8.044 Statistical Physics I
Spring 2008

Simple Quantum Paramagnet, Canonical Ensemble

Origin of magnetic moments:

Electron spin and orbital angular momentum

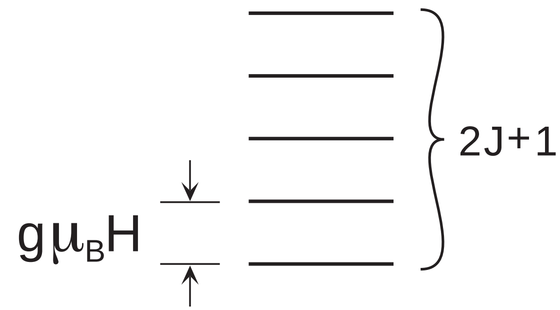
$$\vec{S} + \vec{L} \equiv \vec{J} \qquad \vec{\mu} = g_J \mu_B \vec{J} \qquad \mu_B \equiv e\hbar/2m_e c$$

Nuclear angular momentum

$$\vec{I} \qquad \vec{\mu} = g_I \mu_N \vec{I} \qquad \mu_N \equiv e\hbar/2m_p c$$

$$\epsilon_m = -g\mu_B H\, m$$

$$m = J, J-1, \cdots - J$$



$$Z_1(T,H) = \sum_{m=-J}^{J} e^{-\epsilon_m/kT} = \sum_{m=-J}^{J} (e^{\eta})^m = \frac{\sinh[(J+\frac{1}{2})\eta]}{\sinh[\frac{1}{2}\eta]}$$

$$\eta \equiv \frac{g\mu_B H}{kT} = \frac{\text{level spacing}}{kT}$$

Note $Z_1 = Z_1(\eta)$ $Z = Z_1(\eta)^N = Z(\eta)$ at fixed N

$$p(m) = e^{-\epsilon_m/kT}/Z_1 = e^{\eta\, m}/Z_1$$

$$< \mu > = \sum_m \frac{(g\mu_B m)e^{\eta\, m}}{Z_1} = g\mu_B \left(\frac{1}{Z_1}\frac{\partial Z_1}{\partial \eta}\right)$$

$$\equiv g\mu_B J B_J(\eta) \qquad M = N < \mu >= g\mu_B N J B_J(\eta)$$

$$B_J(\eta) = \frac{1}{J}\left(\frac{1}{Z_1}\frac{\partial Z_1}{\partial \eta}\right)$$

$$= \underline{\frac{1}{J}\left\{(J+\tfrac{1}{2})\coth[(J+\tfrac{1}{2})\,\eta] - \tfrac{1}{2}\coth[\tfrac{1}{2}\,\eta]\right\}}$$

This is called the "Brillouin Function".

$$\coth x \to \frac{1}{x} + \frac{x}{3} \qquad x \ll 1 \qquad \lim_{\eta \to 0} B_J(\eta) = \frac{J+1}{3}\eta$$

$$\coth x \to 1 + 2e^{-2x} \quad x \gg 1 \qquad \lim_{\eta \to \infty} B_J(\eta) = 1 - \frac{e^{-\eta}}{J}$$

$$M \;\to\; N\,\frac{(g\mu_B)^2 J(J+1)}{3}\,\frac{H}{kT}$$ High T (Curie Law)

$$\to\; N\,g\mu_B J \left(1 - \frac{1}{J}\,e^{-\eta}\right)$$ Low T (Energy Gap)

$$\chi_T \equiv \left(\frac{\partial M}{\partial H}\right)_T = Ng\mu_B J\, B'_J(\eta) \left(\frac{\partial \eta}{\partial H}\right)_T = N\,\frac{(g\mu_B)^2 J}{kT}\, B'_J(\eta)$$

Note: T and H enter only through $\eta \equiv \dfrac{g\mu_B H}{kT}$

$$\left(\frac{\partial \eta}{\partial H}\right)_T = \frac{\eta}{H} \qquad\qquad \left(\frac{\partial \eta}{\partial T}\right)_H = -\frac{\eta}{T}$$

We now show that this $\Rightarrow U = 0$.

$$
\begin{aligned}
dU &= TdS + HdM \\
&= T\left(\left(\frac{\partial S}{\partial T}\right)_H dT + \left(\frac{\partial S}{\partial H}\right)_T dH\right) + H\left(\left(\frac{\partial M}{\partial T}\right)_H dT + \left(\frac{\partial M}{\partial H}\right)_T dH\right) \\
&= \underbrace{\left(T\left(\frac{\partial S}{\partial T}\right)_H + H\left(\frac{\partial M}{\partial T}\right)_H\right)}_{0} dT + \underbrace{\left(T\left(\frac{\partial S}{\partial H}\right)_T + H\left(\frac{\partial M}{\partial H}\right)_T\right)}_{0} dH \\
&= 0 \quad \text{for all paths} \Rightarrow U = 0
\end{aligned}
$$

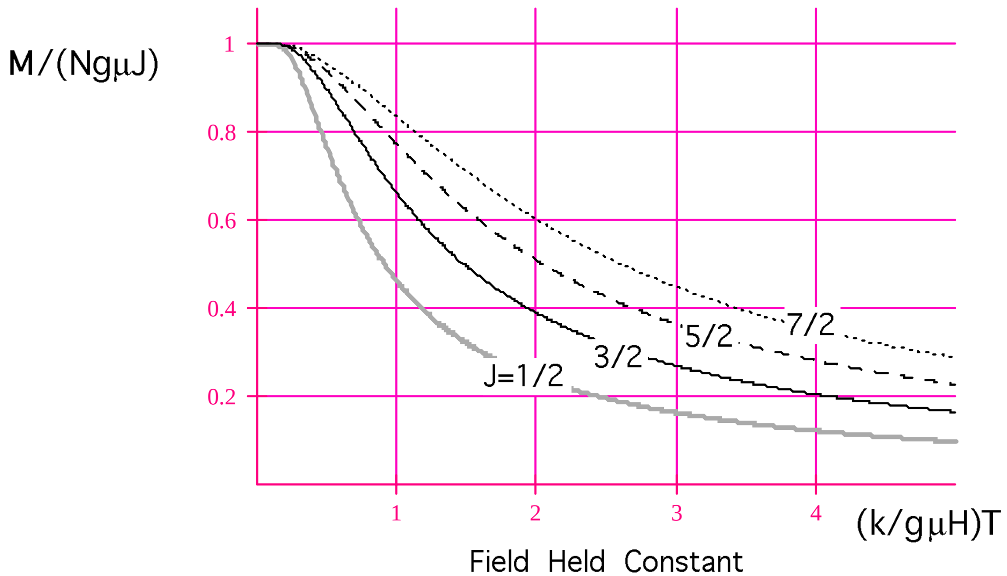
$$
\begin{aligned}
T\left(\frac{\partial S}{\partial T}\right)_H + H\left(\frac{\partial M}{\partial T}\right)_H &= T \underbrace{\left(\frac{\partial S}{\partial T}\right)_H}_{S'(\eta)(-\eta/T)} + H \underbrace{\left(\frac{\partial S}{\partial H}\right)_T}_{S'(\eta)(\eta/H)} \\
&= -\eta S'(\eta) + \eta S'(\eta) \\
&= 0
\end{aligned}
$$

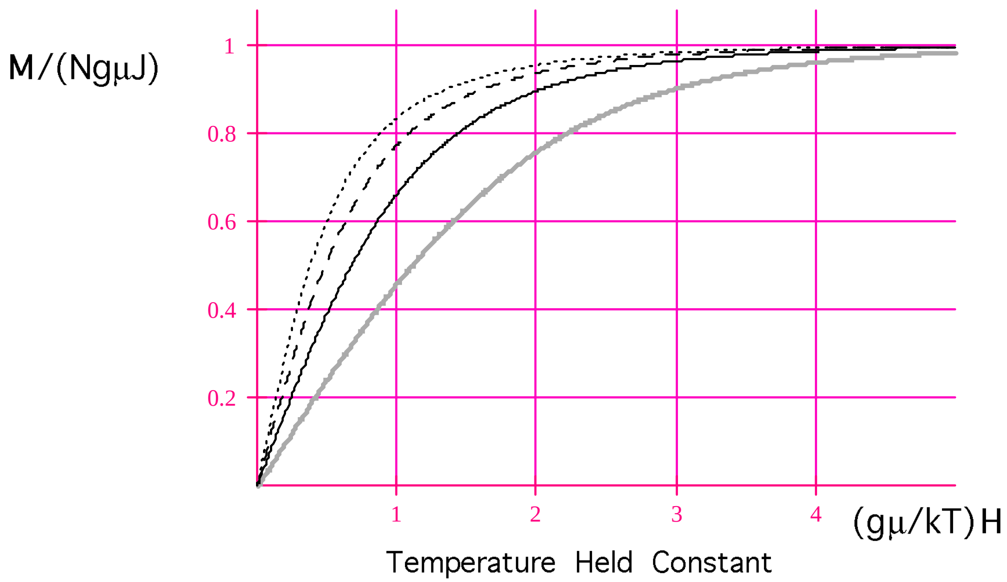
A similar expansion shows that the other term is also zero.

$$C_H \equiv \left(\frac{đQ}{dT}\right)_H = \frac{1}{dT}\left(\underbrace{dU}_{0} - H\,dM\right)_H = -H\left(\frac{\partial M}{\partial T}\right)_H$$

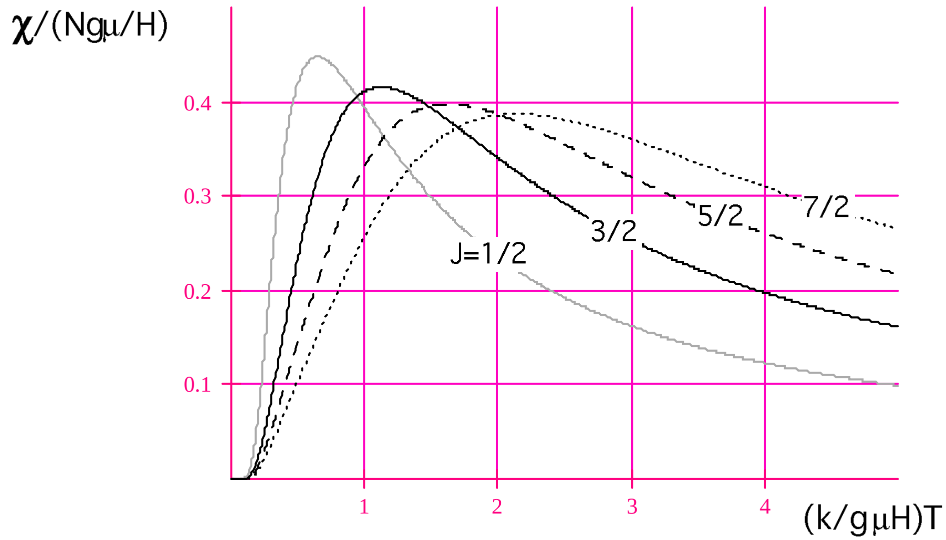
$$= \underline{NkJ\eta^2 B'_J(\eta)}$$

MAGNETIZATION OF A QUANTUM PARAMAGNET

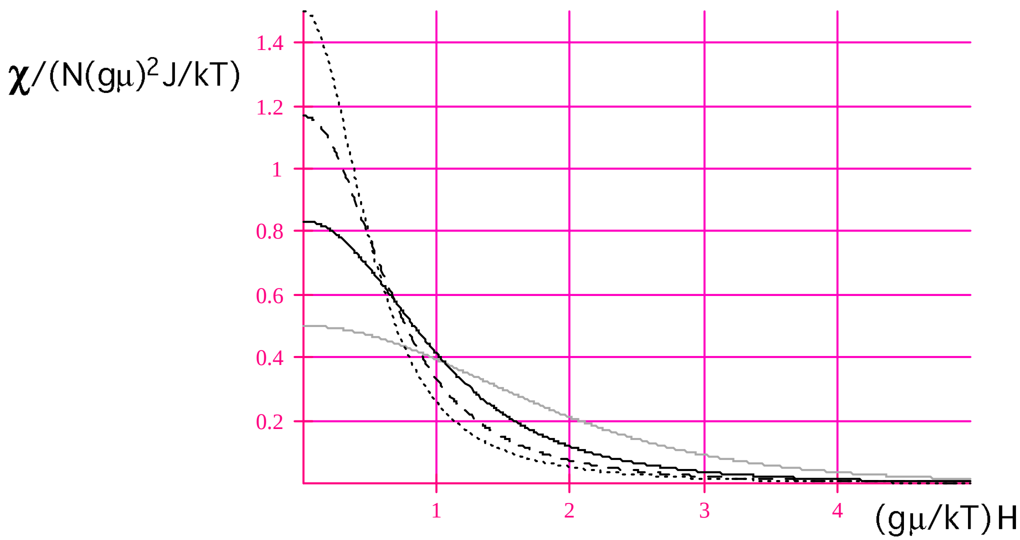




SUSCEPTIBILITY OF A QUANTUM PARAMAGNET

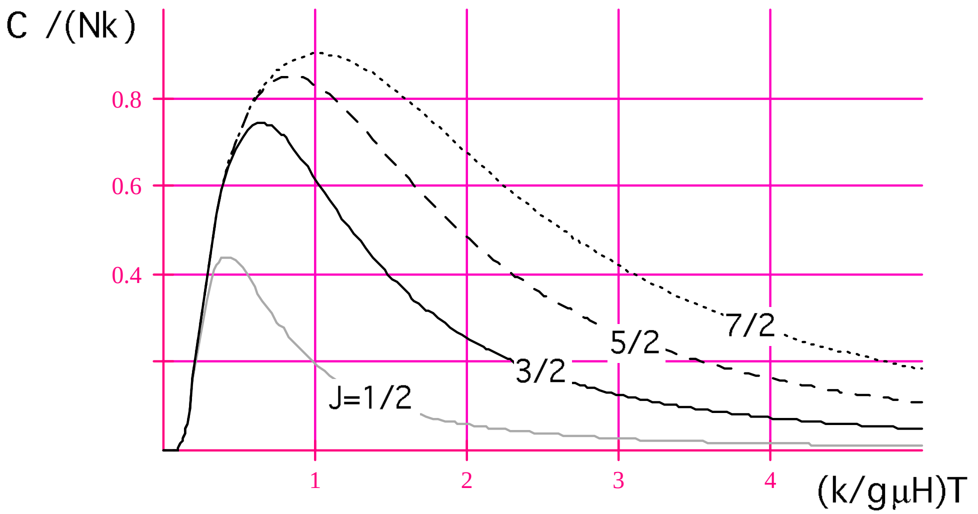


Field Held Constant

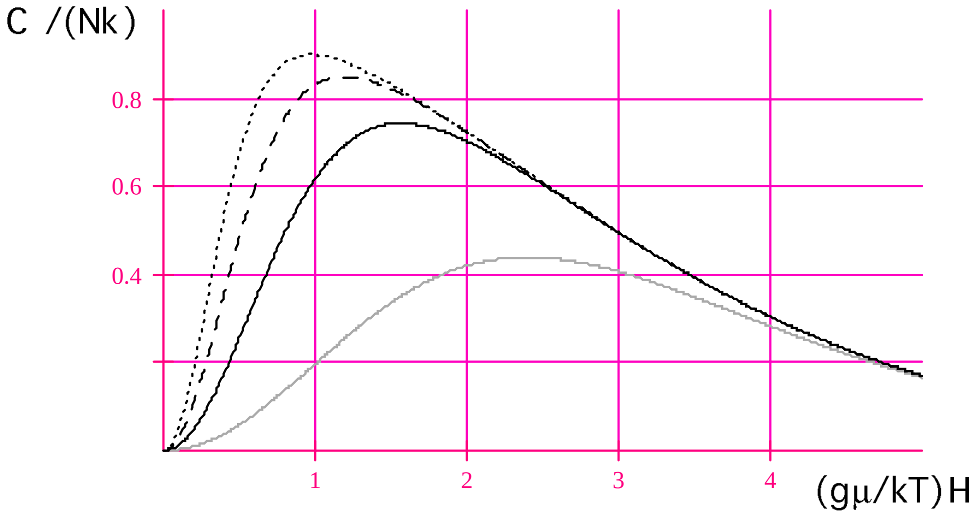


Temperature Held Constant

HEAT CAPACITY OF A QUANTUM PARAMAGNET



Field Held Constant



Temperature Held Constant

ENTROPY OF A QUANTUM PARAMAGNET