8.044 Statistical Physics I
Spring 2008

Sums of Random Variables

Consider n RVs x_i and let $s \equiv \sum_{i=1}^{n} x_i$.

If the RVs are statistically independent, then

$$< s > \; = \; \sum_i < x_i >$$

$$\text{Var}(s) \; = \; \sum_i \text{Var}(x_i)$$

- The individual $p(x_i)$ could be quite different

- Both continuous and discrete RVs could be present

- True for any n

- Even if one RV dominates the sum

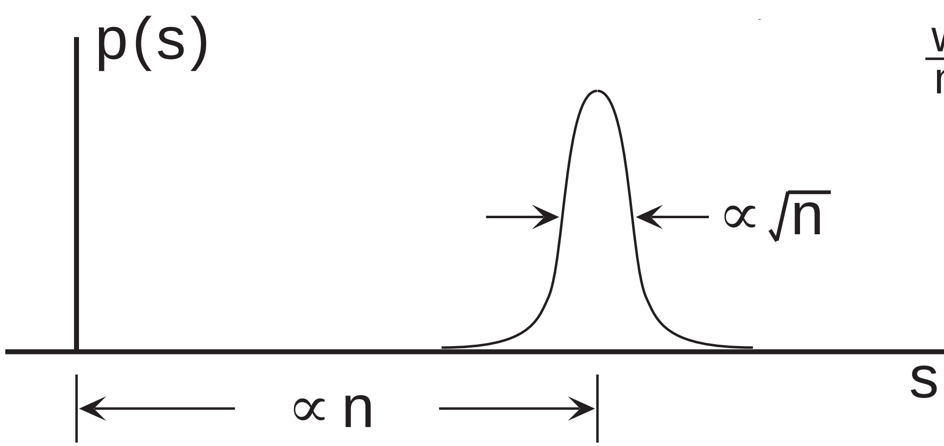
Results have a special meaning when

1) The means are finite ($\neq 0$)

2) The variances are finite ($\neq \infty$)

3) No subset dominates the sum

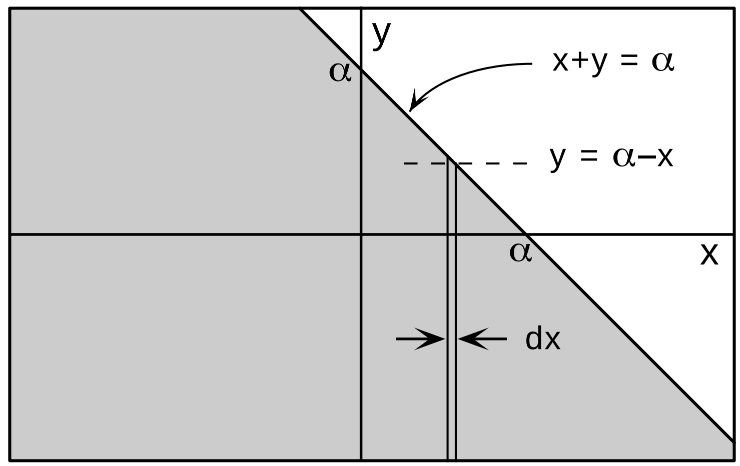
4) n is large



$$\frac{\text{width}}{\text{mean}} \propto \frac{1}{\sqrt{n}}$$

Given $p(x,y)$, find $p(s \equiv x+y)$

A



B

$$P_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\alpha-\zeta} d\eta \; p_{x,y}(\zeta,\eta)$$

C $$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \; p_{x,y}(\zeta, \alpha - \zeta)$$

This is a general result; x and y need not be S.I.

Application to the Jointly Gaussian RVs in Section 2 shows that $p(s)$ is a Gaussian with zero mean and a Variance $= 2\sigma^2(1 + \rho)$.

In the special case that x and y are S.I.

$$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta\, p_x(\zeta)\, p_y(\alpha - \zeta) = \int_{-\infty}^{\infty} d\zeta'\, p_x(\alpha - \zeta')\, p_y(\zeta')$$

The mathematical operation is called "convolution".

$$p \otimes q \equiv \int_{-\infty}^{\infty} p(z) q(x - z) dz = f(x).$$

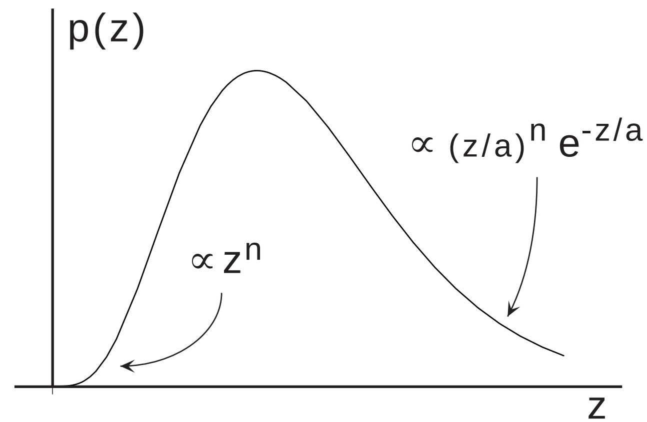
Example

Given:

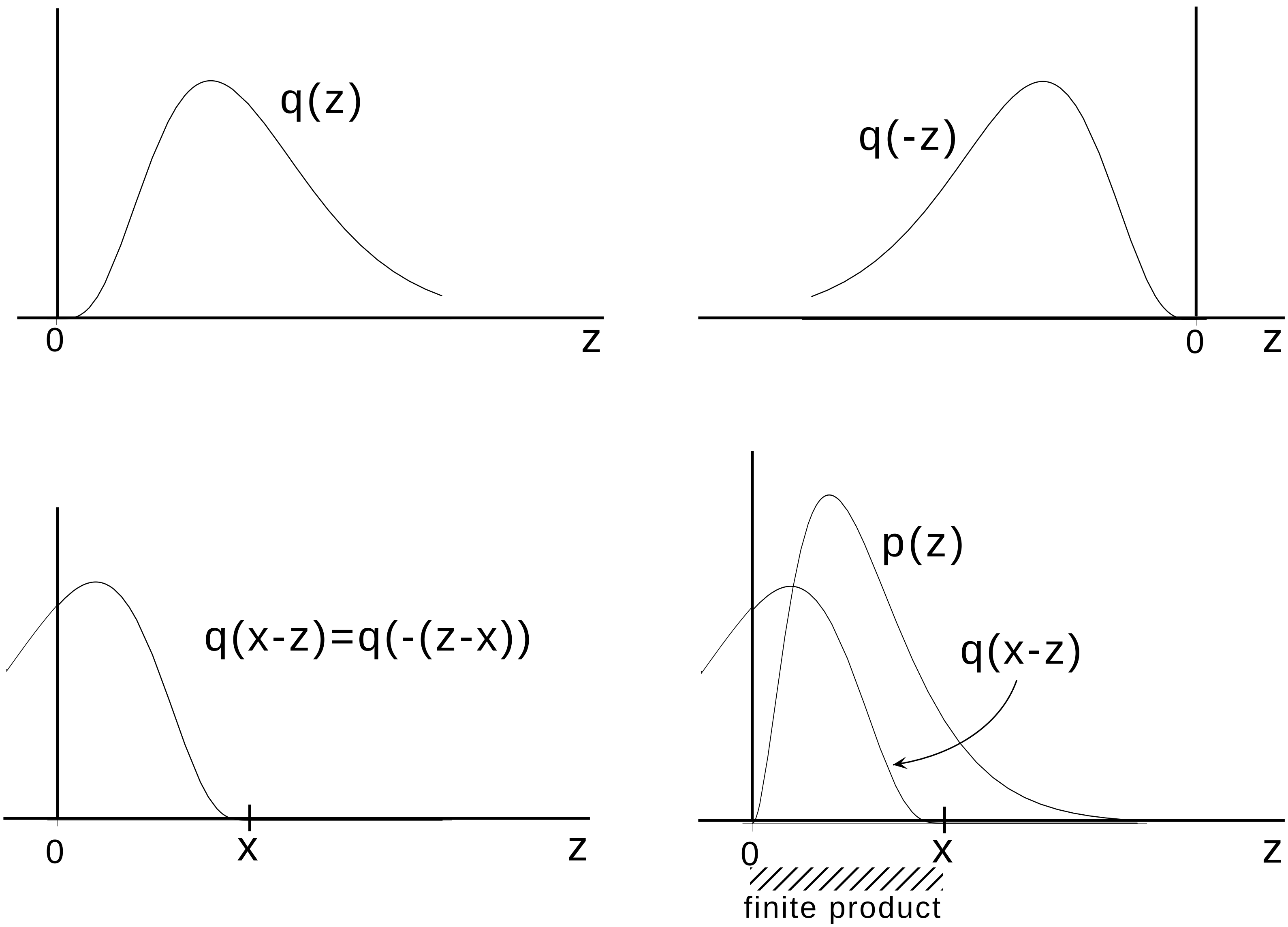
$$p(z) = \frac{1}{n!a}(z/a)^n \exp(-z/a)$$

$$q(z) = \frac{1}{m!a}(z/a)^m \exp(-z/a)$$

$$0 < z \text{ and } n, m = 0, 1, 2, \cdots$$



Find: $p \otimes q$

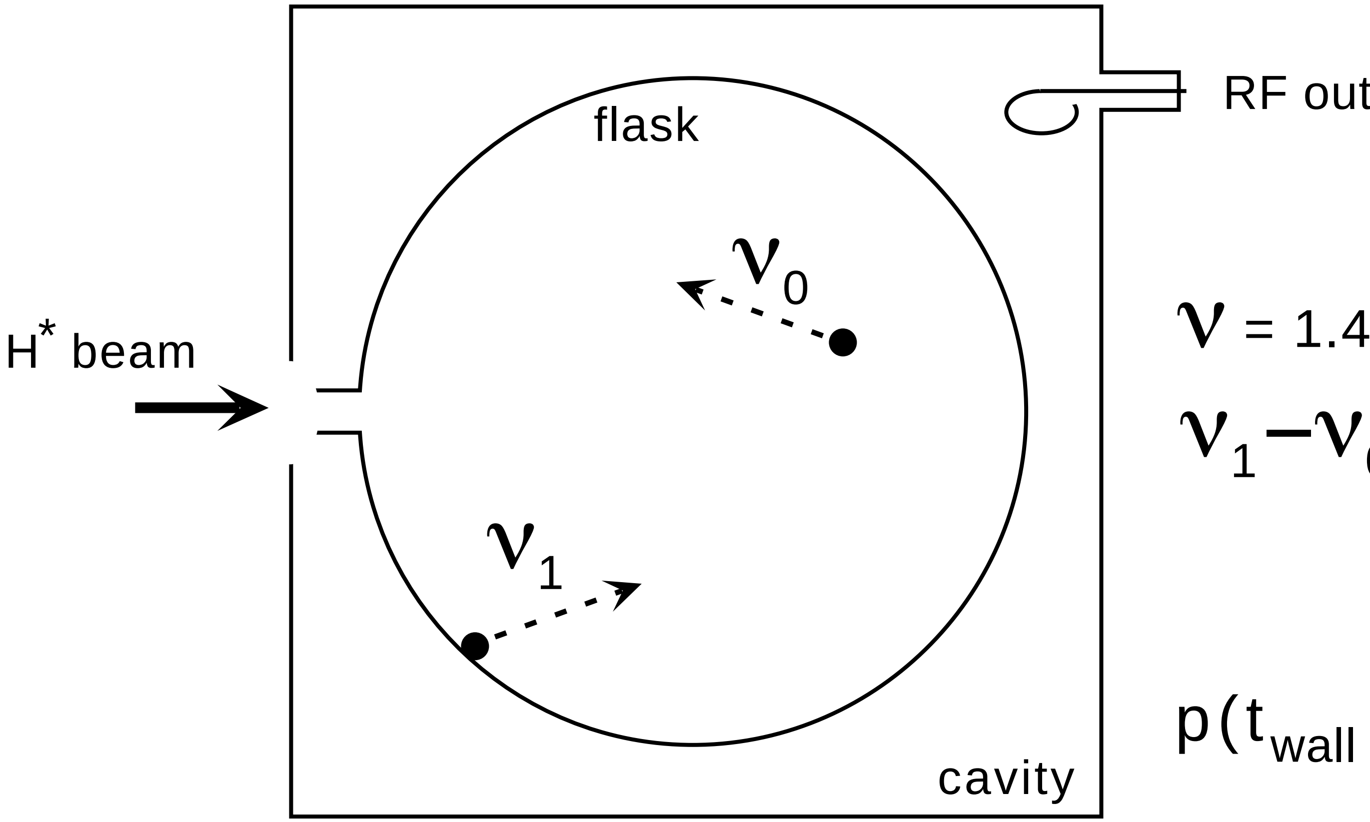

q(z)
0
z
q(-z)
0
z
q(x-z)=q(-(z-x))
0
x
z
p(z)
q(x-z)
0
x
z
finite product

$$
\begin{aligned}
p \otimes q &= \frac{1}{n!m!} \frac{1}{a^2} \int_0^x \left(\frac{z}{a}\right)^n \left(\frac{x-z}{a}\right)^m e^{-z/a} e^{-(x-z)/a} \, dz \\
&= \frac{1}{n!m!} \frac{1}{a} \left(\frac{1}{a}\right)^{n+m+1} e^{-x/a} \int_0^x z^n (x-z)^m \, dz \\
&= \frac{1}{n!m!} \frac{1}{a} \left(\frac{x}{a}\right)^{n+m+1} e^{-x/a} \underbrace{\int_0^1 \zeta^n (1-\zeta)^m \, d\zeta}_{\frac{n!m!}{(n+m+1)!}}
\end{aligned}
$$

$$p \otimes q = \frac{1}{(n+m+1)!}\,\frac{1}{a}\,\left(\frac{x}{a}\right)^{n+m+1}\,e^{-x/a}$$

a function of the same class

Example Atomic Hydrogen Maser



ν = 1.4....... GHz

$\nu_1 - \nu_0$ about 10 KHz

$p(t_{wall} \,|\, n \text{ stays}) = ?$

$$t_{\text{wall}} = \sum_{i=1}^{n} t^n, \qquad \text{Each stay is S.I.}$$

$$p(t\,|1) = (1/\tau)\, e^{-t/\tau}$$

$$p(t\,|2) = p(t\,|1) \otimes p(t\,|1) = (1/\tau)(t/\tau)\, e^{-t/\tau}$$

$$p(t\,|3) = p(t\,|2) \otimes p(t\,|1) = (1/2)(1/\tau)(t/\tau)^2\, e^{-t/\tau}$$

$$p(t\,|n) = \frac{1}{(n-1)!}\,\frac{1}{\tau}\,\left(\frac{t}{\tau}\right)^{n-1}\,e^{-t/\tau}$$