha})s \in \mathfrak{p}.V_\delta \otimes S \subseteq H \otimes S.$$

This gives a non-zero map

$$\mathfrak{p} \otimes V_\delta \xrightarrow{\ \sigma\ } \mathfrak{p}\cdot V_\delta.$$

So $\mathrm{Hom}_{\mathfrak{k}}(\mathfrak{p} \otimes V_\delta, H) \neq 0$. Let $E = \mathrm{Im}\ \sigma$. Since $\langle\ ,\ \rangle_S$ is positive definite this means that $\langle\ ,\ \rangle$ is indefinite on $V_\delta \oplus E$.

This proves a) of the lemma.

For b) simply observe that $\mu - \rho_n^- = \mu + \rho_n^+$; $\rho_n^+ = \rho(\mathfrak{p}^+)$ and $\mathfrak{p}^+$ is a representation of $\mathfrak{k}$. Hence if $\beta \in \Delta(\mathfrak{k})$

$$\langle\rho_n^+, \beta\rangle = 0.$$

So $V_{\rho_n^+}$ is one-dimensional. Since $\rho_n^+ + \alpha$ is not a weight of S, for $\alpha \in \Delta(\rho^+)$, $V_{\rho_n^+}$ is killed by $\gamma(X_\alpha)$ and (2.7.3) becomes, for $\xi \in V_\delta \otimes V_{\rho_n^-}$

$$D\xi = \sum_{\alpha\in\Delta(\mathfrak{p}^+)} \pi(X_\alpha)v \otimes \gamma(X_{-\alpha})s$$

so $D\xi \in (\mathfrak{p}^+)\cdot V_\delta \otimes S \subseteq H \otimes S$. Similarly for ρ_n^-.

q.e.d.

Lemma 2.7.4. Let G be a connected, reductive linear Lie group. Assume that

$$\text{rank } G = \text{rank } K.$$

Then, any representation with real infinitesimal character has a Hermitian form.

Proof. By Proposition 2.6.3 it is enough to prove the lemma for G quasisplit and a Langlands subrepresentation of a principal series $I(\delta\otimes\nu)$ with $\delta \otimes \nu$ a character of a maximally split Cartan subgroup $H^s = T^sA^s$.

Since G is equal rank there is a subset $B = \{\alpha_1, \ldots, \alpha_k\}$ of strongly orthogonal simple real roots such that, since H^s is the maximally split Cartan subgroup of G, then B spans $\mathfrak{a}_0^s = \mathrm{Lie}(A^s)$.

Hence if $\omega = s_{\alpha_1} \ldots s_{\alpha_k}$ is the product of simple reflections s_{α_i}, ω acts by -1 on A^s and by the identity on T_0^s.

Recall from Definition 2.4.3 the maps $\phi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{m}_0^\alpha$. Consider the exponentiated map

$$\Phi_\alpha : SL(2,\mathbb{R}) \longrightarrow M^\alpha$$

set

$$m_\alpha = \Phi_\alpha \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in M.$$

(cfr. Vogan [1981] page 172). Then, since G is connected, T^s is generated by $T_0^s \cup \{m_\alpha \mid \alpha \text{ real}\}$. Let $\omega \in M'/M = W$, then there is $\sigma \in M'$ such that

$$\omega \cdot m_\alpha = \sigma m_\alpha \sigma^{-1} = m_{\omega \cdot \alpha}.$$

But $m_{\omega\cdot\alpha} = m_{-\alpha} = m_\alpha$.

Recall the elements $\delta \in \hat{M}$ and $\nu \in \hat{A}$. Then

$$(\omega\delta)(m_\alpha) = \delta(\omega\cdot m_\alpha) = \delta(m_\alpha)$$

and
$$\omega\cdot\delta\big|_{T_0} = \delta.$$

Hence $\omega\delta = \delta$. Since $I(\delta\otimes\nu)$ is assumed to have real infinitesimal character, ν is real.

Also since $\omega\big|_A = -1$ then $\omega\cdot\nu = -\nu = -\bar{\nu}$.

This is the condition of Proposition 2.6.3 for the existence of a Hermitian form.

q.e.d.

Chapter 3. $G = SL(n,\mathbb{R})$

3.1. Preliminary Notation.

To fix notation consider $G = SL(2n,\mathbb{R})$; the odd case is similar.

$$G = \{g \in GL(2n,\mathbb{R}) \mid \det g = 1\}.$$

The maximal compact subgroup K of G is

$$K = SO(2n,\mathbb{R}) = \{g \in G \mid g^t g = I\}.$$

The corresponding Lie algebras are

$$\mathfrak{g}_0 = \{X \in \mathfrak{gl}(2n,\mathbb{R}) \mid \text{trace } X = 0\}$$

$$\mathfrak{k}_0 = \{X \in \mathfrak{g}_0 \mid X + {}^tX = 0\} = \mathfrak{so}(2n,\mathbb{R}).$$

If θ is the Cartan involution defined by $\theta(X) = -{}^tX$, then

$$\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 \mid X = {}^tX\}.$$

The Cartan subgroup of K is

$$T^c = \left\{ g = \begin{bmatrix} r(\theta_1) & & & & \\ & r(\theta_2 & & & \\ & & \ddots & & \\ & & & & r(\theta_n) \end{bmatrix} \middle| \ \theta_i \in \mathbb{R}; \quad r(\theta_i) = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \right\}$$

and if

$$A^c = \left\{ g = \begin{bmatrix} r_1 & & & & & & \\ & r_1 & & & & & \\ & & r_2 & & & & \\ & & & r_2 & & & \\ & & & & \ddots & & \\ & & & & & r_n & \\ & & & & & & r_n \end{bmatrix} \middle| \ r_i \in \mathbb{R} \ \det g = 1 \right\}$$

then $H^c = T^c A^c$ is a maximally compact Cartan of G.

$$t_0^c = \left\{ g = \begin{bmatrix} 0 & \theta_1 & & & & & \\ -\theta_1 & 0 & & & & & \\ & & 0 & \theta_2 & & & \\ & & -\theta_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \theta_n \\ & & & & & -\theta_n & 0 \end{bmatrix} \middle| \ \theta_i \in \mathbb{R} \right\}$$

$$\mathfrak{a}_0^c = \left\{ X = \begin{bmatrix} a_1 & & & & & & & \\ & a_1 & & & & & & \\ & & a_2 & & & & & \\ & & & a_2 & & & & \\ & & & & \ddots & & & \\ & & & & & & a_n & \\ & & & & & & & a_n \end{bmatrix} \;\middle|\; \begin{matrix} \Sigma\, a_i = 0 \\ a_i \in \mathbb{R} \end{matrix} \right\}$$

$$\mathfrak{h}_0^c = \mathfrak{t}_0^c + \mathfrak{a}_0^c$$

Complexifying:

$$\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$$

and

$$\mathfrak{k} = \mathfrak{so}(2n, \mathbb{C}).$$

Define $e_j \in i(\mathfrak{t}_0^c)^*$ $j = 1, 2, \ldots, n$, by

$$e_j \begin{bmatrix} 0 & \theta_1 & & & & & \\ -\theta_1 & 0 & & & & & \\ & & 0 & \theta_2 & & & \\ & & -\theta_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \theta_n \\ & & & & & -\theta_n & 0 \end{bmatrix} = i\theta_j.$$

Then the roots of $\mathfrak{t}^c$ in $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{g}$ are, respectively:

$$\Delta(\mathfrak{k}, \mathfrak{t}^c) = \{\pm(e_j \pm e_k) \mid 1 \leq j < k \leq n\}$$

$$\Delta(\mathfrak{p}, \mathfrak{t}^c) = \{\pm 2e_\ell;\ (e_j \pm e_k) \mid 1 \le \ell \le n;\ 1 \le j < k \le n\}$$

$$\Delta(\mathfrak{g}, \mathfrak{t}^c) = \{\pm 2e_\ell;\ \pm(e_j \pm e_k) \mid 1 \le \ell \le n;\ 1 \le j < k \le n\}.$$

The multiplicity of $\pm e_j \pm e_k$ as a root in $\mathfrak{g}$ is 2.

Choose $\Delta^+(\mathfrak{k}, \mathfrak{t}^c) = \{e_j \pm e_k \mid 1 \le j < k \le n\}$. Then $\hat{K}$ can be identified with the set

$$\{\mu = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 \ge a_2 \ge \ldots \ge a_{n-1} \ge |a_n|\}.$$

3.2. Computation of $\ell_V(X)$ for a Harish-Chandra module X.

Let $\mu = (a_1, a_2, \ldots, a_n) \in i\mathfrak{t}_0^*$ be the highest weight of a LKT of X. Fix a positive root system $\Delta^+(\mathfrak{k})$ so that

$$a_1 \ge a_2 \ge \ldots \ge |a_n|$$

as in 3.1.

To obtain $\ell_V(X) = \ell_V(\mu)$ as in 2.4.8 we need:

$$2\rho_c = (2n-2, 2n-4, \ldots, 2, 0).$$

Let $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ be a θ-stable positive system making $\mu + 2\rho_c$ dominant. After conjugating by an outer automorphism of K we may assume that $a_n \ge 0$. Then the restriction of $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$ to $\mathfrak{t}^c$ is

$$\Delta^+(\mathfrak{g}, t^c) = \{e_j \pm e_k;\ 2e_\ell \mid 1 \le j,k,\ell \le n;\ j < k\}.$$

Write $\phi(\mathfrak{g}, t^c)$ for the set of simple roots restricted to t^c. Then

$$\phi(\mathfrak{g}, t^c) = \{e_1 - e_2;\ e_2 - e_3;\ldots;\ e_{n-1} - e_n;\ 2e_n\}.$$

Let

$$\mu + 2\rho_c = (x_1 x_2 \ldots x_n).$$

We can form an array with the coordinates of $\mu + 2\rho_c$ by grouping them into maximal blocks of elements decreasing by 2. That is, if

(3.2.1) $$\mu = (\underbrace{a_1,\ldots,a_1}_{r_1 \text{ times}},\underbrace{a_2,\ldots,a_2}_{r_2 \text{ times}}\ldots\underbrace{a_t,\ldots,a_t}_{r_t \text{ times}},\underbrace{0,\ldots,0}_{R \text{ times}})$$

where $a_1 > a_2 > \ldots > a_t > 0$.

Then, since the coordinates of $2\rho_c$ decrease by two, the array would look like

(3.2.2) $$\boxed{m_1\ \ m_1-2\ \ \ldots\ \ m_1-2p_1+1}\ \boxed{m_2\ \ m_2-2\ \ \ldots\ \ m_2-2p_2+1}\ \ldots$$

$$\boxed{2R-2,\ldots,2,0}\ .$$

Proposition 3.2.3. Suppose that $\mu \in i t_0^*$ is the highest weight of a representation of K and that the picture for $\mu + 2\rho_c$ is as in 3.2.2. Then

a) $\lambda_V(\mu) = \lambda_V$ has the form

$$\lambda_V(\mu) = (\underbrace{\lambda_1 \ldots \lambda_1}_{r_1}, \underbrace{\lambda_2 \ldots \lambda_2}_{r_2} \ldots \underbrace{\lambda_t \ldots \lambda_t}_{r_t}, \underbrace{0 \ldots 0}_{R})$$

where

i) $\lambda_j = a_j - 1$

ii) $\lambda_1 > \lambda_2 > \ldots > \lambda_t \geq 0.$

So $\lambda_t = 0$ if $a_t = 1$.

b) $[\mathfrak{l}_V(\mu)]^d$ is isomorphic to, either

$$\mathfrak{sl}(r_1, \mathbb{C}) \oplus \mathfrak{sl}(r_2, \mathbb{C}) \oplus \ldots \oplus \mathfrak{sl}(r_t, \mathbb{C}) \oplus \mathfrak{sl}(2R, \mathbb{R}),$$

if $\lambda_t > 0$ or

$$\mathfrak{sl}(r_1, \mathbb{C}) \oplus \ldots \oplus \mathfrak{sl}(r_{t-1}, \mathbb{C}) \oplus \mathfrak{sl}(2(R+r_t), \mathbb{R}),$$

if $\lambda_t = 0$.

Proof. Notice that $2\rho_c - \rho = (-1, -1, \ldots, -1)$. Define

$$\{\beta_j\} = \{2e_{n-j+1} \mid \langle \mu + 2\rho_c - \rho, 2e_{n-j+1}\rangle = -c_j \le 0\}.$$

Then $\lambda_V(\mu) = \mu + 2\rho_c - \rho + \frac{1}{2}\Sigma c_j\beta_j$ has the form (3.2.4), and the conditions a) - e) of Proposition 2.4.7 are obvious. Moreover the subset of simple roots orthogonal to $\lambda_V(\mu)$ is

$$\{e_1-e_2; e_2-e_3; \ldots e_{r_1-1}-e_{r_1}\} \cup \{e_{r_1}-e_{r_1+1} \ldots e_{r_1+r_2-1}-e_{r_1+r_2}\}$$

$$\cup \ldots \cup \{\ldots; e_{n-1}-e_n; e_{2n}\}.$$

This spans the root system

$$(A_{r_1-1} \oplus A_{r_1-1}) \oplus (A_{r_2-1} \oplus A_{r_2-1}) \oplus \ldots \oplus (A_{r_s-1} \oplus A_{r_s-1}) \oplus A_{2k},$$

since the roots $e_i - e_j$ involved are restrictions of complex roots and therefore occur twice in $\Delta(\mathfrak{g}, t^c)$. Now the proposition is clear.

q.e.d.

3.3. Lowest K-types of the modules $A_{\mathfrak{q}}(\lambda)$.

We will give some criteria to determine when a representation of K is the LKT of one of the $(\mathfrak{g}, K)$ modules $A_{\mathfrak{q}}(\lambda)$.

Recall from 2.3 that to construct a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ we need a weight $x \in i\mathfrak{t}_0^*$. Suppose

$$(3.3.1) \quad x = (\underbrace{x_1,\ldots,x_1}_{r_1 \text{ times}},\underbrace{x_2,\ldots,x_2}_{r_2 \text{ times}},\ldots,\underbrace{x_t,\ldots,x_t}_{r_t \text{ times}},\underbrace{0,\ldots,0}_{R \text{ times}})$$

where $\qquad x_1 > x_2 > \ldots > x_t > 0.$

Write $\mathfrak{q} = \mathfrak{q}(x) = \mathfrak{l}(x) + \mathfrak{u}(x)$ for the parabolic defined by x as in 2.3.

Clearly

(3.3.2)

$$2\rho(\mathfrak{u}\cap\mathfrak{k}) =$$

$$(\underbrace{s_1 s_1 \ldots s_1}_{r_1},\underbrace{s_2 \ldots s_2}_{r_2},\ldots,\underbrace{s_t \ldots s_t}_{r_t},\underbrace{0\ldots0}_{R})$$

with $\qquad s_j = 2(n-r_1-\ldots-r_{j-1}) - r_j - 1$

and $\quad 2\rho(\mathfrak{u}\cap\mathfrak{p}) =$

$$(\underbrace{u_1,\ldots,u_1}_{r_1},\underbrace{u_2,\ldots,u_2}_{r_2},\ldots,\underbrace{u_t,\ldots,u_t}_{r_t},\underbrace{0\ldots0}_{R})$$

$$u_j = 2(n-r_1-\ldots-r_{j-1}) - r_j + 1.$$

Proposition 3.3.3. Let μ be as in (3.2.1) and suppose it is the highest weight of a representation of K. Then V_μ is the LKT of a $(\mathfrak{g},K)$-module $A_{\mathfrak{q}}(\lambda)$ if and only if

a) $a_i - a_{i+1} \geq r_i + r_{i+1}$

and

b) $a_t \geq 2R + r_t + 1$.

Proof. Suppose V_μ is the LKT of an $A_{\mathfrak{q}}(\lambda)$. Then $\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ and λ is the weight of a one-dimensional character of L satisfying 2.5.1 a) and b). Hence λ is orthogonal to the roots of $\mathfrak{t}^c$ in $\mathfrak{l}$ and it is positive in the $\mathfrak{t}^c$-roots in $\mathfrak{u}$. That is,

$$\lambda = (\underbrace{\lambda_1,\ldots,\lambda_1}_{r_1},\underbrace{\lambda_2,\ldots,\lambda_2}_{r_2},\ldots\underbrace{\lambda_t,\ldots,\lambda_t}_{r_t},\underbrace{0,\ldots,0}_{R})$$

and $$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t \geq 0.$$

Then $$a_j = \lambda_j + 2(n-r_1-\ldots-r_{j-1}) - r_j + 1$$

$$a_t = \lambda_t + 2(n-r_1-\ldots-r_{t-1}) - r_t + 1 \geq 2R + r_t + 1$$

$$a_j - a_{j+1} =$$

$$\lambda_j - \lambda_{j+1} - r_j + 1 + 2r_j + r_{j+1} - 1 \geq r_j + r_{j+1}.$$

Conversely, suppose μ is a weight satisfying a) and b) then we can define

$$\mathfrak{q} = \mathfrak{q}(\mu) \quad \text{and}$$

$$\lambda_j = a_j - 2(n-r_1-\ldots-r_{j-1}) + r_j - 1.$$

Then μ will be the LKT of $A_{\mathfrak{q}}(\lambda)$.

q.e.d.

3.4. Proof of Theorem 1.6.7 for $G = SL(n,\mathbb{R})$.

Suppose $X \in \mathcal{A}(\mathfrak{g};K)$ is as in Theorem 2.6.7 with infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$ and $\mu \in (i\mathfrak{t}_0^c)^*$ the highest weight of a LKT of X. Write μ as in 3.3.1.

Considering what the weights in $V_\mu \otimes \mathfrak{p}$ look like, we will study 2 cases:

1. $2R + r_t + 1 > a_t$.
2. $a_t \geq 2R + r_t + 1$.

By the conditions given in 3.3, if V_μ is the LKT of an $A_{\mathfrak{q}}(\lambda)$, then μ is in case 2.

Therefore, the first thing we must do is verify that in case 1 X is not unitary:

Case 1.

We will use the following result.

Lemma 3.4.1. Let μ be as in 3.3.1 and suppose $a_t < 2R + r_t + 1$. Suppose that $a_i - a_{i+1} = 1$. Then Dirac operator inequality fails for $s_n = (n, n-1, \ldots, 1)$.

Proof. The hypotheses on μ imply that

$$\mu = (\underbrace{x+t-1, \ldots, x+t-1}_{r_1}, \underbrace{x+t-2, \ldots, x+t-2}_{r_2} \ldots \underbrace{x, \ldots, x}_{r_t}, \underbrace{0 \ldots 0}_{R})$$

Note that $1 \leq x \leq 2R + r_t$ implies that

$$(*) \qquad R + 1 - x \leq R \quad \text{and} \quad R + r_t - x \geq -R.$$

Now,

$$\rho_n = (n, n-1, \ldots, R+r_t, R+r_t-1, \ldots, R+1, R, R-1, \ldots, 1),$$

so

$$\rho_n - \mu =$$

$$(n-x-t+1, n-x-t, \ldots, R+r_t-x, R+r_t-x-1, \ldots, R+1-x, R, R-1, \ldots, 1).$$

By (*), the sequence of integers

$$R + r_t - x,\ R + r_t - x - 1, \ldots, R + 1 - x$$

overlaps the sequence $R, R-1, R-2, \ldots, -R+1, -R$.

Clearly, the first $n-R$ coordinates of $\rho_n - \mu$ decrease by steps of at most one.

So if $\omega \in W_K$ is such that $\omega(\mu-\rho_n)$ is dominant, then the coordinates of $\omega(\mu-\rho_n)$ will be a sequence of integers decreasing by at most one, ending in 0 or ± 1; and in the latter case, there must be repetitions in the sequence.

Since

$$\rho_c = (n-1, n-2, \ldots, R+1, R, R-1, \ldots, 2, 1, 0)$$

it follows that

$$\langle \omega(\mu-\rho_n), \rho_c \rangle < \langle \rho_n, \rho_c \rangle$$

$$\langle \omega(\mu-\rho_n), \omega(\mu-\rho_n) \rangle < \langle \rho_n, \rho_n \rangle.$$

Hence $\langle \omega(\mu-\rho_n) + \rho_c, \omega(\mu-\rho_n)+\rho_c \rangle < \langle \rho_n+\rho_c, \rho_n+\rho_c \rangle = \langle \rho, \rho \rangle$.

q.e.d.

Now to prove nonunitarity for case 1, take i_0 to be the minimal integer in $\{1, 2, \ldots, t\}$ such that $a_i - a_{i+1} = 1$ for all $i > i_0$.

Let $\quad K = r_{i_0+1} + r_{i_0+2} + \ldots + r_t + R$

and $\quad \mathfrak{l} = \mathfrak{sl}(r_1+r_2+\ldots+r_{i_0}) \oplus \mathfrak{sl}(2K,\mathbb{R}) = \mathfrak{l}_1 \oplus \mathfrak{l}_2.$

Then $\mathfrak{l} \subseteq \mathfrak{l}_V$ and by Proposition 2.4.15, if X is the Langlands quotient of $\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_V}(I_{L_V}(\delta_V \otimes \nu_V))$ and if we set

$$X_L = \mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}_V \cap \mathfrak{l}}[I_{L_V}(\delta_V \otimes \nu_V)),$$

then X is the Langlands quotient of

$$\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L) \cong \mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_V}(I_{L_V}(\delta_V \otimes \nu_V))$$

and a) of Theorem 2.6.7 holds.

Also, by Corollary 2.6.4, X_L^h has a Hermitian form $\langle\ ,\ \rangle^L$.

Write $\mu^L = \mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})$. Then μ^L is a LKT of X^L. By 3.3.2,

$$2\rho(\mathfrak{u} \cap \mathfrak{p}) =$$

$$(\underbrace{2n-(r_1+\ldots+r_{i_0})+1,\ldots,2n-(r_1+\ldots+r_{i_0})+1}_{r_1+r_2+\ldots+r_{i_0}},\underbrace{0,\ldots,0}_{K}).$$

So $\qquad \mu^L\big|_{SL(2K,\mathbb{R})} = \mu\big|_{SL(2K,\mathbb{R})} = \mu^2$

and by Lemma 3.4.1, the Dirac inequality fails on μ^2. By Lemma 2.7.2, there is a K-type V_{η^2} in $V_{\mu^2} \otimes (\mathfrak{l}_2 \cap \mathfrak{p})$ that makes the Hermitian form $\langle\ ,\ \rangle^L$ indefinite.

The roots in $\Delta(\mathfrak{l}_2 \cap \mathfrak{p})$ are

$$\{(\underbrace{0\ldots \pm 1, 0\ldots 0\ \pm 1\ 0\ldots 0}_{K}), (\underbrace{0\ldots 0, \pm 2, 0\ldots 0}_{K})\}.$$

It is clear that if $\eta^2 = \mu^2 + \beta$ is dominant for some $\beta \in \Delta(\mathfrak{l}_2 \cap \mathfrak{p})$ then, since $a_{i_0} - a_{i_0+1} \geq 2$ the K-type $\mu + \beta$ is also dominant for $\Delta(\mathfrak{u} \cap \mathfrak{k})$. Hence Theorem 2.6.7 follows for case 1.

For case 2, note that if $a_j - a_{j+1} \geq r_j + r_{j+1}$, for all $j = 1, \ldots, t$, then we have the LKT of an $A_{\mathfrak{q}}(\lambda)$ and there is nothing to prove.

So, assume that there is $i_0 < t$ such that $a_{i_0} - a_{i_0+1} < r_{i_0} + r_{i_0+1}$.

Set $K = r_1 + r_2 + \ldots + r_t$ and

$$\mathfrak{l} = \mathfrak{sl}(K, \mathbb{C}) \oplus \mathfrak{sl}(2R, \mathbb{R}).$$

Note that $a_t \geq 2$. Hence $\mathfrak{l}_V = \mathfrak{sl}(r_1, \mathbb{C}) \oplus \ldots \oplus \mathfrak{sl}(r_t, \mathbb{C}) \oplus \mathfrak{sl}(2R, \mathbb{R})$, $\mathfrak{l} \supseteq \mathfrak{l}_V$ and we can find X_L s.t. (a) in Theorem 2.6.7 holds. In fact, if $\mathfrak{q}_0 = \mathfrak{l} \cap \mathfrak{q}_V = \mathfrak{l}_V + \mathfrak{l} \cap \mathfrak{u}_V$ and $X_{L_V} = I_{L_V}(\delta_V \otimes v_V)$, as in Definition 2.4.17, we can choose X_L to be

$$X_L = \mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}_0}(X_{L_V}),$$

since then, by Proposition 2.4.15

$$\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L) \cong \mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_V}(X_{L_V})$$

where $$\Delta(\mathfrak{u}) = \Delta(\mathfrak{u}_V)\setminus\Delta(\mathfrak{l})$$

and $$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}.$$

By Corollary 2.6.4, X_L^h admits a non-zero Hermitian form $\langle\ ,\ \rangle^L$.

Write X_L as the exterior tensor product $X_L = X_{L_1} \otimes X_{L_2}$ where X_{L_i} is an $(\mathfrak{l}_i, L_i \cap K)$ module,

$$L_1 = \mathfrak{sl}(K,\mathbb{C}) \quad \text{and} \quad L_2 = SL(2R,\mathbb{R}).$$

By Theorem 6.1 in Enright [1979], and especially its proof (pp. 518-523), if X_{L_1} is not an $A_{\mathfrak{q}'}(\lambda')$ then Dirac inequality fails on the lowest K-type. Write $\mu^L = \mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})$ and

$$\mu^1 = \mu^L\big|_{L_1}.$$

By Lemma 2.7.2 there is an $(L_1 \cap K)$-type V_{η^1} with $\eta^1 = \mu^1 + \beta$ for $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$. If for all $i \neq i_0$

$$a_i - a_{i+1} \geq r_i + r_{i+1} \geq 2. \tag{3.4.1}$$

Then $\mu + \beta$ is dominant.

Otherwise take $K' = \sum_{i \in B} r_i$ with

$$B = \{i = 1, \ldots, t \mid 3.4.1 \text{ holds}\}.$$

Then apply Enright's result to the rest.

q.e.d.

Chapter 4. $G = SU(p,q)$

4.1. Preliminary Notation

Let $n = p + q$. Write I_m for the identity matrix in $GL(m,\mathbb{C})$, and A^* for the conjugate transpose of the matrix A. We define

$$G = \left\{ g \in SL(n,\mathbb{C}) \mid g \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} g^* = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \right\}.$$

Then the maximal compact subgroup K of G is

$$K = \left\{ g \in G \mid g = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix};\quad A \in U(p),\ B \in U(q) \right\}.$$

Also, with the usual notation:

$$\mathfrak{g}_0 = \left\{ X \in \mathfrak{sl}(n,\mathbb{C}) \mid X \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} + \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} X^* = 0 \right\}$$

$$= \left\{ X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(n,\mathbb{C}) \mid A \in \mathfrak{u}(p),\ D \in \mathfrak{u}(q) \text{ skewhermitian};\quad -B^* = C \right\}$$

$$\mathfrak{k}_0 = \left\{ X = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathfrak{g}_0 \mid A \in \mathfrak{u}(p),\ D \in \mathfrak{u}(q) \right\}.$$

If θ is the Cartan involution defined by $\theta(X) = -X^*$, and

$$\mathfrak{p}_0 = \left\{X \in \mathfrak{g}_0 \mid \theta(X) = -X\right\}$$

then

$$\mathfrak{p}_0 = \left\{X = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} \mid B \text{ arbitrary } p \times q \text{ matrix}\right\}.$$

The compact Cartan Subgroup of G is

$$H^c = T^c = \left\{g = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \sum_{j=1}^{n} \theta_j = 0\right\}.$$

Then,

$$\mathfrak{t}_0 = \left\{\operatorname{diag}(i(\theta_1, \ldots, \theta_n)) \in i\mathbb{R}^n \mid \sum_{j=1}^{n} \theta_j = 0\right\}$$

Complexifying everything we have:

$$\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$$

$$\mathfrak{k} = \mathfrak{s}(\mathfrak{gl}(p) \oplus \mathfrak{gl}(q)) = \left\{X = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(p), \right.$$

$$\left. D \in \mathfrak{gl}(q)\right\}$$

$$\mathfrak{t} = \left\{\mathrm{diag}(z_1,\dots,z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} z_j = 0\right\}.$$

$\hat{K}$ can be identified with the space

$$\left\{\mu = (a_1,\dots,a_p \mid a_{p+1},\dots,a_n) \mid a_1 \geq \dots \geq a_p;\right.$$

$$\left. a_{p+1} \geq \dots \geq a_n \quad \sum a_j = 0 \quad a_i - a_j \in \mathbb{Z}\right\}.$$

If we denote by $e_j \in \mathbb{R}^*$, $j = 1,\dots,n$, the elements of the dual basis in $\mathbb{R}^n$, then the roots of $\mathfrak{t}$ in $\mathfrak{g}$ correspond to the set

$$\Delta(\mathfrak{g}) = \Delta(\mathfrak{g},\mathfrak{t}) = \{e_\ell - e_j \mid \ell \neq j;\ 1 \leq \ell,\ j \leq n\}.$$

Also

$$\Delta(\mathfrak{k}) = \Delta(\mathfrak{k},\mathfrak{t}) =$$

$$\{e_\ell - e_j \mid 1 \leq \ell,\ j \leq p\} \cup \{e_k - e_m \mid p < k,\ m \leq n\}$$

the compact imaginary roots of $\mathfrak{k}$ and $\mathfrak{g}$.

$$\Delta(\mathfrak{p}) = \Delta(\mathfrak{p},\mathfrak{t}) = \{\pm (e_{\ell}-e_{p+j}) \mid 1 \leq \ell \leq p;\ 1 \leq j \leq q\}$$

the noncompact imaginary roots of $\mathfrak{t}$ in $\mathfrak{g}$.

4.2. Computation of $\ell_V(X)$ and $\lambda_V(X)$, for a Harish-Chandra module X.

Let $\mu = (a_1,a_2,\dots,a_p|b_1,b_2,\dots,b_q)$ be the highest weight of a lowest K-type of X. Fix the positive system $\Delta^+(\mathfrak{k})$ so that

$$a_1 \geq \dots \geq a_p;\ b_1 \geq \dots \geq b_q.$$

We want to obtain an explicit expression of the parameters λ_V and ℓ_V attached to μ by 2.4.7 and 2.4.8.

$$2\rho_c = (p-1,p-3,\dots,-p+1|q-1,q-2,\dots,-q+1).$$

Let

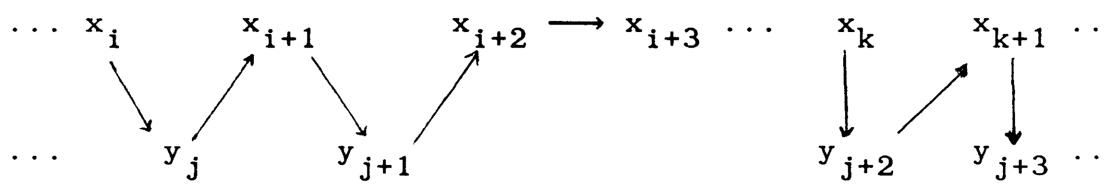
$$\mu + 2\rho_c = (x_1,x_2,\dots,x_p|y_1,y_2,\dots,y_q).$$

We can form an array of two rows with the coordinates of

$\mu + 2\rho_c$ so that they are aligned in decreasing order from left to right as follows: the x_i are in the first rows; the y_j are in the second; and terms decrease from left to right in the array. For example, if we have

$$x_i > y_j > x_{i+1} > y_{j+1} > x_{i+2} > x_{i+3} \cdots > x_k = y_{j+2} > x_{k+1} = y_{j+3} \cdots$$

the array would look like:



This array gives a choice of positive roots $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$, compatible with $\Delta^+(\mathfrak{k})$. That is, the simple roots are given by the arrows. In the preceding example, they would be

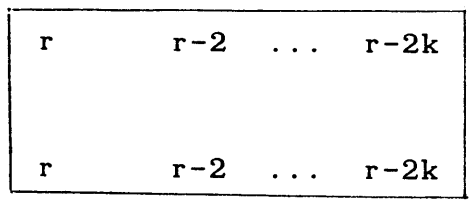
$$\ldots e_i - e_{p+j};\ e_{p+j} - e_{i+1};\ e_{i+1} - e_{p+j+1} - e_{i+2};$$

$$e_{i+2} - e_{i+3}; \ldots$$

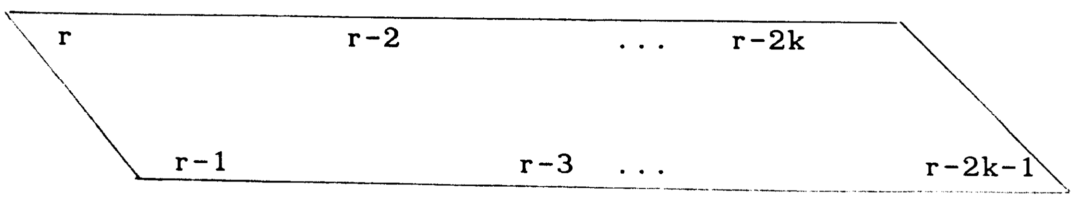
$$\ldots\ e_k - e_{p+j+2};\ e_{p+j+2} - e_{k+1};\ e_{k+1} - e_{p+j+3}; \ldots$$

Because the terms in each row decrease by at least 2, the entire array is a union of blocks of the following five types.

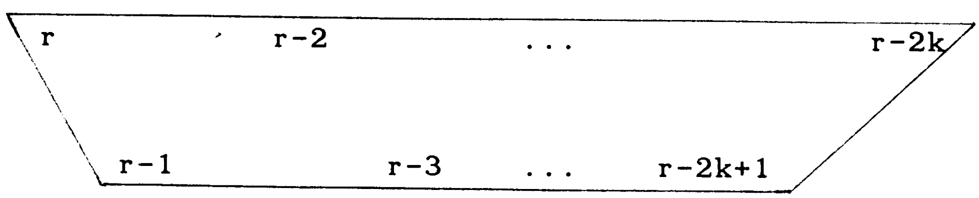
1.



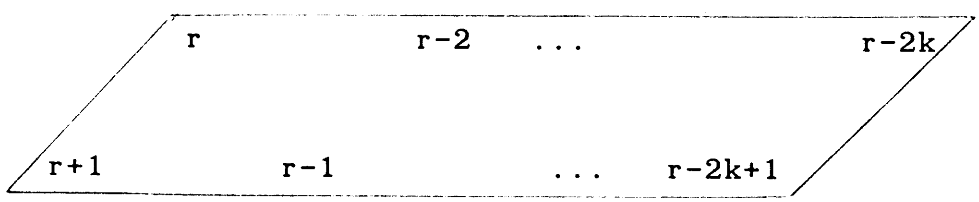
2.



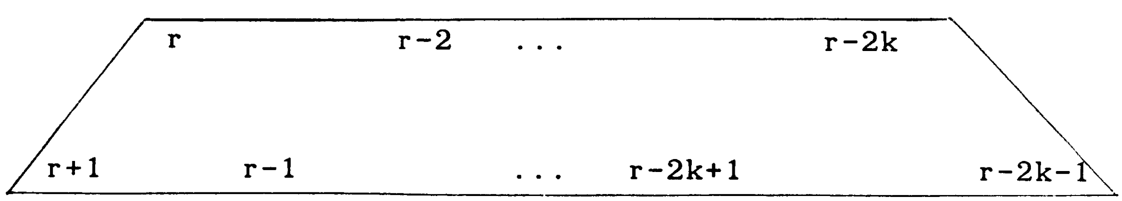
3.

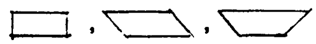


4.



5.



From now on we will drop the arrows in the pictures, since the ordering of the roots is clear from the arrangement of the coordinates of $\mu + 2\rho_c$, provided we agree on choosing the order prescribed in block 1. We will also refer to them as , etc.

Example. Let $p = 7$, $q = 8$, and

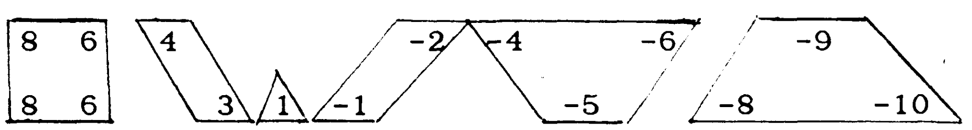
$$\mu = (2,2,2,-2,-2,-2,-3 \mid 1,1,0,0,0,-2,-3,-3)$$

then,

$$2\rho_c = (6,4,2,0,-2,-4,-6 \mid 7,5,3,1,-1,-3,-5,-7)$$

$$\mu + 2\rho_c = (8,6,4,-2,-4,-6,-9 \mid 8,6,3,1,-1,-5,-8,-10).$$

We obtain the following picture



Using the picture of $\mu + 2\rho_c$ we can split the coordinates of μ as follows

$$\mu = (\underbrace{g_1 \ldots g_1}_{r_1 \text{ times}} \ldots \underbrace{g_t \ldots g_t}_{r_t \text{ times}} \mid \underbrace{f_1 \ldots f_1}_{s_1 \text{ times}} \ldots \underbrace{f_t \ldots f_t}_{s_t \text{ times}})$$

where r_i is the number of p-coordinates and s_i the number of q-coordinates making up the i-th block of the array of $\mu + 2\rho_c$, and

$$g_1 \geq g_2 \geq \dots \geq g_t, \quad f_1 \geq f_2 \geq \dots \geq f_t$$

$$r_i \geq 0, \quad s_i \geq 0, \quad i = 1,\dots,t.$$

Write

$$\lambda_V = \lambda_V(\mu) = \lambda_V(X)$$

as in Proposition 2.4.7 in chapter 2.

Proposition 4.2.1. 1. The expression for $\lambda_V(\mu)$ has the form

$$\lambda_V = (\underbrace{\lambda_1 \dots \lambda_1}_{r_1 \text{ times}} \dots \underbrace{\lambda_t \dots \lambda_t}_{r_t \text{ times}} \mid \underbrace{\lambda_1 \dots \lambda_1}_{s_1 \text{ times}} \dots \underbrace{\lambda_t \dots \lambda_t}_{s_t \text{ times}})$$

where

$$\lambda_1 > \lambda_2 > \dots > \lambda_t.$$

2. Let

$$\Delta(\mathfrak{l}_V) = \Delta(\mathfrak{l}_V, \mathfrak{t}^c)$$

$$= \{\alpha \in \Delta \mid \langle \lambda_V, \alpha \rangle = 0\}.$$

Then

$$\mathfrak{l}_V(X) = (\oplus_{\alpha \in \Delta(\mathfrak{l}_V)} \mathbb{C}X_\alpha) \oplus \mathfrak{t}^c$$

$$= \mathfrak{s}(\mathfrak{u}(r_1, s_1) \oplus \ldots \oplus \mathfrak{u}(r_t, s_t)).$$

Example. In our current example we have

$$\mu = (\underbrace{2,2}_{r_1}, \underbrace{2}_{r_2}, \underbrace{-2,-2}_{r_4}, \underbrace{-2,-2}_{r_5}, \underbrace{-3}_{r_6} \mid \underbrace{1,1}_{s_1}, \underbrace{0}_{s_2}, \underbrace{0}_{s_3}, \underbrace{0}_{s_4}, \underbrace{-2}_{s_5}, \underbrace{-3,-3}_{s_6})$$

Note $r_3 = 0$.

Let $\tilde{\lambda}_V = \mu + 2\rho_c - \rho$

$= (1,1,1,-1,-2,-2,-3 \mid 2,2,1,0,-1,-2,-3,-3).$

Then

$$\lambda_V = \left[\underbrace{\tfrac{3}{2},\tfrac{3}{2}}_{r_1}, \underbrace{1}_{r_2}, \underbrace{-1}_{r_4}, \underbrace{-2,-2}_{r_5}, \underbrace{-3}_{r_6} \mid \underbrace{\tfrac{3}{2},\tfrac{3}{2}}_{s_1}, \underbrace{1}_{s_2}, \underbrace{0}_{s_3}, \underbrace{-1}_{s_4}, \underbrace{-2}_{s_5}, \underbrace{-3,-3}_{s_6}\right]$$

and

$$\mathfrak{l}_V(X) \cong \mathfrak{s}(\mathfrak{u}(2,2) \oplus \mathfrak{u}(1,1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1,1) \oplus \mathfrak{u}(2,1) \oplus \mathfrak{u}(1,2)).$$

Proof. Take a block of the form 1. Then coordinatewise we have

$$\mu + 2\rho_c = (\ldots,r,r-2,\ldots,r-2k,\ldots|\ldots,r,r-2,\ldots,r-2k,\ldots)$$

$$\rho = (\ldots,s,s-2,\ldots,s-2k,\ldots|\ldots,s-1,s-3,\ldots,s-2k-1,\ldots)$$

$$\tilde{\lambda}_V = \mu + 2\rho_c - \rho$$

$$= (\ldots,r-s,r-s,\ldots,r-s\ldots|$$

$$\ldots,r-s+1,r-s+1,\ldots,r-s+1,\ldots)$$

Let

$$\{e_m-e_{p+\ell};e_{p+\ell}-e_{m+1};\ldots;e_{m+k}-e_{p+\ell+k}\}$$

be the set of simple roots making up this block. Choose

$$\{\beta_i = e_{m+i-1}-e_{p+\ell+i-1} \mid i = 1,\ldots,k+1\}.$$

Then

$$-c_i = -\langle\tilde{\lambda}_V,\beta_i\rangle = -1.$$

Also

$$\langle \beta_i, \beta_j \rangle = \delta_{ij}$$

and

$$\langle \tilde{\lambda}_V + \frac{c_i}{2} \beta_i, \beta_i \rangle = 0.$$

Let $\{\beta_1, \ldots, \beta_{k_1}, \ldots, \beta_{k_r}\}$ be the subset of imaginary noncompact positive roots chosen this way from all blocks of the form 1. For blocks 2 we can choose the same kind of subset of simple roots. In this case

$$c_i = -\langle \tilde{\lambda}_V, \overset{\vee}{\beta}_i \rangle = 0$$

since coordinatewise, we have

$$\mu + 2\rho_c = (\ldots, r, r-2, \ldots, r-2k, \ldots |$$

$$\ldots, r-1, r-3, \ldots, r-2k+1, \ldots)$$

$$\rho = (\ldots, s, s-2, \ldots, s-2k, \ldots | \ldots, s-1, s-3, \ldots, s-2k+1, \ldots)$$

$$\tilde{\lambda}_V = (\ldots, r-s, r-s, \ldots, r-s, \ldots | \ldots, r-s, r-s, \ldots, r-s, \ldots)$$

A similar situation occurs in cases 3-5. Choose all the simple roots of the form $e_a - e_{p+b}$ involved in blocks 3. For cases 4 and 5 choose all those of the form $e_{p+b} - e_a$. For all these subsets

$$c_i = -\langle\tilde{\lambda}_V, \beta_i\rangle = 0.$$

It is clear that the union Π of these 5 kinds of subsets is a set of strongly orthogonal simple imaginary noncompact roots. Define

$$\lambda_V = \tilde{\lambda}_V + \frac{1}{2} \sum_{\beta_i \in \Pi} c_i \beta_i .$$

a) - d) of Proposition 2.4.7 are clear.

Let

$$\Delta^+ \cap \beta_1^\perp = \alpha \in \Delta^+ \mid \langle\alpha, \beta_1\rangle = 0\}$$

$$= \Delta^+ \backslash \{e_i - e_m; e_m - e_j; e_i - e_{p+\ell}; e_{p+\ell} - e_j\}$$

$$t_0^1 = t_0^{\beta_1}$$

then t_0^1 can be identified with the set

$$\{(x_1, \ldots, x_p | x_{p+1}, \ldots, x_n) \in t_0 | x_m = x_{p+\ell}\} = \beta_1^\perp .$$

So if $\mu^1 = \mu\Big|_{t^1} = \mu\Big|_{g^1 \cap t}$

$$\mu^1 = (a_1,\ldots,\frac{a_m+b_\ell}{2},\ldots,a_p|b_1,\ldots,\frac{a_m+b_\ell}{2},\ldots,b_q).$$

To verify e) of proposition 2.4.7 in chapter 2 we use the following lemmas.

Lemma 4.2.2 (Schmid [1975]) (see Vogan [1981] p. 247). If $\rho_c^1 = \rho(\Delta^+(k^1,t^1))$,

$$\Delta^+(g^1,h^1) = \Delta^+(g) \cap \beta_1^\perp$$

and

$$\rho^1 = \rho(\Delta^+(g^1,h^1))$$

then

$$2\rho_c^1 - \rho^1 = (2\rho_c-\rho)\Big|_{t^1}.$$

So

$$\tilde{\lambda}_V^1 = \mu^1 + 2\rho_c^1 - \rho^1$$

$$= \tilde{\lambda}_V \Big|_{t^1}$$

$$= \tilde{\lambda}_V + \tfrac{1}{2} c_1 \beta_1 .$$

Lemma 4.2.3 (Vogan [1981], p. 249). $\mu^1 + 2\rho_c^1$ is dominant for $\Delta^+(\mathfrak{g}^1, \mathfrak{h}^1)$.

So 2.4.7, e) is clear.

Now it is straightforward to verify Proposition 4.2.1.

4.3. The Lowest K-types of the Modules $A_{\mathfrak{q}}(\lambda)$.

In this section we obtain necessary and sufficient conditions for a representation of K to be the LKT of one of the $\mathfrak{g}$-modules $A_{\mathfrak{q}}(\lambda)$.

If $x = (x_1, \ldots, x_n) \in i(t_0^c)^*$ we obtain a θ-stable parabolic subalgebra $\mathfrak{q}(x) = \mathfrak{l}(x) + \mathfrak{u}(x)$ as in 2.3. After replacing x by a conjugate under W(K,T), we may assume it is of the form

$$x = (\underbrace{x_1,\ldots,x_1}_{p_1},\underbrace{x_2,\ldots,x_2}_{p_2},\ldots,\underbrace{x_t,\ldots,x_t}_{p_t} \mid \underbrace{x_1,\ldots,x_1}_{q_1},\ldots,\underbrace{x_t,\ldots,x_t}_{q_t})$$

such that

$$x_1 > x_2 > \ldots > x_t; \quad p_i,\ q_j \geq 0 \quad \text{and}$$

$$\sum p_i = p \qquad \text{and} \qquad \sum q_i = q.$$

Then

$$\ell(x) \equiv \mathfrak{s}(u(p_1,q_1) \oplus u(p_2,q_2) \oplus \ldots \oplus u(p_t,q_t)).$$

Clearly

$$2\rho(u \cap p) = (\underbrace{r_1,\ldots,r_1}_{p_1},\underbrace{r_2,\ldots,r_2}_{p_2},\ldots,\underbrace{r_t,\ldots,r_t}_{p_t} \mid \underbrace{s_1,\ldots,s_1}_{q_1},\underbrace{s_2,\ldots,s_2}_{q_2},\ldots\underbrace{s_t,\ldots,s_t}_{q_t})$$

and

$$2\rho(u\cap k) = (\underbrace{s_1,\dots,s_1}_{p_1},\underbrace{s_2,\dots,s_2}_{p_2},\dots,\underbrace{s_t,\dots,s_t}_{p_t}|$$

$$\underbrace{r_1,\dots,r_1}_{q_1},\underbrace{r_2,\dots,r_2}_{q_2},\dots,\underbrace{r_t,\dots,r_t}_{q_t}).$$

where $r_i = -(q_1+q_2+\dots+q_{i-1}) + q_{i+1} + \dots + q_t$

$$s_j = -(p_1+p_2+\dots+p_{j-1}) + p_{j+1} + \dots + p_t.$$

Let $\mu \in i(t_0^c)^*$ be the highest weight of a representation of K.

Write $q \cap k = (q\cap k)(\mu)$. $2\rho(u\cap k)$ as above, for $x = \mu$, and $\mu' = \mu + 2\rho(u\cap k)$. (Note that $q(\mu) \neq q(\mu')$ but their compact parts coincide.)

Write

$$\mu' = (\underbrace{z_1,\dots,z_1}_{k_1},\underbrace{z_2,\dots,z_2}_{k_2},\dots,\underbrace{z_a,\dots,z_a}_{k_a}|$$

$$\underbrace{z_1,\dots,z_1}_{\ell_1},\dots,\underbrace{z_a,\dots,z_a}_{\ell_a}).$$

Proposition 4.3.1. In the above setting, let $n_i = k_i + \ell_i$. Then μ is the LKT of an $A_q(\lambda)$ iff $z_i - z_{i+1} \geq n_i + n_{i+1}$.

Proof. Suppose that $\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ for some $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and $\lambda \in (\mathfrak{t}^c)^*$, the weight of a unitary admissible one-dimensional character of L, satisfying 2.5.1 a), b). Then λ is orthogonal to the roots of $\mathfrak{t}^c$ in $\mathfrak{l}$ and it is positive on the $\mathfrak{t}^c$-roots in $\mathfrak{u}$.

By Proposition 2.5.6 and its proof we may assume that μ determines $\mathfrak{q}' \cap \mathfrak{k}$ and μ' determines $\mathfrak{q}'$. So $\mathfrak{q}' = \mathfrak{q}(\mu')$ and if λ satisfies 2.5.1, then

$$\lambda = (\underbrace{\lambda_1,\ldots,\lambda_1}_{k_1},\underbrace{\lambda_2,\ldots,\lambda_2}_{k_2},\ldots,\underbrace{\lambda_a,\ldots,\lambda_a}_{k_a}|$$

$$\underbrace{\lambda_1,\ldots,\lambda_1}_{\ell_1},\underbrace{\lambda_2,\ldots,\lambda_2}_{\ell_2},\ldots,\underbrace{\lambda_a,\ldots,\lambda_a}_{\ell_a}).$$

with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t.$$

Suppose that $\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p})$, then $\mu' = \lambda + 2\rho(\mathfrak{u})$;

$$z_i = \lambda_i + (-n_1-n_2-\cdots-n_{i-1}) + n_{i+1} + \cdots + n_t$$

and

$$z_i - z_{i+1} = \lambda_i - \lambda_{i+1} + n_i + n_{i+1} \geq n_i + n_{i+1}.$$

On the other hand if μ satisfies $z_i - z_{i+1} \geq n_i + n_{i+1}$, then let $\lambda_i = z_i + n_1 + \ldots + n_{i-1} - (n_{i+1}+\ldots+n_t)$ and $\mathfrak{q}' = \mathfrak{q}(\mu')$,

$$\text{i.e.} \quad \Delta(\mathfrak{q}') = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{t}^c) \mid \langle\mu',\alpha\rangle > 0\} \cup$$

$$\{\alpha \in \Delta(\mathfrak{g},\mathfrak{t}^c) \mid \langle\mu',\alpha\rangle = 0\}$$

$$= \Delta(\mathfrak{u}') \cup \Delta(\mathfrak{l}').$$

Then if

$$\lambda = (\underbrace{\lambda_1,\ldots,\lambda_1}_{k_1},\ldots,\underbrace{\lambda_t,\ldots,\lambda_t}_{k_t} \mid$$

$$\underbrace{\lambda_1,\ldots,\lambda_1}_{\ell_1},\ldots,\underbrace{\lambda_t,\ldots,\lambda_t}_{\ell_t}).$$

$$\langle\lambda,\Delta(\mathfrak{u}')\rangle \geq 0$$

$$\langle\lambda,\Delta(\mathfrak{l}')\rangle = 0$$

and $\mu = \mu' - 2\rho(\mathfrak{u}'\cap\mathfrak{k}) = \lambda + 2\rho(\mathfrak{u}'\cap\mathfrak{p})$. q.e.d.

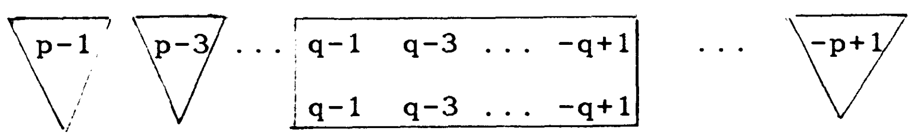
4.4. The parameters $\lambda_V(A_{\mathfrak{q}}(\lambda))$ and $\ell_V(A_{\mathfrak{q}}(\lambda))$

By definition $A_{\mathfrak{q}}(\lambda) = \mathcal{R}^s_{\mathfrak{q}}(\mathbb{C}_\lambda)$, where $s = \dim \mathfrak{u} \cap \mathfrak{k}$. By definition 2.5.1, since λ is perpendicular to the roots of $\mathfrak{l}$, $\lambda \in [\text{center}(\mathfrak{l})]^*$. The LKT of $A_{\mathfrak{q}}(\lambda)$ has highest weight $\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p})$.

By Lemma 2.4.23, $\lambda_V^G(A_{\mathfrak{q}}(\lambda)) = \lambda_V^L(\mathbb{C}_\lambda) + \rho(\mathfrak{u})$. Assume first that

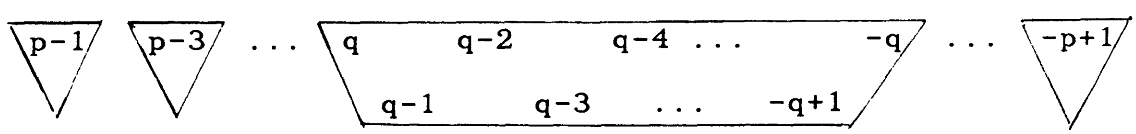
$$\lambda = (\underbrace{0\ldots0}_{p}|\underbrace{0\ldots0}_{q}) \qquad \text{and} \qquad \mathfrak{l} = \mathfrak{g}$$

then $\lambda + 2\rho_c = (p-1,\ldots-p+1|q-1,\ldots,-q+1)$ which gives a picture (say $p \geq q$)



(4.4.1) (if $p = q+2k$)

or else:



(if $p = q+2k+1$)

(4.4.1)

Then

$$\lambda_V(\mathbb{C}) = \begin{cases} \left[\frac{p-q-1}{2}, \frac{p-q-3}{2}, \ldots, \frac{1}{2}\ \underbrace{0\ldots0}_{q}\ -\frac{1}{2}, \ldots, \frac{-p+q+1}{2}\ \middle|\ \underbrace{0\ldots0}_{q}\right]; & (p \equiv q \pmod 2) \\ \left[\frac{p-q-1}{2}, \ldots, 1\ \underbrace{0\ldots0}_{q+1}\ -1, \ldots, \frac{-(p-q-1)}{2}\ \middle|\ \underbrace{0\ldots0}_{q}\right]; & (p \equiv q+1 \pmod 2) \end{cases}$$

So if $\lambda = (a,\ldots,a|a,\ldots,a)$ and $\mathfrak{l} = \mathfrak{g}$, $\lambda + 2\rho_c$ will give the same picture and

$$\lambda_V(\mathbb{C}_\lambda) = \left[a + \frac{p-q-1}{2}, a + \frac{p-q-3}{2}, \ldots, a + \frac{1}{2} + \frac{\epsilon}{2}\ \underbrace{a\ldots a}_{q+\epsilon},\right.$$

$$\left. a - \frac{1}{2} - \frac{\epsilon}{2}, \ldots, a - \frac{(p-q-1)}{2}\middle|\underbrace{a\ldots a}_{q}\right] \qquad p \equiv q+\epsilon \qquad \epsilon = 1, 0.$$

Now suppose

$$a_1 \geq a_2 \geq \ldots \geq a_t, \quad \mathfrak{l} = \mathfrak{s}(\mathfrak{u}(p_1,q_1) \oplus \ldots \oplus \mathfrak{u}(p_t,q_t))$$

and $$\lambda = (\underbrace{a_1,\dots,a_1}_{p_1},\underbrace{a_2,\dots,a_2}_{p_2},\dots,\underbrace{a_t,\dots,a_t}_{p_t} \mid$$

$$\underbrace{a_1,\dots,a_1}_{q_1},\underbrace{a_2,\dots,a_2}_{q_2},\dots,\underbrace{a_t,\dots,a_t}_{q_t}).$$

Clearly

$$2\rho_{\ell\cap k} = (p_1-1,\dots,-p_1+1,p_2-1,\dots,-p_2+1,\dots,p_t-1,\dots,-p_t+1 \mid$$

$$q_1-1,\dots,-q_1+1,\dots,q_t-1,\dots,-q_t+1).$$

So on the (p_i,q_i)-coordinates we have a similar picture and

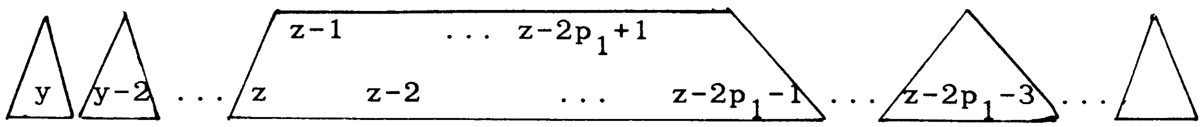
$$\lambda_V(\mathbb{C}_\lambda) =$$

$$\Big[\dots a_i+\frac{p_i-q_i-1}{2},\dots,a_i+\frac{1+\epsilon_i}{2},\underbrace{a_i\dots a_i}_{q_i+\epsilon_i},a_i-\left[\frac{1+\epsilon_i}{2}\right]\dots a_i-\frac{(p_i-q_i-1)}{2}$$

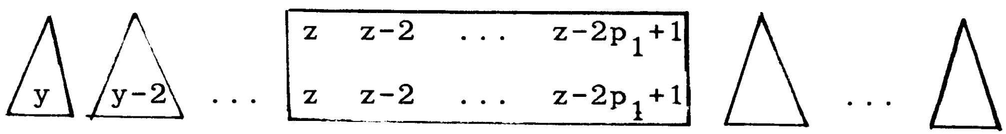
$$p_i$$

$$\dots \mid \underbrace{\dots a_i \dots a_i \dots}_{q_i}\Big]$$

if say, $p_i = q_i + 2k + \epsilon_i$, $\epsilon_i = 0,1$, $k \geq 0$. Also the picture for $\lambda + 2\rho_{\ell\cap k}$ will have pictures like (4.4.1) or, if $q_i \geq p_i$,



(4.4.2)



Now

$$\rho(u) = (\underbrace{u_1,\dots,u_1}_{p_1},\dots,\underbrace{u_t,\dots,u_t}_{p_t} \mid \underbrace{u_1,\dots,u_1}_{q_1},\dots,\underbrace{u_t,\dots,u_t}_{q_t})$$

So

$$\ell_V(A_{\mathfrak{q}}(\lambda)) = \mathfrak{s}\left[\prod_{i=1}^{t}\left[(u(1)^{d_i} \times u(r_i,s_i) \times (u(1))^{d_i}\right]\right]$$

$$\subseteq \prod_{i=1}^{t} ((u(p_i,q_i))$$

where

$$r_i = \min(p_i,q_i) + \epsilon_i$$

($\epsilon_i = 1$ if $p_i \equiv q_i + 1 \pmod 2$ and $p_i > q_i$; $\epsilon_i = 0$

otherwise)

$$s_i = \min(p_i, q_i) + \delta_i$$

($\delta_i = 1$ if $p_i \equiv q_i + 1$ and $q_i > p_i$; $\delta_i = 0$ otherwise)

$$2d_i + r_i + s_i = p_i + q_i.$$

4.5. Proof of Theorem 2.6.7. for $G = SU(p,q)$.

Suppose $X \in \mathcal{A}(\mathfrak{g},K)$ is as in Theorem 2.6.7, with infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$, and let $\mu \in (\mathfrak{t}_0^c)^*$ be the highest weight of a LKT of X.

Let's consider a slightly different splitting of the coordinates of μ than that of Section 4.2:

$$\mu = (\underbrace{x_1,\ldots,x_1}_{p_1},\underbrace{x_2,\ldots,x_2}_{p_2},\ldots\underbrace{x_t,\ldots x_t}_{p_t} \mid$$

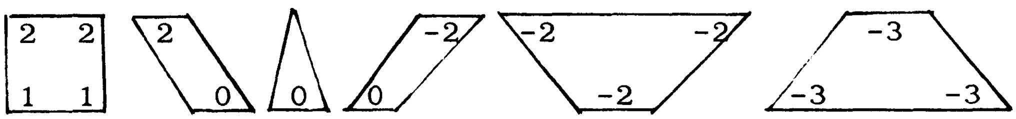
$$\underbrace{y_1,\ldots,y_1}_{q_1},\underbrace{y_2,\ldots,y_2}_{q_2},\ldots,\underbrace{y_s,\ldots,y_s}_{q_s})$$

so that $x_1 > x_2 > \ldots > x_t$
$y_1 > y_2 > \ldots > y_s$

but here $p_i, q_j > 0$, that is, this splitting is not necessarily compatible with the blocks given by $\mu + 2\rho_c$.

It is convenient to draw a picture of the coordinates of μ with the same blocks obtained from $\mu + 2\rho_c$.

In the example of section 4.2 this means



But now, the splitting of μ is

$$\mu = (\underbrace{2\ 2\ 2}_{p_1}\quad \underbrace{-2\ -2\ -2}_{p_2}\quad \underbrace{-3}_{p_3}\ |\ \underbrace{1\ 1}_{q_1}\quad \underbrace{0\ 0\ 0}_{q_2}\quad \underbrace{-2}_{q_3}\quad \underbrace{-3\ -3}_{q_4})\ .$$

We are going to study what happens around the first p_1 coordinates of μ.

We may assume that either

$$x_1 + p-1 > y_1 + q-1$$

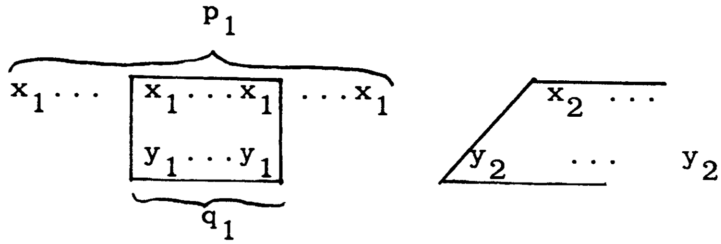
or

$$x_1 + p-1 = y_1 + q-1 \qquad \text{and} \qquad p_1 \geq q_1,$$

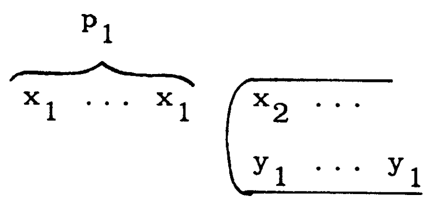
otherwise we can interchange p and q.

If $p_1 < p$ we can have the following configurations for μ

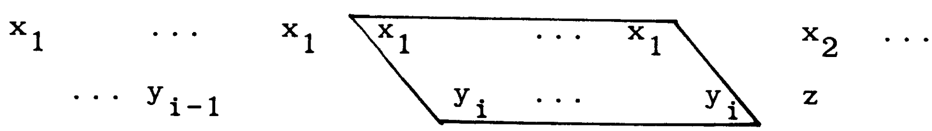
1.



or



2.



where $y_{i-1} > y_i \geq z$.

3.

$$\begin{array}{lllll} x_1 \cdots x_1 & x_1 \cdots x_1 & x_2 \cdots \\ \cdots\ y_{i-1} & y_i \cdots y_i & z \end{array}$$

with $y_{i-1} > y_i \geq z$.

4. $x_1 \; \cdots \; x_1 \; \cdots \; \boxed{\begin{matrix} x_j \; \cdots \; x_j \\ y_1 \; \cdots \; y_1 \end{matrix}} \; z \; \cdots$

$\qquad\quad y_1 \; \cdots \; y_1 \qquad\qquad\qquad\quad y_2$

$x_1 > x_j \geq z$

5. $x_1 \; \cdots \; x_1 \; \cdots \; x_j \; \cdots \; x_j \; z \; \cdots$

$\qquad\quad y_1 \; \cdots \; y_1 \; y_1 \; \cdots \; y_1 \qquad y_2$

$x_1 > x_j \geq z.$

The blocks in these pictures are simple factors of $\mathfrak{l}_V(X)$.

Note that if μ is the LKT of an $A_{\mathfrak{q}}(\lambda)$, we must be in case 1. So we have to check that we can find a reductive subgroup $L \subseteq G$, and embed X as the Langlands submodule of a Zuckerman module coming from a representation of L; but in such a way that, in the case of an $A_{\mathfrak{q}}(\lambda)$, the signature of the Hermitian form is preserved under the derived functor, and in the case when we don't have an $A_{\mathfrak{q}}(\lambda)$, Dirac inequality fails on μ^L, the LKT of the representation of L, and the $(L\cap K)$-types involved in the indefiniteness of the form on L, and occurring in $V_{\mu^L} \otimes (\mathfrak{l}\cap\mathfrak{p})$ with highest weight $\eta^L = \mu^L + \beta$ will be such that $\eta^L + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ is dominant.

On the other hand, for cases 2. - 5. we need to prove non-unitarity. In each case a group L will be found as in 1, making sure that a) - c) of Theorem 2.6.7 hold.

All this will reduce the problem to the case $p_1 = p$.

In this case we have two configurations

6. $$x_1 \begin{pmatrix} x_1 & \cdots & x_1 \\ y_1 & \cdots & y_1 \end{pmatrix} \cdots\ x_1 \cdots \begin{pmatrix} x_1 & \cdots & x_1 \\ y_k & \cdots & y_k \end{pmatrix} \ z \ \cdots$$

$y_1 > y_k \geq z$.

7. $$x_1 \begin{pmatrix} \cdots & & x_1 \\ y_1 & \cdots & y_1 \end{pmatrix} \begin{matrix} \cdots \\ \cdots \end{matrix} \begin{pmatrix} & x_1 & \cdots \\ y_s & \cdots & y_s \end{pmatrix} x_1 \ \cdots \ x_1 \ .$$

Case 6. can be included in either 2. or 3. and case 7. will be dealt with similarly.

Note that as soon as we have shown that a) of Theorem 2.6.7 holds, then by Lemma 2.7.4 the representation of L in question, as well as its Hermitian dual, have a Hermitian form.

For 1, let $\mathfrak{l} = \mathfrak{s}(\mathfrak{u}(p_1,q_a) \oplus \mathfrak{u}(p-p_1,q-q_a))$, here q_a is either q_1 or 0.

Then $\mathfrak{l} \supseteq \mathfrak{l}_V$ and by Proposition 2.4.15, if $\mathfrak{q}_0 = \mathfrak{l} \cap \mathfrak{q}_V = \mathfrak{l}_V + \mathfrak{l} \cap \mathfrak{u}_V$,

$$\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}[\mathcal{R}_{\mathfrak{q}_0}(X_{L_V})) \cong \mathcal{R}_{\mathfrak{q}_V}^{\mathfrak{g}}(X_{L_V}).$$

Assume that $X_L = \mathcal{R}_{\mathfrak{q}_0}(X_{L_V}) \cong A_{\mathfrak{q}_1}(\lambda)$. Define $\mathfrak{q}_2$ by $\mathfrak{u}_2 = \mathfrak{u} + \mathfrak{u}_1$

$$\mathfrak{q}_2 = \mathfrak{q}_1 + \mathfrak{u} = \mathfrak{l}_1 + \mathfrak{u}_1 + \mathfrak{u}.$$

Then

$$\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}(A_{\mathfrak{q}_1}(\lambda)) \cong \mathcal{R}_{\mathfrak{q}_2}^{\mathfrak{g}}(\mathbb{C}_\lambda),$$

Proposition 2.4.15 again.

To see that $\mathcal{R}_{\mathfrak{q}_2}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ is a module $A_{\mathfrak{q}_2}(\mathbb{C}_\lambda)$ we need to prove that $\langle\lambda,\alpha\rangle \geq 0$ for all $\alpha \in \Delta(\mathfrak{u}_2)$. But, by Lemma 2.4.22,

$$\lambda_V = \lambda + \rho(\mathfrak{u}) + \rho_{\mathfrak{m}^L}$$

$$= \lambda + \rho(\mathfrak{u}) + \rho_{\mathfrak{l}} - \rho_{\mathfrak{l}}^{res}$$

Hence $$\lambda + \rho = \lambda_V + \rho_{\mathfrak{l}}^{res}.$$

By Proposition 2.4.16, the infinitesimal character of $\mathcal{R}_{\mathfrak{q}}(X_L)$ is $\lambda + \rho$, and it is regular.

Hence, λ is dominant $\Leftrightarrow$ $\lambda + \rho$ is dominant $\Leftrightarrow$ $\langle\lambda+\rho,\alpha\rangle = \langle\lambda_V + \rho_L^{res},\alpha\rangle \geq 0$ for all α positive. In particular, if $\alpha \in \Delta(u_2)$

$$\langle\lambda_V+\rho_{\ell}^{res},\alpha\rangle = \langle\lambda_V,\alpha\rangle + \langle\rho_{\ell}^{res},\alpha\rangle \geq 0.$$

Now

$2\rho(u\cap p) =$

$$(\underbrace{q-q_a \dots q-q_a}_{p_1},\underbrace{-q_a,\dots,-q_a}_{p-p_1}|\underbrace{p-p_1,\dots,p-p_1}_{q_a},\underbrace{-p_1,\dots,-p_1}_{q-q_a})$$

Set

$$\mu^L = \mu-2\rho(u\cap p) =$$

$$(\underbrace{u_1,\dots,u_1}_{p_1},\dots\underbrace{u_t,\dots,u_t}_{p_t}|\underbrace{v_1,\dots,v_1}_{q_1},\dots,\underbrace{v_s,\dots,v_s}_{q_s})$$

with $u_i > u_{i+1}$, $v_j > v_{j+1}$.

By c) of Lemma 2.4.23 μ^L is the highest weight of a LKT of the module X_L.

Since $p_1 < p$ then $L \neq G$ and $\dim L < \dim G$. Hence, by induction, Theorem 1.3 implies that there exists an $L \cap K$-type V_{η^L} in $V_{\mu^L} \otimes (\mathfrak{l} \cap \mathfrak{p})$ such that, on $V_{\mu^L} \oplus V_{\eta^L}$ the Hermitian form is indefinite.

Now, the weights in $\mathfrak{l} \cap \mathfrak{p}$ are the roots

$$\Delta(\mathfrak{l}\cap\mathfrak{p}) = \Big\{\pm(\underbrace{0\ldots0\ 1\ 0\ldots0}_{p_1}\ \underbrace{0\ldots0}_{p-p_1}|\underbrace{0\ldots0\ -1\ 0\ldots0}_{q_a}\ \underbrace{0\ldots0}_{q-q_a})\Big\}$$

$$\cup \Big\{\pm(\underbrace{0\ldots0}_{p_1}\ \underbrace{0\ldots0\ 1\ 0\ldots0}_{p-p_1}|\underbrace{0\ldots0}_{q_a}\ \underbrace{0\ldots0\ -1\ 0\ldots0}_{q-q_a})\Big\}$$

if

$$q_a = q_1$$

or

$$\Delta(\mathfrak{l}\cap\mathfrak{p}) = \Big\{\pm(\underbrace{0\ldots0}_{p_1}\ \underbrace{0\ldots0\ 1\ 0\ldots0}_{p-p_1}|\underbrace{0\ldots0\ -1\ 0\ldots0}_{q})\Big\}$$

if $q_a = 0$.

A highest weight of an $L \cap K$-type in $V_{\mu^L} \otimes (\mathfrak{l}\cap\mathfrak{p})$ is then of the form $\mu^L + \beta$ for some $\beta \in \Delta(\mathfrak{l}\cap\mathfrak{p})$. So the candidates for highest weights are the weights

$$\eta^L = (u_1,\dots,u_1,\dots,u_j+1,\underbrace{u_j,\dots,u_j}_{p_j-1}\dots|\dots\underbrace{v_k\dots v_k}_{q_k-1}\,v_k-1\dots)$$

$p_j \neq p_1 \qquad q_k \neq q_a$

or

$$\eta^L = (u_1+1,\underbrace{u_1\dots u_1}_{p_1-1},\dots|\underbrace{v_a\dots v_a}_{q_a-1}\,v_a-1\dots).$$

So

$$\eta = (x_1\dots x_1,\dots x_j+1,\underbrace{x_j\dots x_j}_{p_j-1}\dots|\dots\underbrace{y_k\dots y_k}_{q_k-1}\,y_k-1,\dots)$$

or

$$\eta = (x_1+1\ \underbrace{x_1,\dots,x_1}_{p_1-1},x_2\dots|\underbrace{y_1\dots y_1}_{q_a-1}\,y_1-1\dots)$$

are dominant.

This completes the proof of Theorem 2.6.7 for this case.

To prove the result for cases 2. - 5. we need the following lemmas.

Lemma 4.5.1. If $G = U(m,m)$ and $\mu = (a+1,\ldots,a+1|a\ldots a)$, then the Dirac operator inequality fails for

$$\rho_n^+ = (\frac{m}{2},\ldots,\frac{m}{2}|\frac{-m}{2},\ldots,\frac{-m}{2}). \quad \text{(Cfr. 2.7.1.)}$$

Proof. Write μ as $\mu_c + \mu_s$ with

$$\mu_c \in (\text{center } \mathfrak{g})^* \qquad \mu_s \in \mathfrak{g}^d = [\mathfrak{g},\mathfrak{g}].$$

And $\mu_s = (\underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{m}|\underbrace{-\frac{1}{2},\ldots,-\frac{1}{2}}_{m}) \qquad \mu_c = (\underbrace{x\ldots x}_{m}|\underbrace{x\ldots x}_{m})$

$$\omega(\mu_s-\rho_n^+) + \rho_c = (0,-1,\ldots,-m+1|m-1,m-2,\ldots,1,0)$$

$$\langle\omega(\mu-\rho_n^+) + \rho_c,\omega(\mu-\rho_n^+)+\rho_c\rangle =$$

$$\langle\mu_c,\mu_c\rangle + \langle\omega(\mu_s-\rho_n^+)+\rho_c,\omega(\mu_s-\rho_n^+)+\rho_c\rangle.$$

If X is a $(\mathfrak{g},K)$-module with infinitesimal character γ, then

$$\langle\gamma,\gamma\rangle \geq \langle\mu_c,\mu_c\rangle + \langle\rho,\rho\rangle$$

And $\langle\omega(\mu_s-\rho_n^+)+\rho_c,\omega(\mu_s-\rho_n^+)+\rho_c\rangle < \langle\rho,\rho\rangle.$

q.e.d.

Lemma 4.5.2. If $G = U(m+1,m)$, $\mu = (\underbrace{b+1,\dots,b+1}_{m+1}|\underbrace{b\dots b}_{m})$. Then Dirac operator inequality fails for

$$\rho_n^+ = (\underbrace{\tfrac{m}{2},\dots,\tfrac{m}{2}}_{m+1}|\underbrace{\tfrac{-m-1}{2},\dots,\tfrac{-m-1}{2}}_{m}).$$

Proof. Write

$$\mu_s = \left[1-\frac{m+1}{2m+1},\dots,1-\frac{m+1}{2m+1}\ \middle|-\frac{m+1}{2m+1},\dots,-\frac{m+1}{2m+1}\right]$$

$$\mu_c = \left[b+\frac{m+1}{2m+1},\dots,b+\frac{m+1}{2m+1}\ \middle|\ b+\frac{m+1}{2m+1},\dots,b+\frac{m+1}{2m+1}\right]$$

and $\mu = \mu_s + \mu_c$ as in Lemma 4.5.1. By the same argument as in the preceding Lemma we only need to show that

$$\langle\omega(\mu_s-\rho_n^+)+\rho_c,\omega(\mu_s-\rho_n^+)+\rho_c\rangle < \langle\rho,\rho\rangle$$

but

$$\rho_c = \left[\frac{m}{2},\frac{m-2}{2}\dots\frac{-m}{2}\ \middle|\ \frac{m-1}{2},\dots,\frac{-m+1}{2}\right];$$

$$\rho = (m,m-1,\dots,-m)$$

$$\mu_s - \rho_n^+ + \rho_c = (1,0,-1,\dots,-m+1 \mid m,m-1,\dots,1)$$

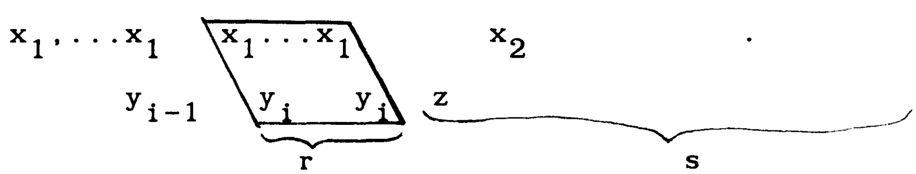
$$+ \left[\frac{-(m+1)}{2m+1}\cdots \middle| \cdots \left[\frac{-m+1}{2m+1}\right]\right]$$

this proves the lemma.

q.e.d.

Now, for each case 2 - 5 we will choose a subgroup L so that we can reduce the problem to the cases discussed in Lemmas 4.5.1, 4.5.2.

2. Remember that we have the following picture:



where $y_{i-1} > y_i \geq z$.

Choose $\mathfrak{l} = \mathfrak{s}(\mathfrak{u}(p_1-r, q_1+\dots+q_{i-1})\oplus\mathfrak{u}(r,r)\oplus\mathfrak{u}(p-p_1,s))$ obviously $\mathfrak{l}_V \subseteq \mathfrak{l}$. Let $\mathfrak{u}' = \mathfrak{u}_V \cap \mathfrak{l}$

$$\mathfrak{u} = \mathfrak{u}_V - \mathfrak{u}' \quad \text{and} \quad \mathfrak{q} = \mathfrak{l} + \mathfrak{u}.$$

Let L = Normalizer of $\mathfrak{q}$ in G, then

$$L \cong S(U(p_1-r, q_1+\dots+q_{i-1})\mathrm{x}U(r,r)\mathrm{x}U(p-p_1,s)).$$

By Proposition 2.4.15 there is a representation X_L of L such that X is the Langlands subrepresentation of the standard module $\mathcal{R}_{\mathfrak{q}}^{\dim \mathfrak{u}\cap\mathfrak{k}}(X_L)$.

Now,

$2\rho(\mathfrak{l}\cap\mathfrak{p})$ =

$$(\underbrace{a,\ldots,a}_{p_1-r}\ \underbrace{b,\ldots,b}_{r}\ \underbrace{c,\ldots,c}_{p-p_1}\ \Big|\ \underbrace{d,\ldots,d}_{q_1+\ldots+q_{i-1}}\ \underbrace{e,\ldots,e}_{r}\ \underbrace{f,\ldots,f}_{s}).$$

By Lemma 2.4.23 $\mu^L = \mu-2\rho(\mathfrak{u}\cap\mathfrak{p})$ is the highest weight of a LKT of X_L.

We claim now that

$$\mu_L\Big|_{U(r,r)} = (\underbrace{x+1,\ldots,x+1}_{r}\ |\underbrace{x,\ldots,x}_{r}).$$

In fact

$\mu_L|_{U(r,r)} =$

$(x_1,\ldots,x_1|y_i,\ldots,y_i) - (-q_1-\ldots-q_{i+1}+s,\ldots$

$-q_1-\ldots-q_{i-1}+s|p-p_1+r\ldots)$

$= (x_1+q_1+\ldots+q_{i-1}-s,\ldots,$

$x_1+q_1+\ldots+q_{i-1}-s|y_i-p+2p_1-r,\ldots,y_i-p+2p_1-r).$

We know from the picture for μ that

$$x_1 + p - 2p_1 = y_i + q - 2(q_1+\ldots+q_{i-1}+r) + 1 =$$

$$y_i + s - (q_i+\ldots+q_{i-1}+r) + 1.$$

So $x_1 + q_1 +\ldots+ q_{i-1} - s - y_i + p - 2p_1 + r = 1.$

Hence, by Lemma 4.5.1 and 2 of Lemma 2.7.2, $\exists\ \beta \in \Delta(\mathfrak{u}(r,r)\cap\mathfrak{p}^-)$ such that the Hermitian form $\langle\ ,\ \rangle^L$ on $V_{\mu^L} \oplus V_{\mu^L+\beta}$ is indefinite. Now if $\mu^L + \beta$ is dominant, necessarily

$$\beta = (0\ldots0\ -1\ |\ 1\ 0\ldots0).$$

Also $\mu + \beta = (x_1, \ldots, x_1, x_1 - 1, x_2, \ldots | y_1, \ldots, y_{i-1} y_i + 1, y_i \ldots)$ is dominant for $\Delta(u \cap k)$.

This proves Theorem 2.6.7 for case 2.

For 3, the picture that we had is

$$\begin{array}{ccccc} \overbrace{x_1 \;\cdots\; x_1}^{p_1 - r} & \boxed{\begin{array}{c} x_1 \quad x_1 \cdots x_1 \\ y_i \quad y_i \cdots y_i \end{array}} & \overbrace{\qquad x_2 \qquad}^{p - p_1} & y_{i-1} > y_i > \geq z. \\ \underbrace{\quad y_{i-1} \quad}_{d} & \underbrace{\qquad\qquad\qquad}_{r} & \underbrace{z \qquad\qquad}_{q-d-r} & \end{array}$$

Let $x = (\underset{p_1 - r - 1}{a \ldots a} \;\; \underset{r+1}{b \ldots b} \;\; \underset{p - p_1}{c \ldots c} \mid \underset{d}{a \ldots a} \;\; \underset{r}{b \ldots b} \;\; \underset{q-d-r}{c \ldots c}) \in it_0^*$

define a θ-stable parabolic subalgebra as in 2.3, and L = Normalizer of q in G, then

$$l \cong u(p_1 - r - 1, d) \oplus u(r+1, r) \oplus u(p - p_1, q-d-r).$$

Note that $l \not\supseteq l_V$. However, Proposition 8.2.15 of Vogan [1981] p. 545 gives us that $Y = \mathcal{R}_q^s[\mathcal{R}_{q_V \cap l}^l(X_{L_V})) = \mathcal{R}_{q_V}^{s_V}(X_{L_V})$, where $s_V = \dim u_V \cap k$. In fact, all we need to check is that if $(q_V, H_V, \lambda_V, \nu_V)$ is the θ-stable data attached to $\mathcal{R}_{q_V}^{s_V}(X_{L_V})$ then $L \supseteq H_V$ and that q contains some Borel subalgebra of q_V.

This is clear by the picture of $\mu + 2\rho_c$. Then the infinitesimal character of Y is $\gamma = (\lambda_V, \nu) \in h_V^*$. Since

$\gamma + \theta\gamma = (2\lambda_V, 0)$, it is enough to show that $\langle\lambda_V, \alpha\rangle \geq 0$ for $\alpha \in \Delta(u)$. As before, it is straightforward to verify that if $\mu^L = \mu - 2\rho(u \cap p)$ then

$$\mu^L\Big|_{U(r+1,r)} = (\underbrace{a+1,\ldots,a+1}_{r+1}|\underbrace{a\ldots a}_{r}).$$

Lemma 4.5.2 and 2) of Proposition 2.7.2 imply that the Hermitian form $\langle\ ,\ \rangle^L$ is indefinite on $V_{\mu^L} \oplus V_{\mu^L+\beta}$ with

$$\beta = (0,\ldots,0\ -1|1,0,\ldots,0) \in \Delta(u(r+1,r) \cap p^-).$$

Also, $\mu + \beta$ is again dominant for $\Delta(u \cap k)$. Hence Theorem 2.6.7 also holds for this case.

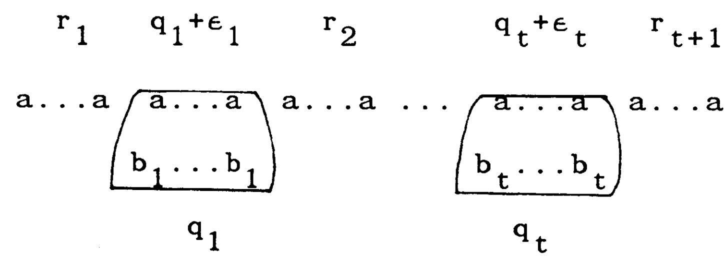
4 and 5 are solved in exactly the same way as 2 and 3, using $\rho_{\bar{n}}$ and p^+.

So I have reduced the problem to the case

$$p_1 = p.$$

6 can be included in either 2 or 3.

For 7 write $\mu = (a\ a\ldots a|b_1\ldots b_1, b_2\ldots b_2\ldots b_t\ldots b_t)$ the picture for μ is



with $\epsilon_j = 0,\ 1$

$$p = \sum_{1}^{t+1} r_i + \sum_{1}^{t} q_j + \epsilon_j .$$

$$q = \sum_{1}^{t} q_j .$$

Then $\mu = (\underbrace{a \ldots a}_{p} | \underbrace{a+s_1 \ldots a+s_1}_{q_1}, \underbrace{a+s_2 \ldots a+s_2}_{q_2} \ldots \underbrace{a+s_t \ldots a+s_t}_{q_t})$

$$s_k = -(r_1 + \ldots + r_k) + (r_{k+1} + \ldots + r_{t+1})$$

$$- (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{k-1}) + (\epsilon_{k+1} + \ldots + \epsilon_t) .$$

In fact, from the above picture for μ, we know that:

$$a + p-2(r_1+\dots+r_k) - 2(q_1 +\epsilon_1+\dots+q_{k-1}+\epsilon_{k-1})-1$$

$$= b_k + q-2(q_1+\dots+q_{k-1})-1+\epsilon_k$$

$$\Rightarrow b_k = a + p-q - 2\sum_1^k r_i - 2\sum_1^{k-1} \epsilon_j - \epsilon_k$$

$$= a - (r_1+\dots+r_k) + (r_{k+1}+\dots+r_{t+1})$$

$$- (\epsilon_1+\dots+\epsilon_{k-1}) + (\epsilon_{k+1}+\dots+\epsilon_t).$$

Note that if $q_i = 0$ for all $i > 1$ this is case 1. So assume $t \geq 2$.

As before, I want to find a group L to which I can apply some reduction argument.

(*) Suppose $r_{t+1} > r_1$ set $s = r_1 + q_1 + \epsilon_1 + r_2$.

Then let $L = U(s,q_1) \times U(p-s),q-q_1)$. Note that $L_V = U(r_1) \times U(q_1+\epsilon_1,q_1) \times U(r_2) \times\dots\times U(q_t+\epsilon_t,q_t) \times U(r_{t+1})$.

So $L \supseteq L_V$, and again, we can use Lemma 2.4.15 to verify a) of Theorem 2.6.7.

$$2\rho(u\cap p) = (\underbrace{q-q_1,\dots,q-q_1}_{s},\underbrace{-q_1,\dots,-q_1}_{p-s}|\underbrace{p-s\dots p-s}_{q_1},\underbrace{-s\dots-s}_{q-q_1}).$$

By Lemma 2.4.22 and 2.4.23 if $\gamma + (\lambda_V, \nu)$ is the infinitesimal character of $\mathcal{R}_{\mathfrak{q}}(X_L)$, then $\gamma^L = (\lambda_V - \rho(u), \nu)$ is the infinitesimal character of X_L.

In fact, by definition of $\Delta(u_V)$,

$$\text{(4.5.3)} \qquad \langle \lambda_V, \alpha \rangle > 0 \quad \text{for all} \quad \alpha \in \Delta(u) \subseteq \Delta(u_V).$$

Write $L_1 = U(s, q_1)$, then

$$\mu^L\big|_{L_1} = (a-q+q_1, \ldots, a-q+q_1 \,|\, a+s_1-p+s \ldots a+s_1-p+s).$$

For some values of r_1, r_2 it could be possible to prove the failure of Dirac inequality as we have done before; that is, by simply using the minimal value of the restriction of ν to the split part of the Cartan of L_1 that makes $\gamma_L\big|_{L_1}$ regular integral.

However, this is not possible for all values of r_1, r_2. Therefore, we need to involve all of ν instead.

If $\lambda_V = \lambda_V(\mu)$ then

$$\lambda_V = (\underbrace{x, x-1, \ldots, x-r_1+1}_{r_1}, \underbrace{w, w, \ldots, w}_{q_1+\epsilon_1}, x-r_1-1, x-r_1-2 \ldots z \;|$$

$$\underbrace{x-r_1, \ldots, x-r_1}_{q_1} \ldots)$$

where $x = a + p - 1 - \frac{(n-1)}{2} = a + \frac{s_1+\epsilon_1-1}{2} + r_1$

$$w = a + \frac{s_1}{2}$$

$$z = a - p + 1 + \frac{(n-1)}{2} = a - \frac{s_1+\epsilon_1-1}{2} - r_1.$$

If $H^s = T^sA^s$ is a maximally split Cartan subgroup of L_V and $\upsilon \in A^s$, then

$$\upsilon = (0\ldots0\ \upsilon_1^1\upsilon_2^1\ldots\upsilon_{q_1}^1\ \underbrace{0\ldots0}_{\epsilon_1+r_2}\ \upsilon_1^2\ldots\upsilon_{q_2}^2\ \underbrace{0\ldots0}_{\epsilon_2+r_3}\ |$$

$$-\upsilon_{q_1}^1,\ldots,-\upsilon_1^1,-\upsilon_{q_2}^1,\ldots,-\upsilon_1^2\ldots)$$

To make (λ_V,υ) regular integral we need:

$$\left.\begin{array}{l} a + \frac{s_1}{2} + \upsilon_1 > \frac{a+s_1+\epsilon_1-1+r_1}{2}+r_1 \\ \\ a + \frac{s_1}{2} - \upsilon_1 < \frac{a-(s_1+\epsilon_1-1)}{2}-r_1 \end{array}\right\} \Rightarrow \upsilon_1 > \max\left[\frac{\epsilon-1}{2}+r,\ s_1+\frac{\epsilon_1-1}{2}+r_1\right]$$

$s_1 = -r_1 + r_2 +\ldots+ r_{t+1} + \epsilon_2 +\ldots+ \epsilon_t.$

By (*), $s_1 \geq 0$. So $\upsilon_1 > s_1 + \frac{\epsilon_1-1}{2} + r_1.$

Let $\nu_1 = s_1 + \frac{\epsilon_1 - 1}{2} + r_1 + \delta \quad \delta = \frac{1}{2}, 1$

$$\nu_j = \nu_1 + j - 1.$$

Now, $\rho(u)\big|_{L_1} = \left[\frac{n-s-q_1}{2}, \ldots, \frac{n-s-q_1}{2} \,\middle|\, \frac{n-s-q_1}{2}, \ldots, \frac{n-s-q_1}{2}\right]$

$(\lambda_V - \rho(u))\big|_{L_1} =$

$$\Bigg[\underbrace{a+\frac{s_1}{2}+\frac{\epsilon_1-1}{2}+r_1+\frac{s+q_1-n}{2}, \ldots, a+\frac{s_1}{2}+\frac{\epsilon_1-1}{2}+1+\frac{s+q_1-n}{2}}_{r_1},$$

$$\underbrace{a+\frac{s_1}{2}+\frac{s+q_1-n}{2}, \ldots, a+\frac{s_1}{2}+\frac{s+q_1-n}{2}}_{q_1}$$

$$\underbrace{a+\frac{s_1}{2}+\frac{\epsilon_1-1}{2}+\frac{s+q_1-n}{2}, a+\frac{s_1}{2}+\frac{s+q_1-n}{2}-1 \ldots a+\frac{s_1}{2}+\frac{s+q_1-n}{2}-r_2}_{\epsilon_1+r_2} \Bigg|$$

$$\underbrace{a+\frac{s_1}{2}+\frac{s+q_1-n}{2}, \ldots, a+\frac{s_1}{2}+\frac{s+q_1-n}{2}}_{q_1}\Bigg].$$

We would like to prove that the K-type with Highest weight

$$\eta = (a+1,\underbrace{a,\dots,a}_{p-1} \mid$$

$$\underbrace{a+s_1,\dots,a+s_1}_{q_1-1},a+s_1-1,\underbrace{a+s_2\dots a+s_2}_{q_2}\dots\underbrace{a+s_t\dots a+s_t}_{q_t})$$

occurs in the representation X and that the Hermitian form is indefinite on

$$V_\mu \oplus V_\eta .$$

It is enough to prove the failure of Dirac operator inequality on

$$\mu^L\big|_{L_1} \quad \text{and} \quad \rho_n^-(\ell_1) = \rho(\ell\cap k) = \Big[\underbrace{\frac{-q_1}{2},\dots,\frac{-q_1}{2}}_{s}\Big|\underbrace{\frac{s}{2},\dots,\frac{s}{2}}_{q_1}\Big].$$

So

$$\mu^L\big|_{L_1} - \rho_n^-(\ell_1) =$$

$$\Big[a+q_1-q+\frac{q_1}{2},\dots,a+q_1-q+\frac{q_1}{2}\Big|a+s_1+s-p-\frac{s}{2},\dots,a+s_1+s-p-\frac{s}{2}\Big].$$

Since

$$\rho_c(\ell_1) = \rho_{\ell_1 \cap k} = \left[\frac{s-1}{2}, \frac{s-3}{2}, \ldots, \frac{-s+1}{2} \middle| \frac{q_1-1}{2}, \ldots, \frac{-q_1+1}{2}\right],$$

$$\mu^L\big|_{L_1} - \rho_n^-(\ell_1) + \rho_c(\ell_1) =$$

$$\left[a+q_1-q+\frac{q_1+s-1}{2}, \ldots, a+q_1-q+\frac{q_1-s+1}{2} \right|$$

$$\left. a+s_1-p+\frac{s+q_1-1}{2}, \ldots, a+s_1-p+\frac{s-q_1+1}{2}\right].$$

Let $y = a + \frac{s_1}{2} + \frac{s+q_1-n}{2}$. Then if $\lambda_1 = (\lambda_V - \rho(u))\big|_{L_1}$ and $\gamma_1 = (\gamma-\rho(u))\big|_{L_1}$,

$$\lambda_1 = (\underbrace{y+\frac{\epsilon_1-1}{2}+r_1, y+\frac{\epsilon_1-1}{2}+r_1-1, \ldots, y+\frac{\epsilon_1-1}{2}+1}_{r_1},$$

$$\underbrace{y,\ldots,y}_{q_1}\ \underbrace{y+\frac{\epsilon_1-1}{2}}_{\epsilon_1}\ \underbrace{y+\frac{\epsilon_1-1}{2}-1 \ldots y+\frac{\epsilon_1-1}{2}-r_2}_{r_2} \Big| \underbrace{y \ldots y}_{q_1})$$

If $w \in W_K$ such that $\omega\gamma$ is dominant, then

$$\omega\gamma_1 = \Big[\underbrace{y+s_1+\frac{\epsilon_1-1}{2}+r_1+\delta+q_1-1\ldots y+s_1+\frac{\epsilon_1-1}{2}+r_1+\delta}_{q_1},$$

$$\underbrace{y+\frac{\epsilon_1-1}{2}+r_1,\ldots,y+\frac{\epsilon_1-1}{2}+1}_{r_1},\underbrace{y+\frac{\epsilon_1-1}{2}}_{\epsilon_1},\underbrace{y+\frac{\epsilon_1-1}{2}-1,\ldots,y+\frac{\epsilon_1-1}{2}-r_2}_{r_2}\Big|$$

$$\underbrace{y-s_1-\frac{\epsilon_1-1}{2}-r_1-\delta,y-s_1-\frac{\epsilon_1-1}{2}-r_1-\delta-q_1+1}_{q_1}\Big].$$

Also

$$\mu^L\big|_{L_1} - \rho_n^-(\ell_1) + \rho_{\ell_1\cap k} = \Big[\underbrace{y+\frac{\epsilon_1-1}{2}+r_1+q_1,\ldots,y+\frac{\epsilon_1-1}{2}+r_1+1}_{q_1},$$

$$\underbrace{y+\frac{\epsilon_1-1}{2}+r_1,\ldots,y+\frac{\epsilon_1-1}{2}+1}_{r_1},\underbrace{y+\frac{\epsilon_1-1}{2}}_{\epsilon_1},\underbrace{\ldots,y+\frac{\epsilon_1-1}{2}-r_2}_{r_2}\Big|$$

$$y-\frac{\epsilon_1-1}{2}-r_1-1,\ldots,y-\frac{\epsilon_1-1}{2}-r_1-q_1\Big].$$

To prove what we want it is enough to prove that

(**) $$\langle \omega\gamma_1, \omega\gamma_1 \rangle - \langle \mu^L|_{L_1} - \rho_n^-(\ell_1) + \rho_{\ell_1 \cap k},$$

$$\mu^L|_{L_1} - \rho_n^-(\ell_1) + \rho_{\ell_1 \cap k} \rangle > 0.$$

But this is equivalent to

$$\sum_{j=1}^{q_1} \left[\left[y+s_1+\frac{\epsilon_1-1}{2}+r_1+\delta+j-1\right]^2 + \left[y-s_1-\frac{\epsilon_1-1}{2}-\delta-j+1\right]^2 \right.$$

$$\left. - \left[y+\frac{\epsilon_1-1}{2}+r_1+j\right]^2 - \left[y-\frac{\epsilon_1-1}{2}-r_1-j\right]^2 \right] > 0.$$

Using that $(b+c)^2 + (b-c)^2 - (b+d)^2 - (b-d)^2 > 0$ if $|c| > |d|$ we conclude that (**) holds if

$$\left| s_1+\frac{\epsilon_1-1}{2}+r_1+\delta+j-1 \right| > \left| \frac{\epsilon_1-1}{2}+r_1+j \right|.$$

Since $q_1 > 0$ then $j > 0$. Hence $\frac{\epsilon_1-1}{2} + r_1 + \delta + j - 1 \geq 0$ and $s_1 > 0 \Rightarrow s_1 + \frac{\epsilon_1-1}{2} + r_1 + j > \frac{\epsilon_1-1}{2} + r_1 + \delta + j - 1$.

Now if $r_1 > r_{t+1}$, we choose

$$L = U(r_t+q_t+\epsilon_t+r_{t+1}, q_t) \times U(p-(r_t+q_t+\epsilon_t+r_{t+1}), q-q_t)$$

and repeat the same argument for this case.

Note that, since $s_1 = r_1 + r_2 + \ldots + r_{t+1} + \epsilon_2 + \ldots + \epsilon_t$, then (**) will also hold if $r_1 = r_{t+1}$ and some $\epsilon_i > 0$ or some $r_j > 0$; $1 < j, i \leq t$.

So this reduces to the case

$$\begin{array}{c} \overbrace{a\ldots a}^{r}\ \overbrace{a\ldots a}^{q_1+\epsilon_1}\ \overbrace{a\ldots a}^{q_2}\ \overbrace{a\ldots a}^{r} \\ \underbrace{b_1\ldots b_1}_{q_1}\ \underbrace{b_2\ldots b_2}_{q_2} \end{array} \qquad q_1,\ q_2 > 0$$

But by symmetry, using the case $r_1 > r_{t+1}$, we can conclude that $\epsilon_1 = 0$.

But then we have

$$\begin{array}{c} \overbrace{a\ldots a}^{r}\ \overbrace{a\ldots a}^{q}\ \overbrace{a\ldots a}^{r}\ . \\ \underbrace{b\ldots b}_{q} \end{array}$$

With which we have dealt before. This is solved in the same way as case 1. for $p_1 < p$.

This proves Theorem 2.6.7. for $G = SU(p,q)$.

q.e.d.

Chapter 5. $G = SP(n,\mathbb{R})$

5.1. Preliminary Notation

Let I_m be the identity matrix in $GL(m,\mathbb{C})$. We define

$$G = SP(n,\mathbb{R}) = \left\{ g \in SL(2n,\mathbb{R}) \;\middle|\; {}^t g \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right\}.$$

The maximal compact subgroup K of G is

$$K = SP(n,\mathbb{R}) \cap U(2n) \cong U(n).$$

Also

$$\mathfrak{g}_0 = \mathfrak{sp}(n,\mathbb{R}) = \left\{ X \in \mathfrak{sl}(2n,\mathbb{R}) \;\middle|\; {}^t X \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} X = 0 \right\}$$

that is

$$\mathfrak{g}_0 = \left\{ X = \begin{bmatrix} A & B \\ C & -{}^t A \end{bmatrix} \;\middle|\; A, B, C \in \mathfrak{gl}(n,\mathbb{R}),\ B,\ C \text{ symmetric} \right\}$$

and if θ is the Cartan involution defined by $\theta(x) = -{}^t X$,

$$k_0 = \{X \in sp(n,\mathbb{R}) \mid -{}^tX = X\}.$$

$$= \left\{X = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \middle| A = -{}^tA \quad B = {}^tB\right\} \cong u(n)$$

$$p_0 = \left\{X \in g_0 \mid \theta(X) = -X\right\}$$

$$= \left\{X = \begin{bmatrix} A & B \\ B & -{}^tA \end{bmatrix} \mid A,\ B \text{ symmetric}\right\}.$$

If $d(\theta_1,\ldots,\theta_n) = \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix}$, then the compact Cartan subgroup of G is

$$H^c = \left\{\begin{bmatrix} \cos\theta_1 & & 0 & \sin\theta_1 & & 0 \\ 0 & \ddots & \cos\theta_n & 0 & \ddots & \sin\theta_n \\ -\sin\theta_1 & \ddots & & \cos\theta_1 & \ddots & 0 \\ & & -\sin\theta_1 & 0 & & \cos\theta_n \end{bmatrix} \middle| \theta_i \in \mathbb{R}\right\} = T^c$$

and its Lie algebra is

$$h_0^c = \left\{X = \begin{bmatrix} 0 & d(\theta_1\ldots\theta_n) \\ -d(\theta_1\ldots\theta_n) & 0 \end{bmatrix} \middle| \theta_i \in \mathbb{R}\right\} = t_0^c$$

$t_0^c \leftrightarrow \mathbb{R}^n$.

The complexification of these Lie Algebras gives

$$\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C}) = \left\{ X \in \mathfrak{sl}(2n,\mathbb{C}) \;\middle|\; {}^tX\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} X \right\}$$

$$= \left\{ X = \begin{bmatrix} A & B \\ C & -{}^tA \end{bmatrix} \;\middle|\; B = {}^tB,\ C = {}^tC \right\}$$

$$\mathfrak{k} \cong \mathfrak{gl}(n,\mathbb{C})$$

$$\mathfrak{h}^c = \mathfrak{t}^c = \left\{ X = \begin{bmatrix} 0 & d(z_1 \ldots z_n) \\ d(z_1 \ldots z_n) & 0 \end{bmatrix} \;\middle|\; z_i \in \mathbb{C} \right\} \cong \mathbb{C}^n .$$

$\hat{K}$ can be identified with the space

$$\{\mu = (a_1 \ldots a_n) \mid a_1 \geq a_2 \geq \ldots \geq a_n;\ a_i \in \mathbb{Z}\}.$$

If
$$X = \begin{bmatrix} 0 & \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{matrix} \\ \begin{matrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n \end{matrix} & 0 \end{bmatrix} \in \mathfrak{t}^c .$$

Define, for $j = 1,2,\ldots,n$, $e_j(X) = i\lambda_j$. Then, the roots of $\mathfrak{t}$ in $\mathfrak{g}$ are

$$\Delta(\mathfrak{g}) = \Delta(\mathfrak{g},\mathfrak{t}^c)$$

$$= \{\pm e_j \pm e_k;\ \pm 2e_\ell \mid j,k,\ell = 1,2,\ldots,n;\ j < k\}$$

also

$$\Delta(k) = \Delta(k, t^c) = \{\pm(e_j - e_k) \mid 1 \le j < k \le n\},$$

the compact imaginary roots of t^c in g.

$$\Delta(p) = \Delta(p, t^c) = \{\pm(e_j + e_k);\ \pm 2e_\ell \mid 1 \le j < k \le n;$$
$$1 \le \ell \le n\},$$

the non-compact imaginary roots of t^c in g.

5.2. Computation of $\iota_V(X)$ for any module X.

As for the preceding cases, fix a positive root system $\Delta^+(k)$ so that if

$$\mu = (a_1, a_2, \ldots, a_n) \qquad a_1 \ge a_2 \ge \ldots \ge a_n,$$

then μ is $\Delta^+(k)$-dominant and

$$2\rho_c = (n-1, n-3, \ldots, -n+3, -n+1).$$

Let $\mu + 2\rho_c = (x_1, x_2, \ldots, x_n)$.

Choosing a positive Weyl Chamber for $\Delta(g, h^c)$ corresponds to forming an array of two rows with the absolute value of the coordinates of $\mu + 2\rho_c$ so that they are aligned in decreasing order as follows:

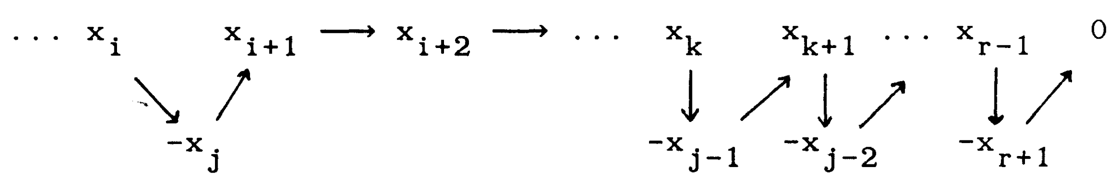
If $x_1 \geq x_2 \geq \ldots \geq x_r \geq 0 > x_{r+1} \geq \ldots \geq x_n$ then $x_1, \ldots, x_r$ are in the first row $-x_n, -x_{n-1}, \ldots, -x_{r+1}$ in the second and they all decrease from left to right in the array.

For example, if we have

$$\ldots x_i > -x_j > x_{i+1} > x_{i+2} > \ldots > x_k =$$

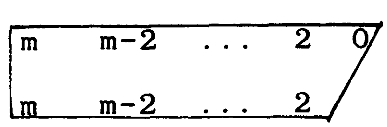
$$-x_{j-1} > x_{k+1} = -x_{j-2} > \ldots > -x_{r+1} = x_{r-1} > x_r = 0$$

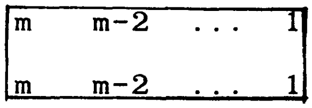
the array would look like

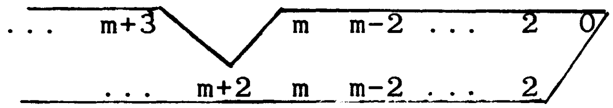


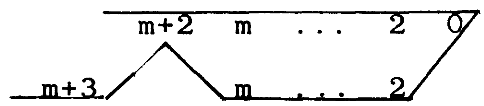
As for the case of SU(p,q), the choice of arrows gives a positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t}^c)$, compatible with $\Delta^+(\mathfrak{k})$.

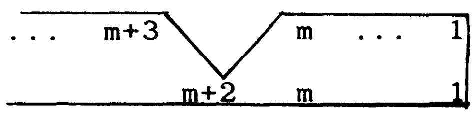
Again, the entire array is a union of blocks of the following types.

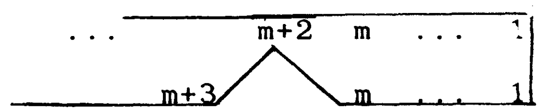
1. 

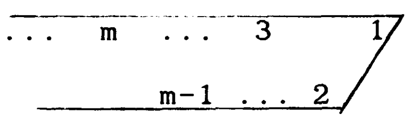
2. 

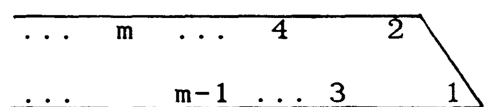
3. 

or 

4. 

or 

5. 

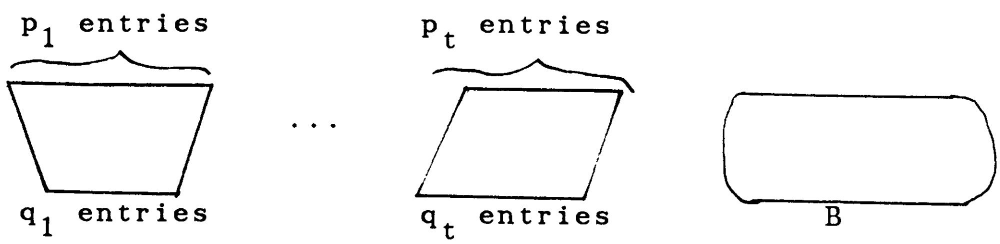
, or 

(5 is a particular case of 4.)

6. All blocks of the five types discussed for SU(p,q), not containing 0 or 1.

Again, using the picture, split the coordinates of μ by the blocks that $\mu + 2\rho_c$ determines as follows.

If $\mu + 2\rho_c$ gives

5.2.1 

with B a block of some type 1-5, set

$$\mu = (\underbrace{a_1 \ldots a_1}_{p_1 \text{ times}} \ldots \underbrace{a_t \ldots a_t}_{p_t \text{ times}} \quad \underbrace{c_1 c_2 \ldots c_m}_{m \text{ entries}} \quad \underbrace{b_t \ldots b_t}_{q_t \text{ times}}, \ldots \underbrace{b_1 \ldots b_1}_{q_1 \text{ times}})$$

where m is the total number of coordinates composing the block B, $c_1 \geq c_2 \geq \ldots \geq c_m$.

Example: If

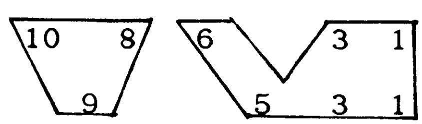
$$\mu = (2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ -1),$$

since

$$2\rho_c = (8\ 6\ 4\ 2\ 0\ -2\ -4\ -6\ -8),$$

$$\mu + 2\rho_c = (10\ 8\ 6\ 3\ 1\ -1\ -3\ -5\ -9)$$

this gives



.

Then

$$\mu = (\underbrace{2\ 2}_{p_1}\ \underbrace{2\ 1\ 1\ 1\ 1\ 1}_{m}\ \underbrace{-1}_{q_1})$$

is the splitting that we want.

Write $\ell_V = \ell_V(\mu)$ $\lambda_V = \lambda_V(\mu)$ as in 2.4.7.

Proposition 5.2.2. If $\mu \in i(t_0^c)^*$ gives figure 5.2.1 then

$$\lambda_V(\mu) =$$

$$(\underbrace{\lambda_1 \ldots \lambda_1}_{p_1 \text{ times}}\ \underbrace{\lambda_2 \ldots \lambda_2}_{p_2 \text{ times}} \ldots \underbrace{\lambda_t \ldots \lambda_t}_{p_2 \text{ times}}\ \underbrace{0 \ldots 0}_{m}\ \underbrace{-\lambda_t \ldots -\lambda_t}_{q_t} \ldots \underbrace{-\lambda_1 \ldots \lambda_1}_{q_1})$$

$$\ell_V(\mu) \cong u(p_1, q_1) \oplus \ldots \oplus u(p_t, q_t) \oplus sp(m, \mathbb{R})$$

with

$$\lambda_1 > \lambda_2 > \ldots > \lambda_t > 0.$$

Proof. Observe that

a) If $\beta_{i_0} = e_{j_0} + e_{k_0} \in \Delta^+(g, h^c)$ is such that $-c_i = \langle \mu + 2\rho_c - \rho, \beta_i \rangle = -1$ then $e_{j_0} - e_{k_0} \in \beta_{i_0}^{\perp}$ and if

$$\langle \mu + 2\rho_c - \rho + \tfrac{1}{2} c_{i_0} \beta_{i_0}, e_{j_0} - e_{k_0} \rangle < 0,$$

then $e_{j_0} - e_{k_0}$ should be included in the set $\{\beta_i\}$ of Proposition 2.4.7.

b) Suppose that $a_1 + p - 1 \geq -b_1 + q - 1$. Then the root system generated by the roots involving the (p_1, q_1) coordinates:

$$\{e_1+e_n; -e_n-e_2; e_2+e_{n-1}; \ldots\}$$

is isomorphic to $A_{p_1+q_1-1}$. Since ℓ_V centralizes an elliptic element then $t^c \subseteq \ell_V$. Hence the real form of $A_{p_1+q_1-1}$ is $U(p_1, q_1)$.

Except for these extra considerations, the proof of this proposition is analogous to the one for the corresponding result for $SU(p,q)$.

5.3. Lowest K types of the modules $A_{\mathfrak{q}}(\lambda)$.

Let $x \in i(t_0^c)^*$. We may assume that it is of the form

$$x = (\underbrace{x_1 \ldots x_1}_{p_1}, \ldots, \underbrace{x_t \ldots x_t}_{p_t} \underbrace{0 \ldots 0}_{m}$$

$$\underbrace{-x_t \ldots -x_t}_{q_t} \ldots \underbrace{-x_1 \ldots -x_1}_{q_1})$$

$$x_1 > x_2 > \ldots > x_t > 0.$$

Write

$$\Delta(\mathfrak{l}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^c) \mid \langle\alpha, x\rangle = 0\},$$

$$\Delta(\mathfrak{u}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^c) \mid \langle\alpha, x\rangle > 0\},$$

as in 2.3. Then

a) $\mathfrak{l} \cong \mathfrak{u}(p_1, q_1) \oplus \mathfrak{u}(p_2, q_2) \oplus \ldots \oplus \mathfrak{u}(p_t, q_t) \oplus \mathfrak{sp}(m, \mathbb{R})$;

b) $2\rho(\mathfrak{u} \cap \mathfrak{p}) = (\underbrace{n-q_1+1 \ldots n-q_1+1}_{p_1}, \underbrace{n-2q_1-q_2+1 \ldots n-2q_1-q_2+1}_{p_2} \ldots$

(5.3.1)

$$\underbrace{n-2(q_1+\ldots+q_{t-1})-q_t+1, \ldots, n-2(q_1+\ldots+q_{t-1})-q_t+1}_{p_t}, \underbrace{p-q, \ldots, p-q}_{m}$$

,

$$\underbrace{-1-n+2(p_1+\ldots+p_{t-1})+p_t \ldots -1-n+2(p_1+\ldots+p_{t-1})+p_t}_{q_t} \ldots$$

$$\underbrace{-1-n+p_1 \ldots -1-n+p_1}_{q_1})$$

c) $2\rho(u \cap k) = (\underbrace{n-p_1 \ldots n-p_1}_{p_1}, \underbrace{n-2p_1-p_2 \ldots n-2p_1-p_2}_{p_2} \ldots$

$\underbrace{n-2(p_1+\ldots+p_{t-1})-p_t, \ldots, n-2(p_1+\ldots+p_{t-1})-p_t}_{p_t}, \underbrace{q-p, \ldots q-p}_{m}$

$\underbrace{-n+2(q_1+\ldots+q_{t-1})+q_t \ldots -n+2(q_1+\ldots+q_{t-1})+q_t}_{q_t} \ldots, \underbrace{-n+q_1 \ldots -n+q_1}_{q_1})$

Set $n_i = p_i + q_i$.

d) $2\rho(u) = (\underbrace{2n-n_1+1 \ldots 2n-n_1+1}_{p_1}, \underbrace{2n-2n_1-n_2+1 \ldots 2n-2n_1-n_2+1}_{p_2} \ldots$

$\underbrace{2n-2(n_1+n_2 \ldots +n_{t-1})-n_t+1, \ldots, 2n-2(n_1+n_2 \ldots +n_{t-1})-n_t+1}_{p_t},$

$\underbrace{0 \ldots 0}_{m}, \underbrace{-2n+2(n_1+\ldots+n_{t-1})+n_t-1 \ldots -2n+2(n_1+\ldots+n_{t-1})+n_t-1}_{q_t} \ldots$

$\underbrace{-2n+n_1-1 \ldots -2n+n_1-1}_{q_1})$

Now suppose that $\mu \in i(t_0^c)^*$ is the highest weight of a representation of K.

By the proof of Proposition 2.5.6 we may use μ to determine a compact parabolic subalgebra $\mathfrak{q} \cap \mathfrak{k} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{u} \cap \mathfrak{k}$.

Set $2\rho(\mathfrak{u}\cap\mathfrak{k}) = 2\rho(\Delta(\mathfrak{u}\cap\mathfrak{k}))$. Suppose that

$$\mu + 2\rho(\mathfrak{u}\cap\mathfrak{k}) =$$

$$(\underbrace{a_1 \ldots a_1}_{r_1} \ldots \underbrace{a_t \ldots a_t}_{r_t}\ \underbrace{0 \ldots 0}_{m}\ \underbrace{-a_t \ldots -a_t}_{s_t} \ldots \underbrace{-a_1 \ldots -a_1}_{s_1}).$$

Proposition 5.3.2. In the above setting, set $n_i = r_i + s_i$ then μ is the LKT of some $A_{\mathfrak{q}}(\lambda)$

$$\Longleftrightarrow \qquad a_i - a_{i+1} \geq n_i + n_{i+1}$$

and $$a_t \geq n_t + 2m + 1.$$

Proof. If $\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p})$, then

$$\lambda = (\lambda_1 \ldots \lambda_1 \ldots \lambda_t \ldots \lambda_t\ 0 \ldots 0\ -\lambda_t \ldots -\lambda_t \ldots -\lambda_1 \ldots -\lambda_1)$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t \geq 0$$

hence the coordinates of $\mu + 2\rho(\mathfrak{u}\cap\mathfrak{k}) = \lambda + 2\rho(\mathfrak{u})$ give

$$\lambda_i + 2n - 2(n_1+\ldots+n_{i-1})-n_i+1-(\lambda_{i+1}+2n-2(n_i+\ldots+n_i)-n_{i+1}+1)$$

$$= \lambda_i - \lambda_{i+1} + n_i + n_{i+1} \geq n_i + n_{i+1}$$

and

$$\lambda_t + 2n - 2(n_1+n_2+\ldots+n_{t-1}) - n_t + 1 \geq n_t + 2m + 1$$

Conversely, suppose we are given μ satisfying the conditions of the theorem. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the parabolic defined by $\mu + 2\rho(\mathfrak{u}\cap\mathfrak{k})$. Set

$$\lambda_i = a_i - 2n + 2(n_1+\ldots+n_{i-1}) + n_i - 1.$$

So
$$\langle\lambda,\Delta(\mathfrak{u})\rangle \geq 0,$$

$$\langle\lambda,\Delta(\mathfrak{l})\rangle = 0,$$

and
$$\mu = \lambda + 2\rho(\mathfrak{u}\cap\mathfrak{p}).$$

q.e.d.

5.4. Proof of Theorem 2.6.7 for $G = SP(n,\mathbb{R})$.

Let X as in Theorem 2.6.7, with infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$ $\mu \in (\mathfrak{t}^c)^*$, the highest weight of a LKT of X. Suppose X is not a module $A_{\mathfrak{q}}(\lambda)$. Let

$\mathfrak{l}_V = \mathfrak{l}_V(\mu) = (\mathfrak{u}(p_1,q_1) \oplus \mathfrak{u}(p_2,q_2) \oplus \ldots \oplus \mathfrak{u}(p_t,q_t)) \oplus \mathfrak{sp}(m,\mathbb{R})$ (cfr. 5.2.2) and $p = \Sigma p_i \quad q = \Sigma q_i$. Set

$$\mathfrak{l}_1 \cong \mathfrak{u}(p,q), \quad \mathfrak{l}_2 = \mathfrak{sp}(m,\mathbb{R})$$

then
$$\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \supseteq \mathfrak{l}_V$$

Define $\mathfrak{u} \subseteq \mathfrak{u}_V$ by $\mathfrak{u}_V = \mathfrak{u} + (\mathfrak{u}_V \cap \mathfrak{l})$.

Then $\mathfrak{q} \supset \mathfrak{q}_V$ and by Proposition 2.4.15, a) of Theorem 2.6.7 holds.

Now let X_L be an $(\mathfrak{l},L\cap K)$-module such that X occurs only as composition factor of $\mathcal{R}_{\mathfrak{q}}(X_L)$. We can see X_L as the exterior tensor product $X_L = X_{L_1} \otimes X_{L_2}$ with X_{L_i} an $(\mathfrak{l}_i,L_i\cap K)$-module.

That X_L^h has a Hermitian form $\langle\ ,\ \rangle^L$ follows from Lemma 2.7.4.

Lemma 5.4.1. $X_{L_1} \cong A_{\mathfrak{q}^0}(\lambda^0)$, for some $\mathfrak{q}^0 \subseteq L_1$; $\lambda^0 : \mathfrak{l}_0 \longrightarrow \mathbb{C}$.

Proof. By Theorem 2.6.7, b) and c) (proved for $SU(p,q)$) and Theorem 2.6.8, if $X_{L_1} \not\cong A_{\mathfrak{q}}(\lambda)$ then there are $\delta_j \in (L_1\cap K)^\wedge$, $j = 1,2$, K-types of X_{L_1} such that

$\langle\ ,\ \rangle^L\big|_{V_{\delta_1}\oplus V_{\delta_2}}$ is indefinite. Moreover, we know that if $\mu^i = \mu^L\big|_{L_i}$ and $\mu^L = \mu - 2\rho(u\cap p)$. Then there is $\beta \in \Delta(\ell_1\cap p)$ such that $\langle\ ,\ \rangle^L$ is indefinite on the sum

$$V_{\mu^1} \oplus V_{\mu^1+\beta}.$$

If $\mu = (x_1,\ldots,x_p,x_{p+1},\ldots,x_{p+m},x_{p+m+1},\ldots x_m)$ and

$$2\rho(u\cap p) = (\underbrace{n-q+1,\ldots,n-q+1}_{p},\underbrace{p-q,\ldots,p-q}_{m},\underbrace{-1-n+p\ldots-1-n+p}_{q})$$

then, since

$$\Delta(\ell_1\cap p) = \pm\{(e_i+e_j) \mid 1 \le i \le p,\ p+m \le j \le n\}$$

it is clear that if $\mu^1 + \beta$ is dominant for $\Delta(\ell_1\cap k)$, then $\mu + \beta$ is dominant for $\Delta^+(k)$, unless $x_p = x_{p+1}$ or $x_{p+m} = x_{p+m+1}$.

Suppose then that $x_p = x_{p+1}$.

Note that $2\rho(u_V\cap p)\big|_{L_2} = 2\rho(u\cap p)\big|_{L_2}$ then $\mu^{L_V}\big|_{L_2} = \mu^2$ and hence μ^2 is fine and X_{L_2} is a principal series.

So $\mu^2 \in \{(0\ldots0);(1\ldots1,0\ldots0);(0,\ldots0,-1,-1\ldots-1)\}$. Suppose that μ^2 is of the form

$$\mu^2 = (\underbrace{1\ldots1}_{a},\underbrace{0\ldots0}_{m-a}) \quad \text{with} \quad a > 0.$$

Then for $\eta^2 = (\underbrace{0\ldots0}_{m-a},\underbrace{-1\ldots-1}_{a})$, V_{η^2} is also a LKT of X_{L_2}, by Frobenius reciprocity.

So if $\eta = \mu^1 + \eta^2 + 2\rho(u\cap p)$, V_η is a LKT of X. Since η is also dominant, it follows that $x_{p+m} - 1 \geq x_{p+m+1}$.

Suppose that $\mu^1 + \beta$ is dominant for some $\beta \in \Delta(\ell_1 \cap p) \cap U(p_t,q_t)$. If

$$\beta = (\underbrace{0\ldots0}_{p-p_t}\ \underbrace{1\ 0\ldots0}_{p_t}\ \underbrace{0\ldots0}_{m}\ \underbrace{1\ 0\ldots0}_{q}),$$

since $x_{p+m} - 1 \geq x_{p+m+1}$, $\mu + \beta$ is dominant. If

$$\beta = (\underbrace{0\ldots0\ 0\ldots0\ 0\ -1}_{p}\ \underbrace{0\ldots0}_{m}\ \underbrace{0\ldots0\ -1}_{q_t}\ \underbrace{0\ldots0}_{q-q_t}),$$

then, since $\mu^L|_{L_1} = \eta^L|_{L_1}$ then $\eta + \beta$ is a dominant candidate.

If $\mu^2 = (\underbrace{0\ldots0}_{m-a},\underbrace{-1\ldots-1}_{a})$, with $a > 0$, a similar argument shows that $x_p - 1 \geq x_{p+1}$.

If $\mu^2 = (0\ldots0)$ then both differences $x_p - x_{p+1}$ and $x_{p+m} - x_{p+m+1}$ must be strictly positive. This is clear from the pictures of $\mu + 2\rho_c$.

q.e.d.

Lemma 5.4.2. In the above setting, assume that $X_{L_1} \cong A_{\mathfrak{q}^0}(\lambda^0)$ for some $\mathfrak{q}^0 \subseteq \mathfrak{l}_1$ and $\lambda^0 : \mathfrak{l}^0 \longrightarrow \mathbb{C}_{\lambda^0}$.

Then, Theorem 2.6.7 is true if we assume that

$$(5.4.3) \qquad \begin{cases} x_p - x_{p+1} \geq 2 \\ \text{and } x_{p+m} - x_{p+m+1} \geq 2. \end{cases}$$

Proof. Suppose first that $\mu^2 = (1,1,\ldots,1,0\ldots0)$. Then if $\rho_n^+ = \left[\frac{m+1}{2},\ldots,\frac{m+1}{2}\right]$, an easy calculation shows

$$\langle \mu^2 - \rho_n^+ + \rho_{\mathfrak{l}_2 \cap k}, \mu^2 - \rho_n^+ + \rho_{\mathfrak{l}_2 \cap k} \rangle < \langle \rho, \rho \rangle.$$

By 2) in Lemma 2.7.2, there is a

$$\beta \in \{(\underbrace{0\ldots0\ -1}_{a}\ \underbrace{0\ldots0\ -1}_{m-a}), (\underbrace{0,0\ldots0\ -2}_{m})\}$$

making $V_{\mu^2} \oplus V_{\mu^2+\beta}$ into a space on which $\langle\ ,\ \rangle^L$ is indefinite.

Moreover $\mu + \beta$ is $\Delta^+(k)$-dominant, by (5.4.3). Similarly if $\mu^2 = (\underbrace{0\ldots0}_{m-a}\ \underbrace{-1,-1,\ldots,-1}_{a})$ then

$$\beta \in \{(\underbrace{1,0\ldots0}_{m-a}\ \underbrace{1\ 0\ldots0}_{a});(2,0\ldots)\}.$$

Now, if $\mu^2 = (0\ldots0)$ then the Dirac operator inequality fails for any choice of $\rho_n = \rho(\Delta^+(\ell_2 \cap \mathfrak{p}))$, unless $\gamma|_{\ell_2} = \rho_{\ell_2}$ in particular, if $\rho_n^+ = \left[\frac{m+1}{2},\ldots,\frac{m+1}{2}\right]$; and, obviously, $\mu + \beta$ is also dominant for $\beta \in \Delta(\mathfrak{p}^+ \cap \ell_2)$.

Now if $\gamma|_{\ell_2} = \rho_{\ell_2}$, then, the Langlands subquotient of X_{L_2} is the trivial representation. (In fact, the representation $X_{L_2} = I(\delta_V^{L_2} \otimes \nu_V^{L_2})$ is a principal series and $\delta_V^{L_2}$ = trivial; $\gamma|_{\ell_2} = \nu_V|_{\ell_2} = \nu_V^{L_2}$.)

Hence the Langlands submodule of

$$\mathcal{R}_{\mathfrak{q}}(X_{L_1} \otimes X_{L_2}) = \mathcal{R}_{\mathfrak{q}}(X_{L_1}) \otimes \mathcal{R}_{\mathfrak{q}}(X_{L_2})$$

is

$$X \cong \mathcal{R}_{\mathfrak{q}}(A_{\mathfrak{q}^0}(\lambda^0)) \otimes \mathcal{R}_{\mathfrak{q}}(\text{trivial representation}).$$

By induction by stages, X is an $A_{\mathfrak{q}}(\lambda)$, contradicting our assumptions on X.

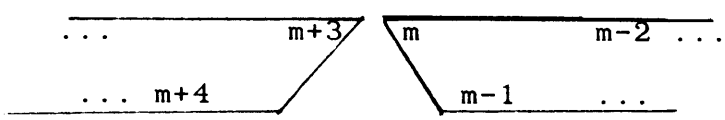
This proves the lemma..

q.e.d.

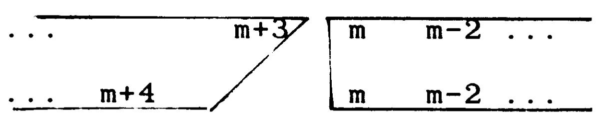
To finish the proof of Theorem 2.6.7, suppose now that $x_p - x_{p+1} \leq 1$.

Lemma 5.4.4. Under the hypothesis of Lemma 5.4.2, if $x_p - x_{p+1} = 1$ and $x_{p+m} - x_{p+m+1} \geq 2$, then Theorem 2.6.7 is true.

Proof. The assumptions on the coordinates of μ imply that the picture of $\mu + 2\rho_c$ around the coordinates involved is either



or



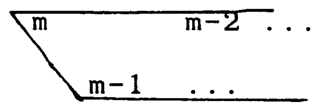
that is,

$$\mu + 2\rho_c = (\ldots m+5 \quad m+3 \mid m \quad m-2 \ \ldots \ -m+1 \mid -m-4 \ \ldots)$$

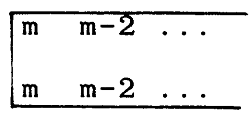
or

$$\mu + 2\rho_c = (\ldots m+5 \quad m+3 \mid m \ \ldots \ -m \mid -m-4 \ \ldots)$$

Observe that $\mu^{L_V} = \mu - 2\rho(\mathfrak{u}_V \cap \mathfrak{p})$ is fine and that the fine K-type that gives the picture



is $\mu^2 = (1,1,\ldots,1,0\ldots0)$ and the fine K-type that gives



is $\mu^2 = (0\ldots0)$.

Arguing as in the proof of Lemma 5.4.2. we can find, in both cases

$$\beta \in \{(0\ldots0\ -1,0\ldots0\ -1);(0\ldots0\ -2)\}$$

as we want.

q.e.d.

Assume now that

$$(5.4.5) \qquad \begin{cases} 0 \leq x_p - x_{p+1} \leq 1 \\ \\ 0 \leq x_{p+m} - x_{p+m+1} \leq 1 \end{cases} .$$

We want to contradict the assumption that the infinitesimal character γ is regular and integral.

Since we have an $A_{\mathfrak{q}}(\lambda)$-module for $L_1 = U(p,q)$, we have some control on γ.

Recall that $L = U(p,q) \times SP(m,\mathbb{R})$ and $L \supseteq L_V = \left[\prod_{i=1}^{t} (U(p_i,q_i))\right] \times SP(m,\mathbb{R})$. We may assume $p_t \geq q_t$. By the computation in 4.3, either

$$\lambda_V \Big|_{U(p_t,q_t)}$$

$$= (\underbrace{\lambda_t+s, \lambda_t+s-1 \ldots \lambda_t+1}_{s}\ \underbrace{\lambda_t \ldots \lambda_t}_{q_t+1}\ \underbrace{\lambda_t-1 \ldots \lambda_t-s}_{s} \,|\, \underbrace{\lambda_t \ldots \lambda_t}_{q_t})$$

or

$$\lambda_V \Big|_{U(p_t,q_t)}$$

$$= (\underbrace{\lambda_t+s, \ldots \lambda_t+\tfrac{1}{2}}\ \underbrace{\lambda_t \ldots \lambda_t}_{q_t}\ \underbrace{\lambda_t-\tfrac{1}{2} \ldots \lambda_t-s} \,|\, \underbrace{\lambda_t \ldots \lambda_t}_{q_t})$$

and

$$\nu \Big|_{U(p_t,q_t)} = (0 \ldots 0\ \nu_1 \ldots \nu_t\ 0 \ldots 0 \mid -\nu_1 \ldots -\nu_t).$$

Inside $SP(n,\mathbb{R})$ this gives

$$(\lambda_t+s, \ldots, \lambda_t \ldots \lambda_t \ldots \lambda_t - s \mid -\lambda_t \ldots -\lambda_t)$$

$$(0 \ldots 0 \ \nu_1 \ldots \nu_{q_t} \ 0 \ldots 0 \mid \nu_1 \ldots \nu_{q_t}).$$

If γ is regular integral

$$\lambda_t + \nu_t > \lambda_t + s,$$

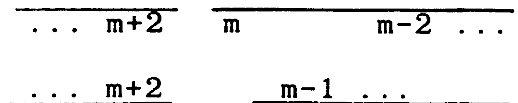
$$\lambda_t - s > 0 > -\lambda_t + \nu_1 \geq -\lambda_t + q_t + \nu_t - 1$$

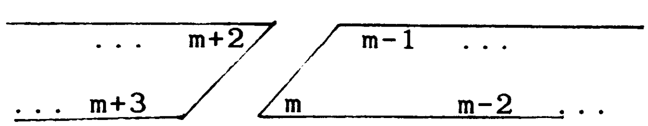
$$\Rightarrow \begin{cases} \nu_{q_t} > s \\ \lambda_t > \nu_{q_t} + q_t^{-1} \end{cases}$$

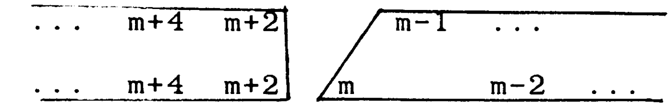
$$\Rightarrow \quad \lambda_t \geq s + q_t$$

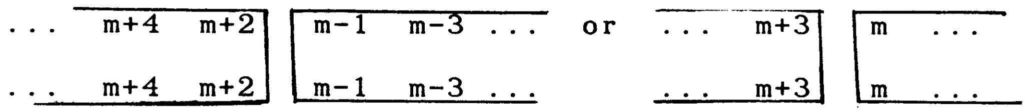
Claim. If μ satisfies (5.4.5) then $\lambda_t - s \leq 1$.

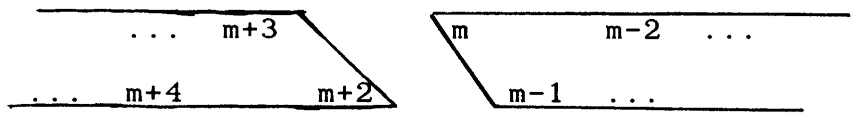
Proof. The picture for $\mu + 2\rho_c$ around these coordinates can be of the following types.

1. 

2. 

3. 

4. 

5. 

So we either have (considering that 5 and 2, and 3 and 1 are symmetric)

$$\mu + 2\rho_c = (\ldots m+k+2, m+k \mid \ldots \mid -m-k\ldots)$$

and
$$\rho = (\ldots m+k+2, m+k \mid \ldots \mid -m-k+1\ldots)$$

or

$$\mu + 2\rho_c = (\quad \ldots\ m+2 \mid \ldots \mid -m-3\ldots),$$

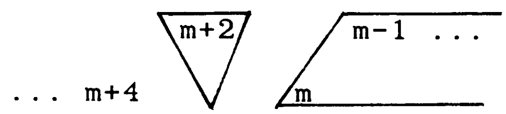
with
$$\rho = (\quad \ldots\ m+1 \mid \ldots \mid -m-2\ldots)$$

In both cases we get

$$\lambda_V = (\ldots 1\ 1 \mid 0\ldots 0 \mid -1\ -1\ldots).$$

This proves the claim.

This reduces to the case when $q_t = 0$. But then, $\mu + 2\rho_c$ gives, at worst,



Because if $q_i = 0$, $p_i = 1$, since $U(p_i, q_i)$ is quasisplit. So, we have $x_{p+m} - x_{p+m+1} \geq 2$!

This concludes the proof of Theorem 2.6.7.

q.e.d.

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