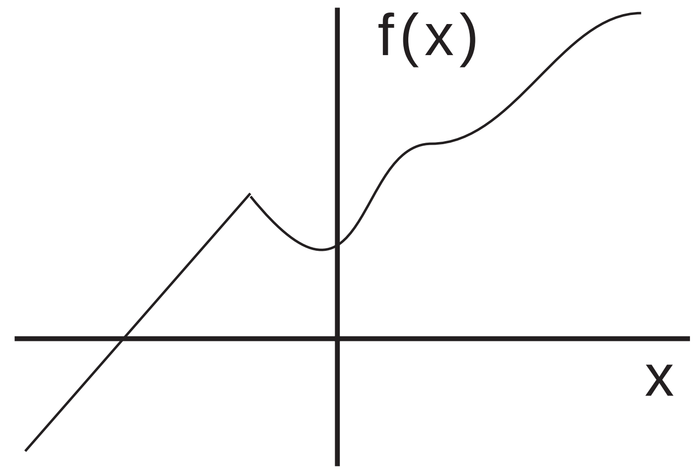
8.044 Statistical Physics I
Spring 2008

Functions of a random variable

Given: $p_x(\zeta)$ and $f(x)$

Find: $p_f(\eta)$



A. Sketch $f(x)$. Find where $f(x) < \eta$

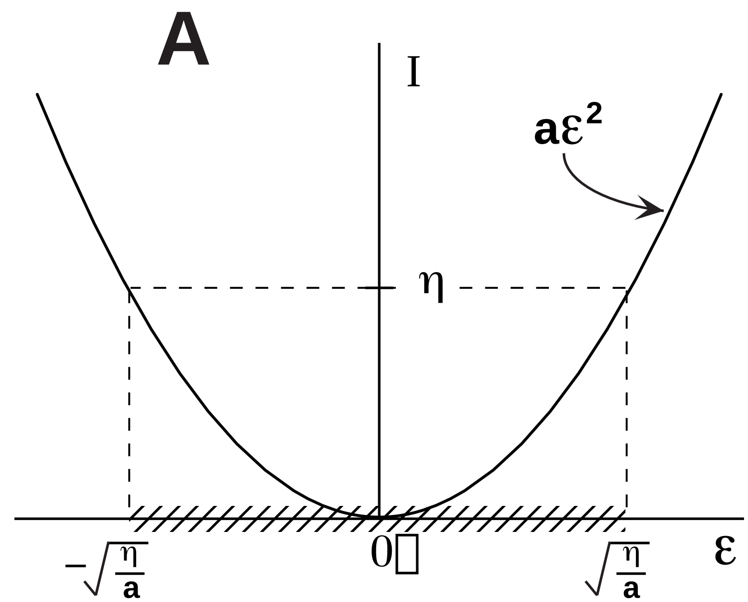
B. Integrate to find $P_f(\eta)$.

C. Differentiate to find $p_f(\eta)$.

Example Intensity of light

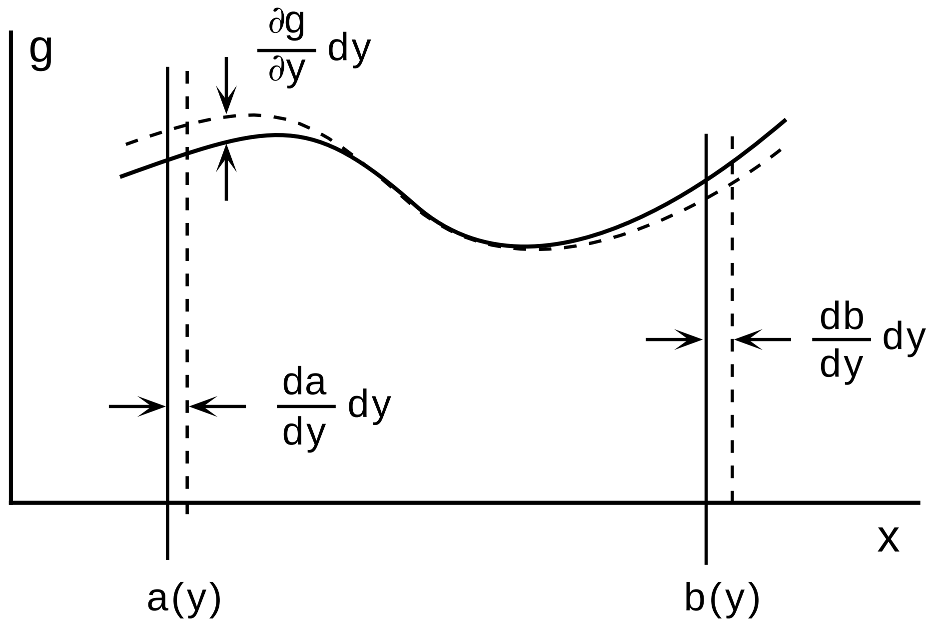
$$I = a\mathcal{E}^2$$

$$p(\mathcal{E}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\mathcal{E}^2/2\sigma^2]$$



B

$$P_I(\eta) = \int_{-\sqrt{\eta/a}}^{\sqrt{\eta/a}} p_{\mathcal{E}}(\zeta)\, d\zeta$$



$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(y,x)\,dx =$$

$$g(y,x=b(y))\frac{db(y)}{dy} - g(y,x=a(y))\frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y,x)}{\partial y}\,dx$$

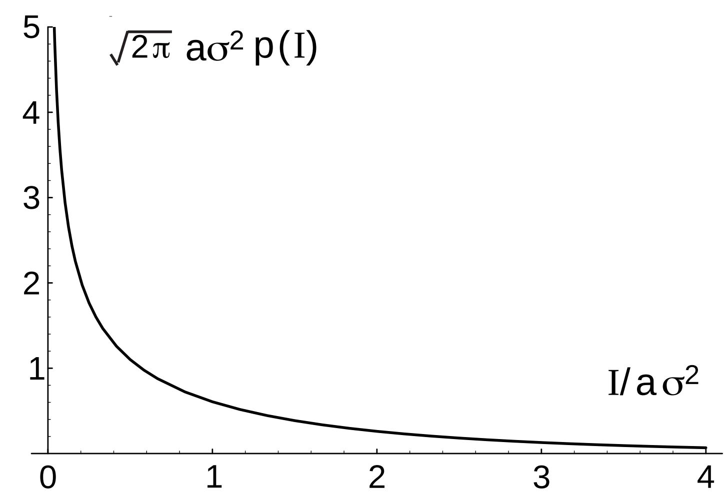
C In general

$$p_I(\eta) = \frac{1}{2}\frac{1}{\sqrt{\eta a}}\, p_{\mathcal{E}}(\sqrt{\eta/a}) - (-\frac{1}{2}\frac{1}{\sqrt{\eta a}})\, p_{\mathcal{E}}(-\sqrt{\eta/a})$$

$$= \frac{1}{2}\frac{1}{\sqrt{\eta a}}\, [p_{\mathcal{E}}(\sqrt{\eta/a}) + p_{\mathcal{E}}(-\sqrt{\eta/a})]$$

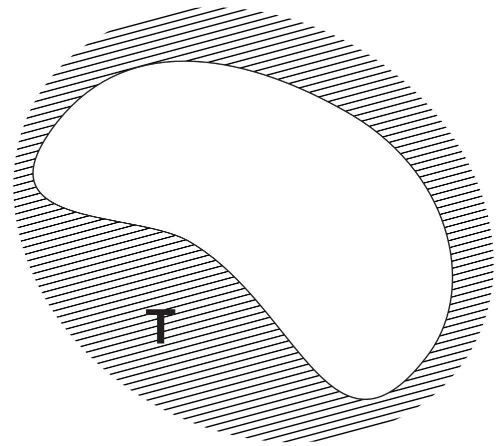
In our particular case

$$p(I) = \frac{1}{\sqrt{2\pi a\sigma^2 I}}\exp[-\frac{I}{2a\sigma^2}]$$

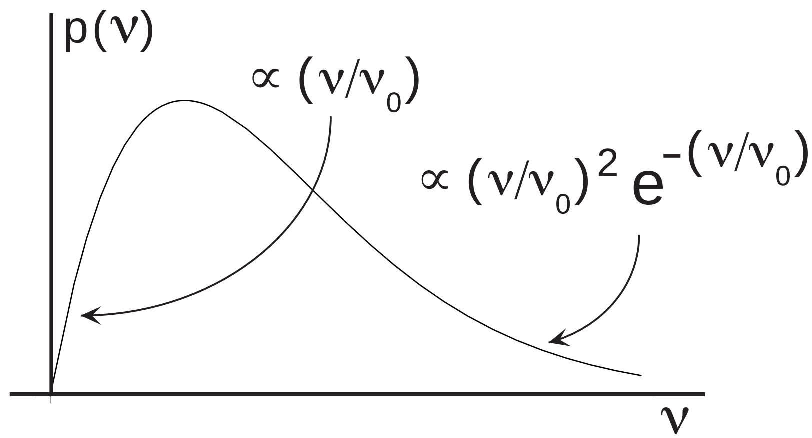


<u>Example</u> Black Body Radiation

$$p(\nu) = \underbrace{\frac{1}{2\zeta(3)}}_{1/2.404} \frac{1}{\nu_0} \frac{(\nu/\nu_0)^2}{\exp[\nu/\nu_0] - 1}$$



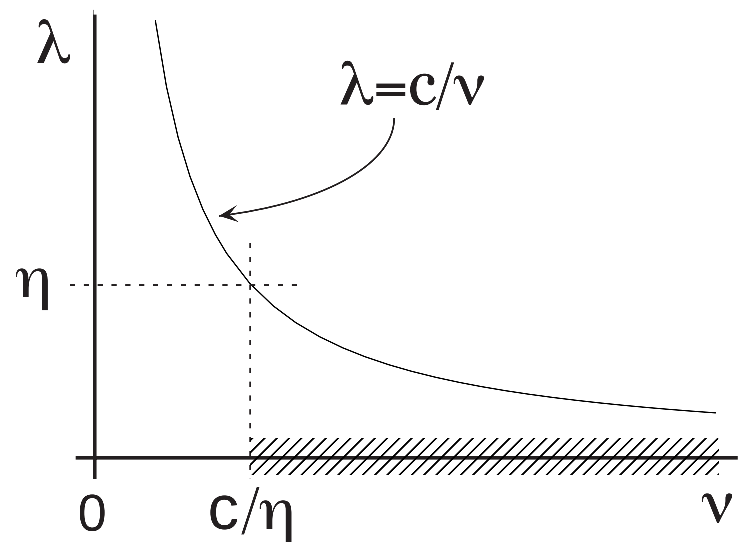
$$\nu_0 = kT/h$$



Given $\lambda = c/\nu$ and $p(\nu)$

Find $p(\lambda)$

A



B

$$P_\lambda(\eta) = \int_{c/\eta}^{\infty} p_\nu(\zeta)\, d\zeta$$

C

In general

$$p_\lambda(\eta) = -(-c/\eta^2)\; p_\nu(c/\eta)$$

In our case

$$p_\lambda(\eta) = \frac{c}{\eta^2}\;\frac{1}{2.404}\;\frac{1}{\nu_0}\;\frac{(c/\eta\nu_0)^2}{\exp[(c/\eta\nu_0)]-1}$$

Let $\lambda_0 \equiv c/\nu_0$, then

$$p_\lambda(\eta) = \frac{1}{2.404}\;\frac{1}{\lambda_0}\left(\frac{\eta}{\lambda_0}\right)^{-4}\frac{1}{\exp[(\eta/\lambda_0)^{-1}]-1}$$

As $(\lambda/\lambda_0)\to 0$ $$\frac{1}{\exp[(\lambda/\lambda_0)^{-1}]-1} \longrightarrow e^{-(\lambda/\lambda_0)^{-1}}$$

As $(\lambda/\lambda_0)\to \infty$ $$\frac{1}{\exp[(\lambda/\lambda_0)^{-1}]-1} \longrightarrow \frac{1}{(1+(\lambda/\lambda_0)^{-1}-1)} \longrightarrow (\lambda/\lambda_0)$$