8.044 Statistical Physics I
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044: Statistical Physics I　　　　Spring Term 2008

Problem Set #2

Problem 1: Examples using the Poisson density.

Suppose a student has written a senior thesis which contains 100 pages in total (each page containing roughly the same number of words). There are also 100 typos randomly scattered throughout the thesis. Estimate the probability that:

a) The first page contains no typos.

b) The first page contains at least 3 typos.

Suppose 30 students in a hallway of a dormitory share a bathroom with 4 showers. They all enter the bathroom to shower between 8:00 AM and 9:00 AM, and they arrive at random during this interval. Assume each student's shower lasts 6 minutes.

c) Estimate the probability that a student who enters the bathroom at 8:06 AM will have to wait (that is, all showers are occupied).

d) Recalculate this probability if an additional shower is built.

Problem 2: Two Quantum Particles

Two quantum particles (1 and 2) move in one dimension along the x-axis. The joint probability density $p(x_1, x_2)$ for finding particle 1 at $x = x_1$ and particle 2 at $x = x_2$ is given by

$$p(x_1, x_2) = \frac{1}{\pi x_0^2} \left(\frac{x_2 - x_1}{x_0} \right)^2 \exp\left[-\frac{x_1^2 + x_2^2}{x_0^2} \right]$$

where x_0 is a characteristic distance.

a) On a sketch of the x_1, x_2-plane, indicate where the joint probability is highest and where it is a minimum.

b) Find the probability density for finding particle 1 at the position x along the line. Do the same for particle 2. Are the positions of the two particles statistically independent?

c) Find the conditional probability density $p(x_1|x_2)$ that particle 1 is at $x = x_1$ given that particle 2 is known to be at $x = x_2$. Sketch the result.

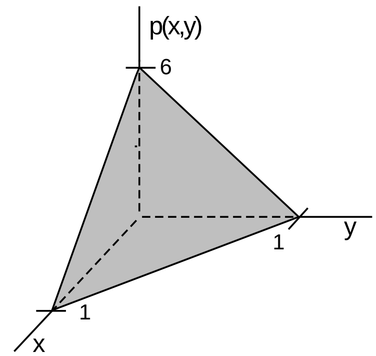
In this case, the motions of the two particles are correlated. One expects such correlation if there is an interaction between the particles. A surprising consequence of quantum mechanics is that the particle motion can be correlated even if there is no physical interaction between the particles.

Problem 3: A Joint Density of Limited Extent

The joint probability density $p(x, y)$ for two random variables x and y is given below.

$$p(x,y) = 6(1-x-y) \quad \text{if} \begin{cases} 0 \le x \\ 0 \le y \\ x+y \le 1 \end{cases}$$

$$= 0 \qquad \text{otherwise}$$

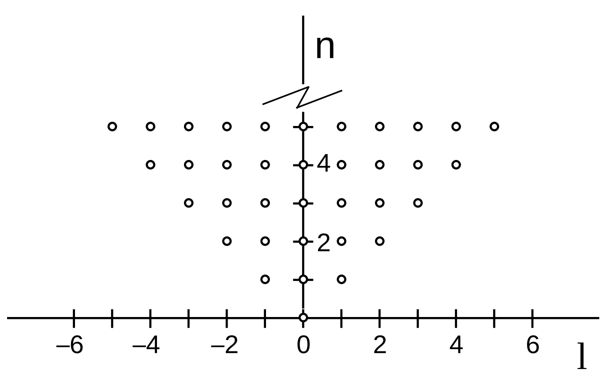


a) Find the probability density $p(x)$ for the random variable x alone. Sketch the result.

b) Find the conditional probability density $p(y|x)$. Sketch the result.

Problem 4: A Discrete Joint Density

A certain system can be completely described by two integers n and l. $n = 0, 1, 2, \cdots$ but the values of l are constrained by n to the $2n+1$ integers $-n \le l \le n$. The figure below indicates the possibilities up to $n = 5$.



The joint probability density for n and l is

$$\begin{aligned} p(n,l) &= c\,e^{-an} \qquad && n \geq 0 \text{ and } |l| \leq n \\ &= 0 && \text{otherwise} \end{aligned}$$

Here a is a parameter with the constraint that $e^{-a} < 1$ and c is a normalization constant with depends on a.

a) Find $p(n)$ and sketch.

b) Find $p(l|n)$ and sketch.

c) Find $p(l)$ and sketch. [Hint: You will need to sum a geometric series which does not begin with $n = 0$; do this by factoring out an appropriate term to reduce the series to conventional form. Don't forget that l can be negative.]

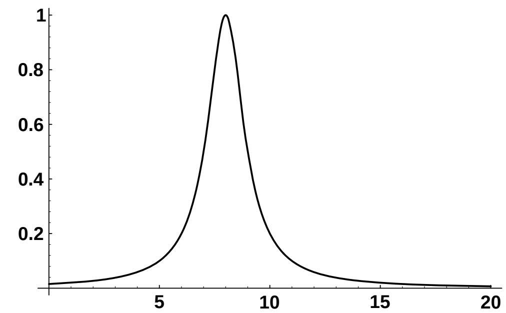
d) Find $p(n|l)$ and sketch.

Problem 5: Shot Noise

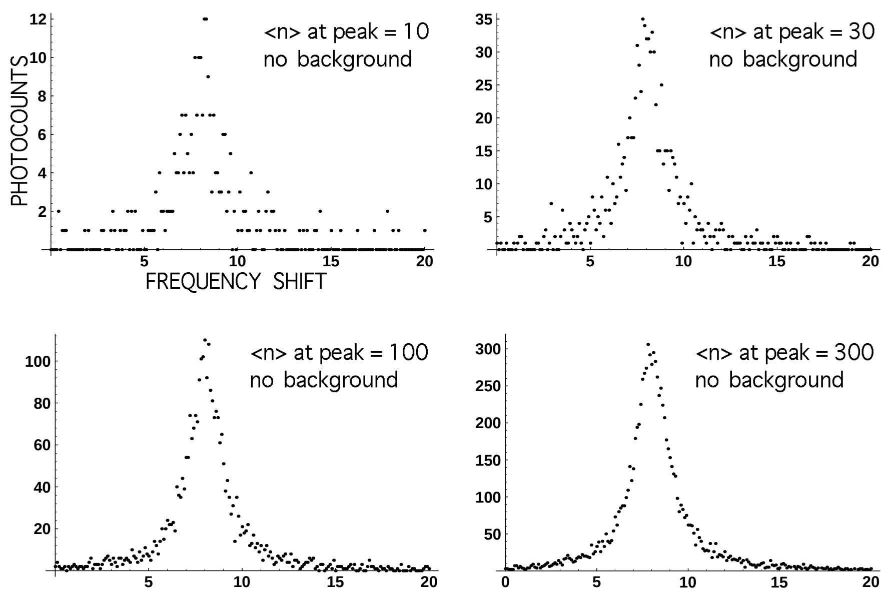
Shot noise is a type of quantization noise. It is the noise that arises when one tries to represent a continuous variable by one that can take on only discrete values. One example of shot noise is the measurement of the intensity of a weak light source using a photomultiplier tube. The light causes photoelectrons to be ejected from the photocathode of the tube. Each photoelectron initiates a cascade of electrons down the dynode chain which results in a well resolved pulse of electrons at the anode of the tube. The pulse may be 5 nanoseconds wide (successive pulses are easily resolved except at very high light levels) and contain 10^6 electrons. For a fixed light intensity these pulses arrive at random with an average rate r_s counts/second.

$$r_s = \eta \frac{IA}{h\nu}$$

where I is the intensity of the light (power per unit area) at the phototube, A the photocathode area, ν the frequency of the light, and h Planck's constant. η is the quantum efficiency of the photocathode (number of photoelectrons per photon) which for a good tube can be 20%. From these facts one expects that the the number of counts detected in a counting interval T will have a Poisson density with mean r_sT.

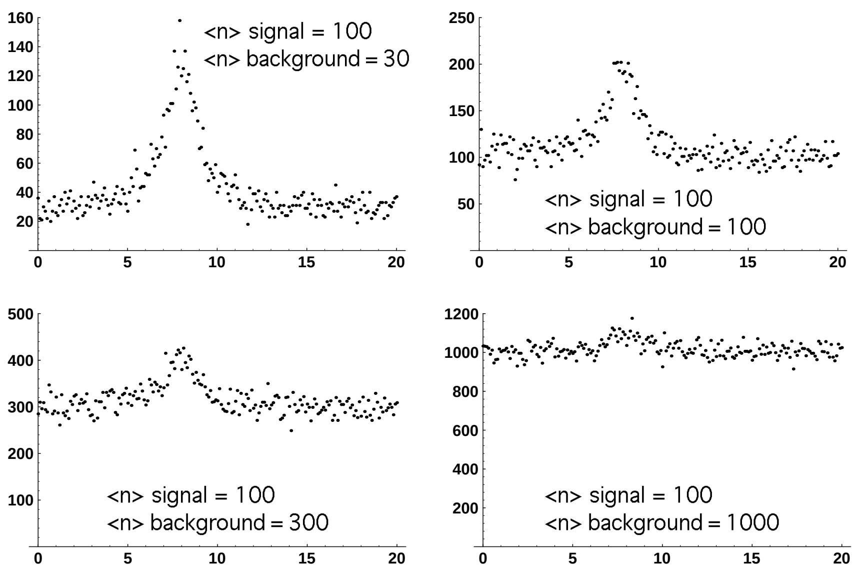


The intensity of light controls the photocurrent, which is quantized in units of about 10^6 electrons. The physical consequences of this type of quantization can be illustrated by imagining that our phototube is used as a detector for an optical spectrometer which is swept across an atomic line. The intrinsic line shape is shown above. It has a half width at half height of one unit. The intensity of the light is measured at frequencies separated by one-tenth of a unit. The mean number of photo-counts in each increment can be increased by increasing the overall intensity of the light or by increasing the amount of counting time per increment, T. The traces below represent spectra measured with differing mean counting rates at the peak of the line.



Observe the unusual characteristics of this type of "noise." The fluctuations are in the measured signal itself. No signal; no fluctuations. Small signal; small fluctuations. Large signal; large fluctuations. But the mean of the signal increases faster than the magnitude of the fluctuations so that the larger the signal, the easier it is to estimate its true strength. Note that in the absence of a background source of counts, even a single measured count is sufficient to establish that a signal is indeed present. A low counting rate only makes the determination of the exact location and width of the line difficult.

The situation is changed in a fundamental way when there is a source of background counts. They could arise from dark counts in the detector (in the case of the phototube, thermal emission from the photocathode which takes place even in the absence of light), or from another contribution to the spectrum of the light which is uniform across frequency interval being studied. In this case, the fluctuations in the measured background signal can be sufficiently large to mask even the presence of the signal one is searching for. The following traces demonstrate this effect by keeping the mean counting rate from the atomic line at its peak at 100, while increasing the mean counting rate due to the background.



Let us now examine shot noise quantitatively using the phototube as a concrete example. Assume one wants to measure light of constant intensity I in the presence of a background caused by dark current.

a) Find the mean number of counts to be expected in the time interval T due to the light signal alone. We shall refer to this as the signal S.

b) If the measurement is repeated in a number of successive time intervals of the same length, different numbers of counts could be recorded each time. What is the standard deviation, $\sigma \equiv (\text{Variance})^{1/2}$, of these measured numbers from the mean. This is referred to as the shot noise N.

c) Find the signal to noise ratio S/N in the absence of background. How does it depend on the intensity I and on the time T?

d) Now consider the case when the background is present. Assume that the dark counts occur at random with a rate r_d. The number of counts recorded in a time T when I is on is compared with the number in an equal time interval when I is off; the difference is the signal S which is proportional to I. We will take the noise N to be the standard deviation in the counts during the interval when the source is on (in actuality the fluctuations in the recorded counts when the source is off also add to the uncertainty in determining the amplitude of the signal). Find the signal to noise ratio for this experiment in the limit when $r_d \gg r_s$. Now how does it depend on I and T?