8.044 Statistical Physics I
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044: Statistical Physics I — Spring Term 2008

Practice Problems

Not to be graded or handed-in. Solutions will be available on the web.

Problem 1: Two Identical Particles

A system consists of two identical, non-interacting, spinless (no spin variables at all) particles. The system has only three single-particle states ϕ_1, ϕ_2, and ϕ_3 with energies $\epsilon_1 = 0 < \epsilon_2 < \epsilon_3$ respectively.

a) List in a vertical column all the two-particle states available to the system, along with their energies, if the particles are Fermions. Use the occupation number notation (n_1, n_2, n_3) to identify each state. Indicate which state is occupied at $T = 0$.

b) Repeat a) for the case of Bose particles.

c) Use the Canonical Ensemble to write the partition function for both Fermi and Bose cases.

d) Using only the leading <u>two</u> terms in the partition function, find the temperature dependence of the internal energy in each case. Contrast the behavior of the internal energy near $T = 0$ in the two cases.

Problem 2: A Number of Two-State Particles

Consider a collection of N non-interacting, spinless Bose particles. There are only two single-particle energy eigenstates: ϕ_0 with energy $\epsilon = 0$ and ϕ_1 with energy $\epsilon = \Delta$.

a) How would you index the possible N-body energy eigenstates in the occupation number representation? What are their energies? How many N-body states are there in all?

b) Find a closed form expression for the partition function $Z(N, T)$ using the Canonical Ensemble.

c) What is the probability $p(n)$ that n particles will be found in the excited state ϕ_1?

d) Find the partition function $Z_d(N, T)$ that would apply if the N particles were distinguishable but possessed the same single particle states as above.

Problem 3: A Fermi System at Low Temperature

A hypothetical system of N fermions has a single-particle density of states given by the linear relation $D(\epsilon) = \epsilon/\epsilon_0^2$, where ϵ_0 is a positive constant with the dimensions of energy. The fermions do not interact among themselves. Calculate:

a) The system's Fermi energy.

b) The chemical potential as a function of T under the condition that $0 \leq T \ll T_F$.

c) The total energy $< E >$ and heat capacity C_V under the same condition.

In parts (b) and (c), it suffices to work through the lowest non-vanishing order in powers of the temperature, that is, through the first temperature-dependent term. Express all final answers in terms of N, ϵ_0, k, and T.

Problem 4: Relativistic Electron Gas

Consider a 3 dimensional non-interacting quantum gas of ultra-relativistic electrons. In this limit the single particle energies are given by $\epsilon = c\hbar|\vec{k}|$. The density of allowed wavevectors in k space is $V/(2\pi)^3$.

a) Find an expression for the Fermi Energy ϵ_F as a function of c, $\hbar$, N and V.

b) Find the density of states as a function of energy $D(\epsilon)$. Sketch the result.

c) Find the total kinetic energy E of the gas at absolute zero as a function of N and ϵ_F.

d) Find the pressure exerted by the gas at $T = 0$. How does it depend on the particle density N/V? Is this a stronger or weaker dependence on density than in the non-relativistic gas?

e) Assume that this gas represents the electrons in a white dwarf star composed of α particles (a bound state of two neutrons and two protons: a ^{4}He nucleus) and electrons. Express the kinetic energy E_K in terms of the total mass of the star M and its radius R (as well as a handful of physical constants including the α particle mass). Recall that the potential energy of a self gravitating star of radius R with uniform density is given by $E_P = -\frac{3}{5}GM^2/R$ where G is the gravitational constant. Proceed as we did in class to find the equilibrium R as a function of M by minimizing the total energy, $E_T = E_K + E_P$. What can you conclude about the stability of the star in this model?

f) We know from observation that white dwarfs of mass less than a certain critical mass (of the order of the Sun's mass) are stable. The results of e) show that determining this critical mass will require a more sophisticated model of the star, certainly taking into account the dependence of the mass density on depth in the star and using a dispersion relation for the electrons, $\epsilon(k)$, valid for all energies. However the calculation in e) allows us to find how the critical mass might depend on the important parameters in the problem: c, $\hbar$, G, and a reference mass such as m_α. Use your results to find an expression for the critical mass of a white dwarf, M_c, in terms of these parameters but neglecting any purely numerical constants of the order of one.