Coxeter Systems, Multiplicity Free Representations, and Twisted Kazhdan-Lusztig Theory

by

Eric Marberg

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Abstract

This thesis considers three topics related to the representations of Coxeter systems, their Hecke algebras, and related groups.

The first topic concerns the construction of generalized involution models, as defined by Bump and Ginzburg. We compute the automorphism groups of all complex reflection groups $G(r, p, n)$ and using this information, we classify precisely which complex reflection groups have generalized involution models.

The second topic concerns the set of "unipotent characters" $\mathrm{Uch}(W)$ which Lusztig has attached to each finite, irreducible Coxeter system (W, S). We describe a precise sense in which the irreducible multiplicities of a certain W-representation can be used to define a function which serves naturally as a heuristic definition of the Frobenius-Schur indicator on $\mathrm{Uch}(W)$. The formula we obtain for this indicator extends prior work of Casselman, Kottwitz, Lusztig, and Vogan addressing the case in which W is a Weyl group.

Finally, we study a certain module of the Hecke algebra of a Coxeter system (W, S), spanned by the set of twisted involutions in W. Lusztig has shown that this module has two distinguished bases, and that the transition matrix between these bases defines interesting analogs of the much-studied Kazhdan-Lusztig polynomials of (W, S). We prove several positivity properties related to these polynomials for universal Coxeter systems, using combinatorial techniques, and for finite Coxeter systems, using computational methods.

Thesis Supervisor: David A. Vogan
Title: Professor

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Contents

List of Figures

List of Tables

Introduction

This thesis studies several related questions concerning the representations of Coxeter systems and their Hecke algebras, and is divided into three self-contained chapters.

The first is devoted to the construction and classification of certain types of multiplicity free representations of complex reflection groups. (By a *multiplicity free representation*, we mean one for which no distinct subrepresentations are isomorphic.) Central to the story told in this section is the notion of a *generalized involution model*, as defined by Bump and Ginzburg in [24]. As our first significant result, we prove that if a finite group H has a generalized involution model, then the wreath product $H \wr S_n$ also has a generalized involution model. We go to compute the automorphism groups of all complex reflection groups $G(r,p,n)$ and using this information, we classify precisely which complex reflection groups have generalized involution models. Our final result is to derive a formulation of this classification in terms of the existence of isomorphisms between certain *projective reflection groups*, as defined by Caselli [25].

The second chapter concerns the set of "unipotent characters" $\mathrm{Uch}(W)$ which Lusztig has attached to each finite, irreducible Coxeter system (W,S). There is a notion of a "Fourier transform" on the space of functions $\mathrm{Uch}(W) \to \mathbb{R}$, due to Lusztig for Weyl groups and to Broué, Lusztig, and Malle in the remaining cases. We show that the irreducible multiplicities of a certain W-representation ϱ_W, defined on the vector space generated by W's involutions, are given by the Fourier transform of a unique function $\epsilon : \mathrm{Uch}(W) \to \{-1,0,1\}$ which for various reasons serves as a heuristic definition of the Frobenius-Schur indicator on $\mathrm{Uch}(W)$. The formula we obtain for ϵ extends prior work of Casselman, Kottwitz, Lusztig, and Vogan addressing the case in which W is a Weyl group. We also prove that a conjecture of Kottwitz connecting the decomposition of ϱ_W to the left cells of W holds in all non-crystallographic types, and observe that a weaker form of Kottwitz's conjecture holds in general. In giving these results, we carefully survey the construction and notable properties of the set $\mathrm{Uch}(W)$ and its attached Fourier transform.

The final chapter concerns some conjectural properties associated with a certain module of the Hecke algebra of a Coxeter system (W,S). Let $w \mapsto w^*$ be an involution of W which preserves the set of simple generators S. Lusztig [70] has recently shown that the set of twisted involutions (i.e., elements $w \in W$ with $w^{-1} = w^*$) naturally generates a module of the Hecke algebra of (W,S) with two distinguished bases. The transition matrix between these bases defines a family of polynomials $P^\sigma_{y,w}$ which one can view as "twisted" analogs of the much-studied Kazhdan-Lusztig polynomials of (W,S). The polynomials $P^\sigma_{y,w}$ can have negative coefficients, but display several conjectural positivity properties of interest. We review Lusztig's construction and then prove three such positivity properties for universal Coxeter systems (i.e., such that st has order 2 or ∞ for all $s, t \in S$), generalizing previous work of Dyer [35]. We present in addition a computational proof of a subset of the same conjectures for finite Coxeter systems, relying on extensions to Fokko du Cloux's algebra system `Coxeter` [34].

The three chapters of this thesis are independent from one other, and can be read in any order. The results included here appear elsewhere in the papers [29, 78, 79, 80, 81, 82], the first of which is joint work with Fabrizio Caselli.

Chapter 1

Generalized involution models of finite complex reflection groups

The material in this chapter is drawn primarily from the articles [78, 79]. Section 1.8 includes results from the paper [29], which is joint work with Fabrizio Caselli.

1.1 Introduction

A *model* for a finite group G is a set $\{\lambda_i : H_i \to \mathbb{C}\}$ of linear characters of subgroups of G, such that the sum of induced characters $\sum_i \mathrm{Ind}_{H_i}^G(\lambda_i)$ is equal to the multiplicity free sum of all irreducible characters $\sum_{\psi\in\mathrm{Irr}(G)} \psi$. Models are interesting because they lead to interesting representations in which live all of the irreducible representations of G. This is especially the case when the subgroups H_i are taken to be the stabilizers of the orbits of some natural G-action. A canonical example of this phenomenon comes from the following model for the symmetric group S_n, described (among other places) by Inglis, Richardson, and Saxl in [49].

Example 1.1.1. The symmetric group S_n acts by conjugation on its *involutions* (elements of order ≤ 2). For each nonnegative integer i with $2i \leq n$, choose an involution ω_i with $n-2i$ fixed points and write H_i for its centralizer. The elements of H_i permute the support of ω_i (i.e., the set of points not fixed by ω_i), inducing a map $\pi_i : H_i \to S_{2i}$. If $\lambda_i \in \mathrm{Irr}(H_i)$ is the linear character $\lambda_i \overset{\text{def}}{=} \mathrm{sgn} \circ \pi_i$ then $\{\lambda_i : H_i \to \{\pm 1\}\}$ is a model for S_n [49].

Generalizing the preceding construction, Adin, Postnikov, and Roichman proved that the following gives a model for the wreath product of a symmetric group and a cyclic group of odd order. We will rederive this result from a more general theorem in Section 1.3.

Example 1.1.2. Let $G = G(r,n)$ be the group of complex $n \times n$ matrices with exactly one nonzero entry, given by an rth root of unity, in each row and column. Assume r is odd. Then G acts on its symmetric elements by $g : X \mapsto gXg^T$, and the distinct orbits of this action are represented by the block diagonal matrices of the form

$$X_i \stackrel{\text{def}}{=} \begin{pmatrix} J_{2i} & 0 \\ 0 & I_{n-2i} \end{pmatrix},$$

where J_n denotes the $n \times n$ matrix with ones on the anti-diagonal and zeros elsewhere. Write H_i for the stabilizer of X_i in G. The elements of H_i preserve the standard copy of $\mathbb{C}^{2i}$ in $\mathbb{C}^n$, inducing a map $\pi_i : H_i \to \mathrm{GL}_{2i}(\mathbb{C})$. If $\lambda_i \stackrel{\text{def}}{=} \det \circ \pi_i$ then $\{\lambda_i : H_i \to \mathbb{C}\}$ is a model for $G(r,n)$ [2, Theorem 1.2].

The following definition of Bump and Ginzburg [24] captures the salient features of these examples. Let τ be an automorphism of G with $\tau^2 = 1$. Then G acts on the set of *generalized involutions*

$$\mathcal{I}_{G,\tau} \stackrel{\text{def}}{=} \{\omega \in G : \omega^{-1} = \tau(\omega)\}$$

by the twisted conjugation $g : \omega \mapsto g \cdot \omega \cdot \tau(g)^{-1}$. We write

$$C_{G,\tau}(\omega) \stackrel{\text{def}}{=} \{g \in G : g \cdot \omega \cdot \tau(g)^{-1} = \omega\}$$

to denote the stabilizer of $\omega \in \mathcal{I}_{G,\tau}$ under this action, and say that a model $\{\lambda_i : H_i \to \mathbb{C}\}$ is a *generalized involution model* (or *GIM* for short) with respect to τ if each H_i is the stabilizer $C_{G,\tau}(\omega)$ of a generalized involution $\omega \in \mathcal{I}_{G,\tau}$, with each twisted conjugacy class in $\mathcal{I}_{G,\tau}$ contributing exactly one subgroup. The model in our second example is a GIM with respect to the inverse transpose automorphism of $G(r,n)$. The model in Example 1.1.1 is a GIM with respect to the identity automorphism; we call such a GIM an *involution model.*

The main object of this chapter is to classify the finite *complex reflection groups* possessing generalized involution models. Recall that a complex reflection group G is a group generated by a set of pseudo-reflections of some finite-dimensional complex vector space V. By *pseudo-reflection* we mean an element of $\mathrm{GL}(V)$ which fixes every point in some hyperplane. Complex reflection groups include Coxeter groups as a special case, and one can view our classification of GIMs for complex reflection groups as a natural extension of the classification of *involution models* for Coxeter groups. The latter classification is due to Baddeley [11] and Vinroot [101]; consult Corollary 1.7.3 below.

We may summarize the main results of this chapter as follows. For positive integers r, p, n with p dividing r, define $G(r,p,n)$ as the subgroup of $G(r,n)$ (as given in Example 1.1.2) consisting of matrices whose nonzero entries, multiplied together, form an (r/p)th root of unity. A complex reflection group $G \subset \mathrm{GL}(V)$ is *irreducible* if V has exactly two G-invariant subspaces (namely, $\{0\}$ and V). Every complex reflection group is isomorphic to the direct product of a list of irreducible groups (which is unique up to permutation of factors). The irreducible finite complex reflection groups are given by the groups $G(r,p,n)$ along with thirty-four exceptional groups labeled $G_4, \dots G_{37}$; see Section 1.4.1 for a more extensive discussion.

We prove the following theorem in Section 1.7.3.

Theorem. A finite complex reflection group has a generalized involution model if and only if each of its irreducible factors is one of the following:

(i) $G(r,p,n)$ with $\gcd(p,n) = 1$.

(ii) $G(r,p,2)$ with r/p odd.

(iii) G_{23}, the Coxeter group of type H_3.

There is an interesting reformulation of this classification, due originally to Caselli and described in the joint work [29], in terms of isomorphisms between *projective reflection groups.* These groups were introduced in [25] and studied, for example, in [15]. They include as an important special case an infinite series of groups $G(r,p,q,n)$ defined as the quotient

$$G(r,p,q,n) \stackrel{\text{def}}{=} G(r,p,n)/C_q$$

where C_q is the cyclic subgroup of scalar $n \times n$ matrices of order q. For this quotient to be well-defined we must have $C_q \subset G(r,p,n)$, which occurs precisely when q divides r and pq divides rn. Observe also that $G(r,n) = G(r,1,n)$ and $G(r,p,n) = G(r,p,1,n)$.

We now have our second main theorem, which we prove in Section 1.8.3.

Theorem. The complex reflection group $G(r,p,n) = G(r,p,1,n)$ has a GIM if and only if $G(r,p,1,n) \cong G(r,1,p,n)$.

On the way to establishing the main theorems just mentioned, our discussion will uncover a number of intermediate results. Most prominently:

(a) Given a finite group H with a generalized involution model, we describe how to construct a GIM for the wreath product $H \wr S_n$; see Section 1.3.

(b) Using (a) we give an alternate proof of the main result [2, Theorem 1.2] in Adin, Postnikov, and Roichman's paper [2]; see Section 1.3.4.

(c) We compute the automorphism group $\mathrm{Aut}(G(r,p,n))$ in Section 1.6.

(d) In Section 1.8, we give necessary and sufficient conditions for two projective reflection groups $G(r,p,q,n)$ and $G(r,p',q',n)$ to be isomorphic.

The results in this chapter completely solve the problem of determining whether a complex reflection group has a generalized involution model. The same problem for projective reflection groups is still not yet fully understood; one can find some partial results and conjectures stated in the followup work [29].

1.2 Preliminaries

Let G be a finite group. Throughout this chapter we employ the following notational conventions:

We let $\mathcal{I}_G = \mathcal{I}_{G,1} = \{g \in G : g^2 = 1\}$ denote the set of involutions in G.

We let $C_G(\omega) = C_{G,1}(\omega) = \{g \in G : g\omega g^{-1} = \omega\}$ denote the centralizer of an element.

We write $1\!\!1 = 1\!\!1_G$ for the trivial character defined by $1\!\!1(g) = 1$ for $g \in G$.

We write $\otimes$ to denote the internal tensor product;

We write $\odot$ to denote the external tensor product.

Thus, if ρ, ρ' are representations of G, then $\rho \otimes \rho'$ is a representation of G while $\rho \odot \rho'$ is a representation of $G \times G$, and similarly for characters.

1.2.1 Facts about models

Fix a finite group G with an automorphism $\tau \in \mathrm{Aut}(G)$ such that $\tau^2 = 1$. We denote the action of τ on an element $g \in G$ by ${}^\tau g$ or $\tau(g)$. For each $\psi \in \mathrm{Irr}(G)$ let ${}^\tau\psi$ denote the irreducible character ${}^\tau\psi = \psi \circ \tau$. Define $\epsilon_\tau : \mathrm{Irr}(G) \to \{-1, 0, 1\}$ as the function with

$$\epsilon_\tau(\psi) = \begin{cases} 1, & \text{if } \psi \text{ is the character of a representation } \rho \text{ with } \rho(g) = \overline{\rho({}^\tau g)} \text{ for all } g \in G, \\ 0, & \text{if } \psi \neq \overline{{}^\tau\psi}, \\ -1, & \text{otherwise.} \end{cases}$$

When $\tau = 1$, this defines the familiar Frobenius-Schur indicator function. Kawanaka and Matsuyama prove in [52, Theorem 1.3] that the twisted indicator ϵ_τ has the formula

$$\epsilon_\tau(\psi) = \frac{1}{|G|} \sum_{g \in G} \psi(g \cdot {}^\tau g), \qquad \text{for } \psi \in \mathrm{Irr}(G).$$

In turn, we have the following result which appears in a slightly different form as [24, Theorems 2 and 3].

Theorem 1.2.1 (Bump, Ginzburg [24]). Let G be a finite group with an automorphism $\tau \in \mathrm{Aut}(G)$ such that $\tau^2 = 1$. The following are equivalent:

(i) The function $\chi : G \to \mathbb{Q}$ defined by

$$\chi(g) = |\{u \in G : u \cdot {}^\tau u = g\}|, \qquad \text{for } g \in G$$

is the multiplicity free sum of all irreducible characters of G.

(ii) Every irreducible character ψ of G has $\epsilon_\tau(\psi) = 1$.

(iii) The sum $\sum_{\psi \in \mathrm{Irr}(G)} \psi(1)$ is equal to $|\mathcal{I}_{G,\tau}| = |\{\omega \in G : \omega \cdot {}^\tau\omega = 1\}|$.

This theorem motivates Bump and Ginzburg's original definition of a generalized involution model. In explanation, if the conditions (i)-(iii) hold, then the dimension of any Gelfand model for G is equal to the sum of indices $\sum_i [G : C_{G,\tau}(\omega_i)]$ where ω_i ranges over a set of representatives of the distinct orbits in $\mathcal{I}_{G,\tau}$. The twisted centralizers of a set of orbit representatives in $\mathcal{I}_{G,\tau}$ thus present an obvious choice for the subgroups $\{H_i\}$ from which to construct a model $\{\lambda_i : H_i \to \mathbb{C}\}$, and one is naturally tempted to investigate whether G has a generalized involution model with respect to the automorphism τ.

Remark. Bump and Ginzburg's definition of a generalized involution model in [24] differs from the one we have given here in the following way: in [24], the set $\mathcal{I}_{G,\tau}$ is defined as $\{\omega \in G : \omega \cdot {}^\tau\omega = z\}$ where $z \in Z(G)$ is a fixed central element with $z^2 = 1$. One can show using the preceding theorem that under this definition, any generalized involution model with respect to τ, z is also a generalized involution model with respect to τ', z', where τ' is given by composing τ with an inner automorphism and $z' = 1$. Thus our definition is equivalent to the one in [24], in the sense that the same models (that is, sets of linear characters) are classified as generalized involution models.

The following observation concerns the relationship between a generalized involution model and a corresponding *Gelfand model*, by which is meant a representation equivalent to the multiplicity free sum of all of a group's irreducible representations. Given $\tau \in \mathrm{Aut}(G)$ with $\tau^2 = 1$ and a fixed subfield $\mathbb{K}$ of the complex numbers $\mathbb{C}$, let

$$\mathcal{V}_{G,\tau} = \mathbb{K}\text{-span}\{C_\omega : \omega \in \mathcal{I}_{G,\tau}\} \tag{1.2.1}$$

be a vector space generated by the generalized involutions of G. We often wish to translate a generalized involution model with respect to $\tau \in \mathrm{Aut}(G)$ into a Gelfand model defined in the space $\mathcal{V}_{G,\tau}$. For this purpose, we repeatedly use the following result.

Lemma 1.2.2. Let G be a finite group with an automorphism $\tau \in \mathrm{Aut}(G)$ such that $\tau^2 = 1$. Suppose there exists a function $\mathrm{sign}_G : G \times \mathcal{I}_{G,\tau} \to \mathbb{K}$ such that the map $\rho : G \to \mathrm{GL}(\mathcal{V}_{G,\tau})$ defined by

$$\rho(g)C_\omega = \mathrm{sign}_G(g,\omega) \cdot C_{g\cdot\omega\cdot{}^\tau g^{-1}}, \qquad \text{for } g \in G,\ \omega \in \mathcal{I}_{G,\tau} \tag{1.2.2}$$

is a representation. Then the following are equivalent:

(i) The representation ρ is a Gelfand model for G.

(ii) The functions

$$\left\{ \begin{array}{rccl} \mathrm{sign}_G(\cdot,\omega): & C_{G,\tau}(\omega) & \to & \mathbb{K} \\ & g & \mapsto & \mathrm{sign}_G(g,\omega) \end{array} \right\},$$

with ω ranging over any set of orbit representatives of $\mathcal{I}_{G,\tau}$, form a generalized involution model for G.

Remark. If G has a generalized involution model $\{\lambda_i : H_i \to \mathbb{K}\}$ with respect to $\tau \in \mathrm{Aut}(G)$, then there automatically exists a function $\mathrm{sign}_G : G \times \mathcal{I}_{G,\tau} \to \mathbb{K}$ such that ρ is a representation and (1) and (2) hold. One can construct this function by considering the standard representation attached to the induced character $\sum_i \mathrm{Ind}_{H_i}^G(\lambda_i)$.

Proof. This proof is an elementary exercise involving the definition of a representation and the formula for an induced character. Fix $\omega \in \mathcal{I}_{G,\tau}$. Since ρ is a representation, the function $\mathrm{sign}_G(\cdot, \omega) : G \to \mathbb{K}$ restricts to a linear character of $C_{G,\tau}(\omega)$. Let $\sigma = x \cdot \omega \cdot {}^\tau x^{-1}$ for some $x \in G$. Then $g \in G$ has $x^{-1}gx \in C_{G,\tau}(\omega)$ if and only if $g \in C_{G,\tau}(\sigma)$, and $\mathrm{sign}_G(x^{-1}gx, \omega) = \mathrm{sign}_G(g, \sigma)$ since by the definition of a representation

$$1 = \mathrm{sign}_G(1, \omega) = \mathrm{sign}_G(x^{-1}x, \omega) = \mathrm{sign}_G(x, \omega) \cdot \mathrm{sign}_G(x^{-1}, \sigma)$$

and so

$$\begin{aligned}\mathrm{sign}_G(x^{-1}gx, \omega) &= \mathrm{sign}_G(x, \omega) \cdot \mathrm{sign}_G(x^{-1}g, \sigma) \\ &= \mathrm{sign}_G(x, \omega) \cdot \mathrm{sign}_G(g, \sigma) \cdot \mathrm{sign}_G(x^{-1}, \sigma) = \mathrm{sign}_G(g, \sigma).\end{aligned}$$

If Ω is a set of orbit representatives of $\mathcal{I}_{G,\tau}$ and χ^ρ is the character of ρ, then it follows that

$$\begin{aligned}\chi^\rho(g) &= \sum_{\substack{\sigma \in \mathcal{I}_{G,\tau} \\ g \in C_{G,\tau}(\sigma)}} \mathrm{sign}_G(g, \sigma) \\ &= \sum_{\omega \in \Omega} \frac{1}{|C_{G,\tau}(\omega)|} \sum_{\substack{x \in G \\ g \in C_{G,\tau}(x \cdot \omega \cdot {}^\tau x^{-1})}} \mathrm{sign}_G(g, x\omega \cdot {}^\tau x^{-1}) \\ &= \sum_{\omega \in \Omega} \frac{1}{|C_{G,\tau}(\omega)|} \sum_{\substack{x \in G \\ x^{-1}gx \in C_{G,\tau}(\omega)}} \mathrm{sign}_G(x^{-1}gx, \omega) = \sum_{\omega \in \Omega} \mathrm{Ind}_{C_{G,\tau}(\omega)}^G \left(\mathrm{sign}_G(\cdot, \omega)\right)(g).\end{aligned}$$

Thus ρ is a Gelfand model if and only if $\{\mathrm{sign}(\cdot, \omega) : C_{G,\tau}(\omega) \to \mathbb{K}\}_{\omega \in \Omega}$ is a generalized involution model. $\square$

1.2.2 An involution model for the symmetric group

Here we review what is known of the generalized involution models for the symmetric groups from [1, 24, 49]. Since the symmetric group typically has a trivial center and a trivial outer automorphism group, the group's generalized involution models are always involution models in the classical sense. In preparation for the next section, we quickly review the proof of [1, Theorem 1.2] using the results of [49].

Klyachko [56, 57] and Inglis, Richardson, and Saxl [49] first constructed involution models for the symmetric group; additional models for the symmetric group and related Weyl groups appear in [3, 7, 8, 10, 91]. More recently, Adin, Postnikov, and Roichman [1] describe a simple combinatorial action which defines a Gelfand model for the symmetric group. Their construction turns out to derive directly from the involution model in [49], and goes as follows. Let S_n be the symmetric group of bijections $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and define $\mathcal{I}_{S_n} = \{\omega \in S_n : \omega^2 = 1\}$. Let

$$\mathcal{V}_n = \mathbb{Q}\text{-span}\{C_\omega : \omega \in \mathcal{I}_{S_n}\}$$

be a vector space with a basis indexed by $\mathcal{I}_{S_n}$. For any permutation $\pi \in S_n$, define two sets

$$\operatorname{Inv}(\pi) = \{(i,j) : 1 \le i < j \le n,\ \pi(i) > \pi(j)\},$$
$$\operatorname{Pair}(\pi) = \{(i,j) : 1 \le i < j \le n,\ \pi(i) = j,\ \pi(j) = i\}.$$

The set $\operatorname{Inv}(\pi)$ is the inversion set of π, and its cardinality is equal to the minimum number of factors needed to write π as a product of simple reflections. In particular, the value of the alternating character at π is $\operatorname{sgn}(\pi) = (-1)^{|\operatorname{Inv}(\pi)|}$. The set $\operatorname{Pair}(\pi)$ corresponds to the set of 2-cycles in π.

Define a map $\rho_n : S_n \to \operatorname{GL}(\mathcal{V}_n)$ by

$$\rho_n(\pi)C_\omega = \operatorname{sign}_{S_n}(\pi,\omega) \cdot C_{\pi\omega\pi^{-1}}, \qquad \text{for } \pi,\omega \in S_n,\ \omega^2 = 1,$$

where

$$\operatorname{sign}_{S_n}(\pi,\omega) = (-1)^{|\operatorname{Inv}(\pi)\cap\operatorname{Pair}(\omega)|}. \tag{1.2.3}$$

Adin, Postnikov, and Roichman [1] prove the following result.

Theorem 1.2.3 (Adin, Postnikov, Roichman [1]). The map ρ_n defines a Gelfand model for S_n.

Kodiyalam and Verma first gave a proof of this theorem in the unpublished preprint [58], but their methods are considerably more technical than the ones used in the later work [1]. We provide a very brief proof of this, using the results of [49], which follows the strategy outlined in the appendix of [1]. This will serve as a pattern for later results.

The fact that ρ_n is a representation appears as [1, Theorem 1.1]. We provide a slightly simpler, alternate proof of this fact for completeness.

Lemma 1.2.4. The map $\rho_n : S_n \to \operatorname{GL}(\mathcal{V}_n)$ is a representation.

Proof. It suffices to show that for $\omega \in \mathcal{I}_{S_n}$ and $\pi_1, \pi_2 \in S_n$,

$$|\operatorname{Inv}(\pi_1\pi_2) \cap \operatorname{Pair}(\omega)| \equiv |\operatorname{Inv}(\pi_1) \cap \operatorname{Pair}(\pi_2\omega\pi_2^{-1})| + |\operatorname{Inv}(\pi_2) \cap \operatorname{Pair}(\omega)| \pmod 2.$$

Let A^c denote the set $\{(i,j) : 1 \le i < j \le n\} \setminus A$. The preceding identity then follows by considering the Venn diagram of the sets $\operatorname{Inv}(\pi_1\pi_2)$, $\operatorname{Pair}(\omega)$, and $\operatorname{Inv}(\pi_2)$ and noting that

$$|\operatorname{Inv}(\pi_1) \cap \operatorname{Pair}(\pi_2\omega\pi_2^{-1})| = |\operatorname{Inv}(\pi_1\pi_2) \cap \operatorname{Pair}(\omega) \cap \operatorname{Inv}(\pi_2)^c| + |\operatorname{Inv}(\pi_1\pi_2)^c \cap \operatorname{Pair}(\omega) \cap \operatorname{Inv}(\pi_2)|$$

since if $i' = \pi_2(i)$ and $j' = \pi_2(j)$, then we have

$$i < j \text{ and } (i',j') \in \operatorname{Inv}(\pi_1) \cap \operatorname{Pair}(\pi_2\omega\pi_2^{-1}) \quad \text{iff} \quad (i,j) \in \operatorname{Inv}(\pi_1\pi_2) \cap \operatorname{Pair}(\omega) \cap \operatorname{Inv}(\pi_2)^c,$$
$$i > j \text{ and } (i',j') \in \operatorname{Inv}(\pi_1) \cap \operatorname{Pair}(\pi_2\omega\pi_2^{-1}) \quad \text{iff} \quad (j,i) \in \operatorname{Inv}(\pi_1\pi_2)^c \cap \operatorname{Pair}(\omega) \cap \operatorname{Inv}(\pi_2).$$

□

The preceding proof shows that as a map

$$\begin{array}{rccl} \mathrm{sign}_{S_n}(\cdot,\omega): & C_{S_n}(\omega) & \to & \mathbb{C} \\ & \pi & \mapsto & (-1)^{|\mathrm{Inv}(\pi)\cap\mathrm{Pair}(\omega)|}, \end{array}$$

the symbol $\mathrm{sign}_{S_n}(\cdot,\omega)$ defines a linear character of the centralizer $C_{S_n}(\omega)$. To name this character more explicitly, observe that elements of $C_{S_n}(\omega)$ permute the support of ω and also permute the set of fixed points of ω. In particular, if $\omega \in \mathcal{I}_{S_n}$ has f fixed points, then $C_{S_n}(\omega)$ is isomorphic to $(S_2 \wr S_k) \times S_f$, where $k = (n-f)/2$ and where the wreath product $S_2 \wr S_k$ is embedded in S_n so that the subgroup $(S_2)^k$ is generated by the 2-cycles of ω. We now have a more intuitive definition of $\mathrm{sign}_{S_n}(\pi,\omega)$.

Corollary 1.2.5. The value of $\mathrm{sign}_{S_n}(\pi,\omega)$ for $\omega \in \mathcal{I}_{S_n}$ and $\pi \in C_{S_n}(\omega)$ is the signature of π as a permutation of the set $\{i : 1 \leq i \leq n,\ \omega(i) \neq i\}$.

Proof. If in cycle notation $\omega = (i_1,j_1)\cdots(i_k,j_k)$ where each $i_t < j_t$, then $C_{S_n}(\omega)$ is generated by permutations of the three forms α,β,γ, where $\alpha = (i_t,i_{t+1})(j_t,j_{t+1})$, $\beta = (i_t,j_t)$, and γ fixes $i_1,j_1,\dots,i_k,j_k$. By inspection, our original definition of $\mathrm{sign}_{S_n}(\pi,\omega)$ agrees with the given formula when π is one of these generators, and so our formula holds for all $\pi \in C_{S_n}(\omega)$ since $\mathrm{sign}_{S_n}(\cdot,\omega) : C_{S_n}(\omega) \to \mathbb{C}^\times$ is a homomorphism. $\square$

That ρ_n is a Gelfand model now comes as a direct result of the following lemma, which appears as [12, Lemma 4.1] and [49, Lemma 2]. In the statement, we implicitly identify partitions with their Ferrers diagrams.

Lemma 1.2.6 (Barbasch and Vogan [12]; Inglis, Richardson, Saxl [49]). Let $\omega \in S_n$ be an involution fixing exactly f points. Then the induced character

$$\mathrm{Ind}_{C_{S_n}(\omega)}^{S_n}\left(\mathrm{sign}_{S_n}(\cdot,\omega)\right)$$

is the multiplicity free sum of the irreducible character of S_n corresponding to partitions of n with exactly f odd columns.

Corollary 1.2.7. The linear characters $\{\mathrm{sign}_{S_n}(\cdot,\omega) : C_{S_n}(\omega) \to \mathbb{C}\}$, with ω ranging over any set of representatives of the conjugacy classes in $\mathcal{I}_{S_n}$, form an involution model for S_n.

Remark. [49, Lemma 2] actually concerns the function $\mathrm{sign}_{S_n}(\cdot,\omega) \otimes \mathrm{sgn}$, whose value at $\pi \in C_{S_n}(\omega)$ is the signature of π as a permutation of the set $\mathrm{Fix}(\omega) \overset{\mathrm{def}}{=} \{i : \omega(i) = i\}$. Our version follows from the fact that tensoring with the alternating character commutes with induction. Specifically, the authors of [49] prove that if $\omega \in \mathcal{I}_{S_n}$ is an involution with no fixed points, then the induction of the trivial character

$$\mathrm{Ind}_{C_{S_n}(\omega)}^{S_n}(\mathbb{1})$$

is equal to the multiplicity free sum of the irreducible characters of S_n corresponding to partitions with all even rows. Proposition 1.3.5 gives a generalization of this result.

1.3 Generalized involution models for wreath products

In this section we fix a finite group H and a positive integer n and let $G_n = H \wr S_n$ denote the *wreath product* of H with S_n. By definition, G_n is then the semidirect product $G_n = H^n \rtimes S_n$ where S_n acts on H^n by permuting the coordinates of elements. We denote the action of $\pi \in S_n$ on $h = (h_1, \dots, h_n) \in H^n$ by

$$\pi(h) \stackrel{\text{def}}{=} \left(h_{\pi^{-1}(1)}, \dots, h_{\pi^{-1}(n)}\right)$$

and write elements of G_n as ordered pairs (h, π) with $h \in H^n$ and $\pi \in S_n$. The group's multiplication is then given by

$$(h,\pi)(k,\sigma) = (\sigma^{-1}(h)\cdot k, \pi\sigma), \qquad \text{for } h,k \in H^n,\ \pi,\sigma \in S_n.$$

Throughout, we identify H^n and S_n with the subgroups $\{(h,1) : h \in H^n\}$ and $\{(1,\pi) : \pi \in S_n\}$ in G_n, respectively.

The main goal of this section is generalize several results [10], in order to show how to construct a generalized involution model for the wreath product $H \wr S_n$ given a generalized involution model for H. From this construction we will give a simple alternate derivation of Adin, Postnikov, and Roichman's proof [2] that the complex reflection group $G(r,n) = G(r,1,n)$ has a GIM.

1.3.1 Irreducible characters of wreath products

Here we review the construction of the irreducible characters of G_n. Our notation mirrors but slightly differs from that in [10]. Given groups H_i and representations $\varrho_i : H_i \to \mathrm{GL}(V_i)$, for $i = 1, \dots, m$, let

$$\bigodot_{i=1}^m \varrho_i : \prod_{i=1}^m H_i \to \mathrm{GL}\left(\bigotimes_{i=1}^m V_i\right)$$

denote the representation defined for $h_i \in H_i$ and $v_i \in V_i$ by

$$\left(\bigodot_{i=1}^m \varrho_i\right)(h_1,\dots,h_m)(v_1 \otimes \cdots \otimes v_m) = \varrho_1(h_1)v_1 \otimes \cdots \otimes \varrho_m(h_m)v_m.$$

If χ_i is the character of ϱ_i, then we let $\bigodot_{i=1}^m \chi_i$ denote the character of $\bigodot_{i=1}^m \varrho_i$.

Given a representation $\varrho : H \to \mathrm{GL}(V)$, we extend $\bigodot_{i=1}^n \varrho$ to a representation of G_n by defining for $h \in H^n$, $\pi \in S_n$, and $v_i \in V$,

$$\left(\widetilde{\bigodot_{i=1}^n \varrho}\right)(h,\pi)(v_1 \otimes \cdots \otimes v_n) = \left(\varrho\left(h_{\pi^{-1}(1)}\right) v_{\pi^{-1}(1)} \otimes \cdots \otimes \varrho\left(h_{\pi^{-1}(n)}\right) v_{\pi^{-1}(n)}\right).$$

Remark. This formula differs from the corresponding formula in [10]: there the right hand side is $\left(\varrho(h_1)v_{\pi^{-1}(1)} \otimes \cdots \otimes \varrho(h_n)v_{\pi^{-1}(n)}\right)$. This is an artifact of our convention for naming elements of G_n, which differs from the one implicitly used in [10], but which will later make some of our formulas nicer.

Let $\mathscr{P}(n)$ denote the set of integer partitions of $n \geq 0$ and let $\mathscr{P} = \bigcup_{n=0}^\infty \mathscr{P}(n)$. Given

$\lambda \in \mathscr{P}(n)$, let ρ^λ denote the corresponding irreducible representation of S_n (as specified, for example, in [10]) and write $\chi^\lambda : S_n \to \mathbb{Q}$ for its character. One extends the representation ρ^λ of S_n to a representation $\widetilde{\rho^\lambda}$ of G_n by setting

$$\widetilde{\rho^\lambda}(h,\pi) = \rho^\lambda(\pi), \qquad \text{for } h \in H^n,\ \pi \in S_n.$$

If ϱ is a representation of H and $\lambda \in \mathscr{P}(n)$, then we define $\varrho \wr \lambda$ as the representation of G_n given by

$$\varrho \wr \lambda \stackrel{\text{def}}{=} \left(\widetilde{\bigodot_{i=1}^n \varrho}\right) \otimes \widetilde{\rho^\lambda}.$$

If ψ is the character of ϱ, then we define $\psi \wr \lambda$ as the character of $\varrho \wr \lambda$. We now have the following preliminary lemma.

Lemma 1.3.1. Let ψ be a character of H and let $\lambda \in \mathscr{P}(n)$. If the cycles of $\pi \in S_n$ are $(i_1^t, i_2^t, \cdots i_{\ell(t)}^t)$ for $t = 1, \ldots, r$, then

$$(\psi \wr \lambda)(h,\pi) = \chi^\lambda(\pi) \prod_{t=1}^r \psi\left(h_{i_{\ell(t)}^t} \cdots h_{i_2^t} h_{i_1^t}\right), \qquad \text{for } h = (h_1, \ldots, h_n) \in H^n.$$

Proof. Suppose ψ is the character of a representation ϱ in a vector space V with a basis $\{v_j\}$. Observe that if $h_{i_1}, h_{i_2}, \ldots, h_{i_\ell} \in H$, then

$$\psi\left(h_{i_\ell} \cdots h_{i_2} h_{i_1}\right) = \sum_{j_1, j_2, \ldots, j_\ell} \left(\varrho\left(h_{i_1}\right) v_{j_1}\big|_{v_{j_2}}\right) \left(\varrho\left(h_{i_2}\right) v_{j_2}\big|_{v_{j_3}}\right) \cdots \left(\varrho\left(h_{i_\ell}\right) v_{j_\ell}\big|_{v_{j_1}}\right).$$

Therefore, it follows by definition that

$$\begin{aligned}(\psi \wr \lambda)(h,\pi) &= \chi^\lambda(\pi) \sum_{j_1,\ldots,j_n} \left(\varrho\left(h_{\pi^{-1}(1)}\right) v_{j_{\pi^{-1}(1)}}\right) \otimes \cdots \otimes \left(\varrho\left(h_{\pi^{-1}(n)}\right) v_{j_{\pi^{-1}(n)}}\right) \big|_{(v_{j_1} \otimes \cdots \otimes v_{j_n})} \\ &= \chi^\lambda(\pi) \sum_{j_1,\ldots,j_n} \prod_{i=1}^n \left(\varrho\left(h_i\right) v_{j_i}\big|_{v_{j_{\pi(i)}}}\right) = \chi^\lambda(\pi) \prod_{t=1}^r \psi\left(h_{i_{\ell(t)}^t} \cdots h_{i_2^t} h_{i_1^t}\right).\end{aligned}$$

□

Let $\mathscr{P}_H$ denote the set of all maps $\theta : \mathrm{Irr}(H) \to \mathscr{P}$ and define

$$\mathscr{P}_H(n) = \left\{\theta \in \mathscr{P}_H : \textstyle\sum_{\psi \in \mathrm{Irr}(H)} |\theta(\psi)| = n\right\}.$$

The following classification appears, among other places, in [10] and as [98, Theorem 4.1]. Stembridge [98] attributes its original proof to Specht [94].

Theorem 1.3.2 (Specht [94]). The set of irreducible characters of G_n is in bijection with

$\mathscr{P}_H(n)$. In particular, each element of $\mathrm{Irr}(G_n)$ is equal to χ_θ for a unique $\theta \in \mathscr{P}_H(n)$, where

$$\chi_\theta \stackrel{\text{def}}{=} \mathrm{Ind}_{S_\theta}^{G_n}\left(\bigodot_{\psi\in\mathrm{Irr}(H)} \psi \wr \theta(\psi)\right) \qquad \text{and} \qquad S_\theta \stackrel{\text{def}}{=} \prod_{\psi\in\mathrm{Irr}(H)} G_{|\theta(\psi)|}.$$

In addition, the degree of the character χ_θ is

$$\deg(\chi_\theta) = n! \prod_{\psi\in\mathrm{Irr}(H)} \frac{\deg(\psi)^{|\theta(\psi)|}\deg\left(\chi^{\theta(\psi)}\right)}{|\theta(\psi)|!}.$$

All products here proceed in the order of a some fixed enumeration of $\mathrm{Irr}(H)$. The character χ_θ is independent of this enumeration because reordering the factors in S_θ yields a conjugate subgroup.

1.3.2 Inducing the trivial character

Fix an automorphism $\tau \in \mathrm{Aut}(H)$ with $\tau^2 = 1$. In this section, we describe the irreducible constituents of the induced character

$$\mathrm{Ind}_{V_k^\tau}^{G_{2k}}(\mathbb{1}),$$

where $\mathbb{1} \in \mathrm{Irr}(V_k^\tau)$ denotes the trivial character of V_k^τ, a subgroup which will be one of the twisted centralizers in our generalized involution model for G_n.

Fix a nonnegative integer k, and define $W_k \subset S_{2k}$ as the subgroup

$$W_k = \xi(S_2 \wr S_k), \tag{1.3.1}$$

where $\xi : S_2 \wr S_k \to S_{2k}$ embeds $S_2 \wr S_k$ in S_{2k} such that the wreath subgroup $(S^2)^k \subset S_2 \wr S_k$ is mapped to the subgroup of S_{2k} generated by the simple transpositions $(2i-1, 2i)$ for $i = 1, \dots, k$. In other words, let W_k be the centralizer in S_{2k} of the involution

$$\omega_k \stackrel{\text{def}}{=} (1,2)(3,4)\cdots(2k-1,2k) \in S_{2k}, \tag{1.3.2}$$

where by convention $\omega_0 = 1$. Next, define $\delta_k^\tau(H)$ as the subgroup

$$\delta_k^\tau(H) = \{(h_1, {}^\tau h_1, h_2, {}^\tau h_2, \dots, h_k, {}^\tau h_k) : h_i \in H\} \subset H^{2k}.$$

Observe that the action of W_k preserves $\delta_k^\tau(H)$, and let V_k^τ denote the subgroup of G_{2k} given by

$$V_k^\tau = \delta_k^\tau(H) \cdot W_k = \{(h, \pi) \in G_{2k} : h \in \delta_k^\tau(H),\ \pi \in W_k\}.$$

The most difficult step in constructing a model for G_n from a model for H will be to determine the irreducible constituents of the character of G_n induced from the trivial character of the subgroup V_k^τ. The following two lemmas address some the calculations needed to compute this. In what follows we write $\langle \cdot, \cdot \rangle_G$ for the L^2-inner product on functions $G \to \mathbb{C}$ given by

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}.$$

Recall that the irreducible characters of G are orthonormal with respect to this inner product.

Lemma 1.3.3. Let ψ be an irreducible character of H with $\epsilon_\tau(\psi) = \pm 1$ and let $\lambda \in \mathscr{P}(2k)$. Then

$$\left\langle \mathbb{1}, \mathrm{Res}^{G_{2k}}_{V_k^\tau} (\psi \wr \lambda) \right\rangle_{V_k^\tau} = \begin{cases} 1 & \text{if } \epsilon_\tau(\psi) = 1 \text{ and } \lambda \text{ has all even rows;} \\ 1 & \text{if } \epsilon_\tau(\psi) = -1 \text{ and } \lambda \text{ has all even columns;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Fix $\pi \in W_k$. The cycles of π are either of the form $(2i_s - 1,\ 2i_s)$ for $s = 1, \ldots, S$, or come in pairs of the form $(i_1^t, i_2^t, \cdots\ i_{\ell(t)}^t)$, $(j_1^t, j_2^t, \cdots\ j_{\ell(t)}^t)$ for $t = 1, \ldots, T$, where (i_a^t, j_a^t) is a cycle of ω_k for each a, t. If $h \in \delta_k^\tau(H)$, then in the former case $h_{k+i_s} = {}^\tau h_{i_s}$ and in the latter case $h_{j_a^t} = {}^\tau h_{i_a^t}$. In addition, note that $\mathrm{sgn}(\pi) = (-1)^S$. By Lemma 1.3.1,

$$\begin{aligned} \sum_{h \in \delta_k^\tau(H)} (\psi \wr \lambda)(h, \pi) &= \chi^\lambda(\pi) \sum_{h \in \delta_k^\tau(H)} \prod_{s=1}^{S} \psi\left(h_{k+i_s} h_{i_s}\right) \prod_{t=1}^{T} \psi\left(h_{i^t_{\ell(t)}} \cdots h_{i_2^t} h_{i_1^t}\right) \chi^\psi\left(h_{j^t_{\ell(t)}} \cdots h_{j_2^t} h_{j_1^t}\right) \\ &= \chi^\lambda(\pi) \prod_{s=1}^{S} \left(\sum_{h \in H} \psi\left(h \cdot {}^\tau h\right) \right) \prod_{t=1}^{T} \left(\sum_{h_1, \ldots, h_{\ell(t)} \in H} \psi\left(h_1 \cdots h_{\ell(t)}\right) \psi\left({}^\tau h_1 \cdots {}^\tau h_{\ell(t)}\right) \right). \end{aligned}$$

We have $\psi(h) = \psi({}^\tau h^{-1})$ since $\epsilon_\tau(\psi) = \pm 1$. Therefore

$$\begin{aligned} \sum_{h_1, \ldots, h_{\ell(t)} \in H} \psi\left(h_1 \cdots h_{\ell(t)}\right) \psi\left({}^\tau h_1 \cdots {}^\tau h_{\ell(t)}\right) &= |H|^{\ell(t)-1} \sum_{h \in H} \psi(h) \overline{\psi\left({}^\tau h^{-1}\right)} \\ &= |H|^{\ell(t)} \langle \psi, \psi \rangle_H = |H|^{\ell(t)}. \end{aligned}$$

Substituting this and $\epsilon_\tau(\psi) = \frac{1}{|H|} \sum_{h \in H} \psi(h \cdot {}^\tau h)$ into our expression above, and noting that $2S + \sum_{t=1}^{T} 2\ell(t) = 2k$, we obtain $\sum_{h \in \delta_k^\tau(H)} (\psi \wr \lambda)(h, \pi) = |H|^k (\epsilon_\tau(\psi))^S \chi^\lambda(\pi)$. Since $\mathrm{sgn}(\pi) = (-1)^S$, applying this identity gives

$$\begin{aligned} \left\langle \mathbb{1}, \mathrm{Res}^{G_{2k}}_{V_k^\tau} (\psi \wr \lambda) \right\rangle_{V_k^\tau} &= \frac{1}{|V_k^\tau|} \sum_{\pi \in W_k} \sum_{h \in \delta_k^\tau(H)} (\psi \wr \lambda)(h, \pi) \\ &= \begin{cases} \langle \mathbb{1}, \mathrm{Res}^{S_{2k}}_{W_k}(\chi^\lambda) \rangle_{W_k} & \text{if } \epsilon_\tau(\psi) = 1; \\ \langle \mathrm{sgn}, \mathrm{Res}^{S_{2k}}_{W_k}(\chi^\lambda) \rangle_{W_k} & \text{if } \epsilon_\tau(\psi) = -1. \end{cases} \end{aligned}$$

Our result now follows from applying Frobenius reciprocity to Lemma 1.2.6. ☐

Define another subgroup of G_{2k} by

$$I_k^\tau = \left\{(h, (\pi, \pi)) \in G_{2k} : h = (h_1, \ldots, h_k, {}^\tau h_1, \ldots, {}^\tau h_k) \in H^{2k},\ \pi \in S_k\right\}$$

where we view $(\pi, \pi) \in S_k \times S_k$ as an element of S_{2k} in the obvious way. We then have a second lemma.

Lemma 1.3.4. Let ψ be an irreducible character of H with $\epsilon_\tau(\psi) = 0$ and let $\lambda, \mu \in \mathscr{P}(k)$. Define $\varpi_k \in S_{2k} \subset G_{2k}$ as the permutation given by

$$\begin{aligned} \varpi_k(2i-1) &= i, \\ \varpi_k(2i) &= i + k, \end{aligned} \qquad \text{for } i = 1, \ldots, k.$$

Then $I_k^\tau = (G_k \times G_k) \cap \varpi_k^{-1}(V_k^\tau)\varpi_k$ and

$$\left\langle 1\!\!1, \operatorname{Res}_{I_k^\tau}^{G_k \times G_k} \left((\psi \wr \lambda) \odot (\overline{{}^\tau\psi} \wr \mu)\right)\right\rangle_{I_k^\tau} = \begin{cases} 1 & \text{if } \lambda = \mu; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first observe that if $\omega = (1, k+1)(2, k+2)\cdots(k, 2k) = \varpi_k \omega_k \varpi_k^{-1} \in S_{2k}$ then

$$\varpi_k^{-1}(V_k^\tau)\varpi_k = \left\{(h, \pi) : \pi \in C_{S_{2k}}(\omega),\ h = (h_1, \ldots, h_k, {}^\tau h_1, \ldots, {}^\tau h_k) \in H^{2k}\right\}.$$

It immediately follows that $I_k^\tau = (G_k \times G_k) \cap \varpi_k^{-1}(V_k^\tau)\varpi_k$. Next note that $\overline{{}^\tau\psi}({}^\tau h) = \overline{\psi(h)}$ for $h \in H$ and that $\chi^\mu \in \operatorname{Irr}(S_n)$ is real-valued. Hence

$$\left(\overline{{}^\tau\psi} \wr \mu\right)({}^\tau h, \pi) = \overline{(\psi \wr \mu)(h, \pi)}, \qquad \text{for } \pi \in S_k \text{ and } h \in H^k,$$

where we let ${}^\tau h = ({}^\tau h_1, \ldots, {}^\tau h_k)$. Therefore by Lemma 1.3.1 and an argument similar to the one used in the previous lemma, if $\pi \in S_k$ then we have

$$\sum_{h \in H^k} (\psi \wr \lambda)(h, \pi) \cdot (\overline{{}^\tau\psi} \wr \mu)({}^\tau h, \pi) = \sum_{h \in H^k} (\psi \wr \lambda)(h, \pi) \cdot \overline{(\psi \wr \mu)(h, \pi)} = |H|^k \chi^\lambda(\pi)\overline{\chi^\mu(\pi)}.$$

Our result now follows from

$$\begin{aligned} \left\langle 1\!\!1, \operatorname{Res}_{I_k^\tau}^{G_k \times G_k} \left((\psi \wr \lambda) \odot (\overline{{}^\tau\psi} \wr \mu)\right)\right\rangle_{I_k^\tau} &= \frac{1}{|I_k^\tau|} \sum_{(\pi, h) \in I_k^\tau} \left((\psi \wr \lambda) \odot (\overline{{}^\tau\psi} \wr \mu)\right)(h, \pi) \\ &= \frac{1}{|I_k^\tau|} \sum_{\pi \in S_k} \sum_{h \in H^k} (\psi \wr \lambda)(h, \pi) \cdot (\overline{{}^\tau\psi} \wr \mu)({}^\tau h, \pi) \\ &= \langle \chi^\lambda, \chi^\mu \rangle_{S_k}. \end{aligned}$$

□

We are now prepared to prove the following instrumental proposition.

Proposition 1.3.5. The induction of the trivial character of V_k^τ to G_{2k} decomposes as the multiplicity free sum

$$\operatorname{Ind}_{V_k^\tau}^{G_{2k}}(\mathbb{1}) = \sum_{\theta} \chi_\theta,$$

where the sum is over all $\theta \in \mathscr{P}_H(2k)$ such that for every irreducible character $\psi \in \operatorname{Irr}(H)$,

(1) $\theta(\psi) = \theta(\overline{{}^\tau\psi})$;

(2) $\theta(\psi)$ has all even columns if $\epsilon_\tau(\psi) = -1$;

(3) $\theta(\psi)$ has all even rows if $\epsilon_\tau(\psi) = 1$.

This result generalizes [10, Proposition 3], which treats the case $\tau = 1$. Our proof derives from a pair of detailed but straightforward calculations using the preceding lemmas. This approach differs signficantly from the inductive method used by Baddeley in [10].

Proof. Choose $\theta \in \mathscr{P}_H(2k)$ satisfying (1)-(3). We first show that χ_θ appears as a constituent of $\operatorname{Ind}_{V_k^\tau}^{G_{2k}}(\mathbb{1})$ and then demonstrate that the given decomposition has the correct degree. To this end, define

$$\eta_\theta = \bigodot_{\psi \in \operatorname{Irr}(H)} \psi \wr \theta(\psi), \qquad \text{so that} \qquad \chi_\theta = \operatorname{Ind}_{S_\theta}^{G_{2k}}(\eta_\theta).$$

Let $s \in S_{2k}$ and define the subgroup $D_s = S_\theta \cap s^{-1}(V_k^\tau)s$. Then by Frobenius reciprocity and Mackey's theorem, we have

$$\begin{aligned}
\left\langle \operatorname{Ind}_{V_k^\tau}^{G_{2k}}(\mathbb{1}), \chi_\theta \right\rangle_{G_{2k}} &= \left\langle \operatorname{Res}_{S_\theta}^{G_{2k}}\left(\operatorname{Ind}_{V_k^\tau}^{G_{2k}}(\mathbb{1})\right), \eta_\theta \right\rangle_{S_\theta} && \text{(by Frobenius reciprocity)},\\
&\geq \left\langle \operatorname{Ind}_{D_s}^{S_\theta}(\mathbb{1}), \eta_\theta \right\rangle_{S_\theta} && \text{(by Mackey's theorem)},\\
&= \left\langle \mathbb{1}, \operatorname{Res}_{D_s}^{S_\theta}(\eta_\theta) \right\rangle_{D_s} && \text{(by Frobenius reciprocity)}.
\end{aligned}$$

Recall from Section 1.2.1 that if $\psi \in \operatorname{Irr}(H)$ then the two irreducible characters $\psi, \overline{{}^\tau\psi}$ of H are distinct if and only if $\epsilon_\tau(\psi) = 0$. Therefore we can list the distinct elements of $\operatorname{Irr}(H)$ in the form

$$\psi_1, \psi_1', \dots, \psi_r, \psi_r', \vartheta_1, \dots, \vartheta_s,$$

where for all i we have $\psi_i' = \overline{{}^\tau\psi_i}$ and $\epsilon_\tau(\psi_i) = \epsilon_\tau(\psi_i') = 0$ and $\epsilon_\tau(\vartheta_i) \neq 0$. Without loss of generality, we can assume that the products defining ρ_θ and S_θ proceed in the order of this list; a different ordering corresponds to a conjugate choice of s in what follows. Since $|\theta(\psi_i)| = |\theta(\psi_i')|$ and $|\theta(\vartheta_i)|$ is even for all i, if we define $s \in S_{2k}$ as the element

$$s = \left(\varpi_{|\theta(\psi_1)|}, \dots, \varpi_{|\theta(\psi_r)|}, 1, \dots, 1\right) \in \prod_{i=1}^{r} S_{|\theta(\psi_i)|+|\theta(\psi_i')|} \times \prod_{i=1}^{s} S_{|\theta(\vartheta_i)|} \subset S_{2k}$$

where ϖ_k for $k = |\theta(\psi_1)|, \ldots, |\theta(\psi_r)|$ is as in Lemma 1.3.4, then $D_s = \prod_{i=1}^r I^{\tau}_{|\theta(\psi_i)|} \times \prod_{i=1}^s V^{\tau}_{|\theta(\vartheta_i)|/2}$. Consequently $\langle 1\!\!1, \mathrm{Res}^{S_\theta}_{D_s}(\eta_\theta)\rangle_{D_s} = \varepsilon_0\varepsilon_{\pm 1}$ where

$$\varepsilon_0 = \prod_{i=1}^r \left\langle 1\!\!1, \mathrm{Res}^{G_{|\theta(\psi_i)|}\times G_{|\theta(\psi_i')|}}_{I^{\tau}_{|\theta(\psi_i)|}} \left((\psi_i \wr \theta(\psi_i)) \odot (\psi_i' \wr \theta(\psi_i'))\right)\right\rangle_{I^{\tau}_{|\theta(\psi_i)|}},$$

$$\varepsilon_{\pm 1} = \prod_{i=1}^s \left\langle 1\!\!1, \mathrm{Res}^{G_{|\theta(\vartheta_i)|}}_{V^{\tau}_{|\theta(\vartheta_i)|/2}} (\vartheta_i \wr \theta(\vartheta_i))\right\rangle_{V^{\tau}_{|\theta(\vartheta_i)|/2}}.$$

We have $\varepsilon_0 = 1$ by Lemma 1.3.4 and $\varepsilon_{\pm 1} = 1$ by Lemma 1.3.3 and so we conclude that if $\theta \in \mathscr{P}_H(n)$ satisfies (1)-(3), then χ_θ appears as a constituent of $\mathrm{Ind}^{G_n}_{V^{\tau}_k}(1\!\!1)$ with multiplicity at least one.

To prove that this multiplicity is exactly one and that these are the only constituents, we show that both sides of the equation in the proposition statement have the same degree. Define $\mathcal{F}$ as the set of functions $f : \mathrm{Irr}(H) \to \mathbb{Z}_{\geq 0}$ which have $f(\psi) = |\theta(\psi)|$ for some $\theta \in \mathscr{P}_H(2k)$ satisfying (1)-(3). Then the sum of the degrees of χ_θ as $\theta \in \mathscr{P}_H(2k)$ varies over all maps satisfying (1)-(3) is

$$\sum_\theta \deg(\chi_\theta) = \sum_\theta (2k)! \prod_{\psi\in\mathrm{Irr}(H)} \frac{\deg(\psi)^{|\theta(\psi)|}\deg\left(\chi^{\theta(\psi)}\right)}{\theta(\psi)!} = \sum_{f\in\mathcal{F}} n!\cdot\Pi_0(f)\cdot\Pi_+(f)\cdot\Pi_-(f)$$

where

$$\Pi_0(f) \stackrel{\text{def}}{=} \prod_{i=1}^r \left(\sum_{\lambda\in\mathscr{P}(f(\psi_i))} \left(\frac{\deg(\psi_i)^{f(\psi_i)}\deg\left(\chi^\lambda\right)}{f(\psi_i)!}\right)\left(\frac{\deg(\psi_i')^{f(\psi_i')}\deg\left(\chi^\lambda\right)}{f(\psi_i')!}\right)\right),$$

and

$$\Pi_+(f) \stackrel{\text{def}}{=} \prod_{\substack{\psi\in\mathrm{Irr}(H)\\ \epsilon_\tau(\psi)=1}} \left(\sum_{\substack{\lambda\in\mathscr{P}(f(\psi))\text{ with}\\ \text{all even rows}}} \frac{\deg(\psi)^{f(\psi)}\deg\left(\chi^\lambda\right)}{f(\psi)!}\right)$$

and

$$\Pi_-(f) \stackrel{\text{def}}{=} \prod_{\substack{\psi\in\mathrm{Irr}(H)\\ \epsilon_\tau(\psi)=-1}} \left(\sum_{\substack{\lambda\in\mathscr{P}(f(\psi))\text{ with}\\ \text{all even columns}}} \frac{\deg(\psi)^{f(\psi)}\deg\left(\chi^\lambda\right)}{f(\psi)!}\right).$$

Note that $\deg(\psi_i) = \deg(\psi_i')$ and $f(\psi_i) = f(\psi_i')$ for all i if $f \in \mathcal{F}$. Therefore

$$\Pi_0(f) = \prod_{i=1}^r \frac{\deg(\psi_i)^{2f(\psi_i)}}{(f(\psi_i)!)^2}\left(\sum_{\lambda\in\mathscr{P}(f(\psi_i))} \deg\left(\chi^\lambda\right)^2\right) = \prod_{i=1}^r \frac{\left(2\deg(\psi_i)^2\right)^{f(\psi_i)}}{2^{f(\psi_i)}f(\psi_i)!}.$$

Next, recall from Lemma 1.2.6 that the sum $\sum_\lambda \deg\left(\chi^\lambda\right)$ as λ varies over the partitions of $2n$ with all even rows is equal to $\frac{(2n)!}{2^n n!}$, and that the sum over λ with all even columns has the same value. Thus

$$\Pi_+(f)\cdot\Pi_-(f) = \prod_{i=1}^{s} \frac{(\deg(\vartheta_i)^2)^{f(\vartheta_i)/2}}{2^{f(\vartheta_i)/2}\,(f(\vartheta_i)/2)!}.$$

As f varies over all elements of $\mathcal{F}$, the numbers $f(\psi_1),\dots,f(\psi_r), f(\vartheta_1)/2,\dots,f(\vartheta_s)/2$ range over all compositions of k. Therefore, following substitutions in the preceding expressions, we obtain by the multinomial formula

$$\begin{aligned}
\sum_\theta \deg(\chi_\theta) &= \frac{(2k)!}{2^k k!}\sum_{f\in\mathcal{F}} k! \prod_{i=1}^{r} \frac{(2\deg(\psi_i)^2)^{f(\psi_i)}}{f(\psi_i)!}\prod_{i=1}^{s}\frac{(\deg(\vartheta_i)^2)^{f(\vartheta_i)/2}}{(f(\vartheta_i)/2)!} \\
&= \frac{(2k)!}{2^k k!}\left(\sum_{i=1}^{r} 2\deg(\psi_i)^2 + \sum_{i=1}^{s}\deg(\vartheta_i)^2\right)^k \\
&= \frac{(2k)!}{2^k k!}\left(\sum_{\psi\in\mathrm{Irr}(H)} \deg(\psi)^2\right)^k \\
&= \frac{|G_{2k}|}{|V_k^\tau|}.
\end{aligned}$$

Since this is precisely the degree of $\mathrm{Ind}_{V_k^\tau}^{G_n}(\mathbb{1})$, the given decomposition now follows by dimensional considerations. □

1.3.3 Construction of a model

With this proposition in hand, we can now construct a generalized involution model for G_n from any generalized involution model for H. As above, we fix an automorphism $\tau \in \mathrm{Aut}(H)$ with $\tau^2 = 1$. Throughout this section, we assume there exists a model for H given by a set of linear characters $\{\lambda_i : H_i \to \mathbb{C}\}_{i=1}^m$ for some positive integer m and some subgroups $H_i \subset H$.

Our notation is intended to coincide with that of [10] when $\tau = 1$. Let $\mathscr{U}_m$ denote the set of vectors $(x_0, x_1, \dots, x_m)$ with all entries nonnegative integers, and define

$$\mathscr{U}_m(n) = \left\{ x \in \mathscr{U}_m : 2x_0 + \sum_{i=1}^{m} x_i = n \right\}.$$

Let $\sigma_k^\tau : V_k^\tau \to \{\pm 1\}$ be the linear character given by

$$\sigma_k^\tau(h,\pi) = \mathrm{sgn}(\pi) \qquad \text{for } (h,\pi)\in V_k^\tau.$$

For each $x \in \mathscr{U}_m(n)$, we define a subgroup $G_x^\tau \subset G_n$ and a linear character $\phi_x^\tau : G_x^\tau \to \mathbb{C}$ by

$$G_x^\tau = V_{x_0}^\tau \times \prod_{i=1}^m (H_i \wr S_{x_i}) \qquad \text{and} \qquad \phi_x^\tau = \sigma_{x_0}^\tau \odot \bigodot_{i=1}^m (\lambda_i \wr (x_i)), \tag{1.3.3}$$

where on the right hand side (x_i) denotes the trivial partition in $\mathscr{P}(x_i)$ and we ignore terms corresponding to i if $x_i = 0$.

Given $x \in \mathscr{U}_m(n)$, define

$$\mathscr{R}(x) = \left\{ \theta \in \mathscr{P}_H(n) : \begin{array}{l} \text{for each } i > 0,\ x_i \text{ is the sum of the number of odd columns in} \\ \theta(\psi) \text{ as } \psi \text{ ranges over the irreducible constituents of } \operatorname{Ind}_{H_i}^H(\lambda_i) \end{array} \right\}.$$

We then have the following extension of [10, Theorem 1], which treats the special case $\tau = 1$.

Theorem 1.3.6. *Suppose $\epsilon_\tau(\psi) = 1$ for every irreducible character ψ of H. Then*

$$\operatorname{Ind}_{G_x^\tau}^{G_n}(\phi_x^\tau) = \sum_{\theta \in \mathscr{R}(x)} \chi_\theta \quad \text{for each } x \in \mathscr{U}_m(n),$$

and $\{\phi_x^\tau : G_x^\tau \to \mathbb{C}\}_{x \in \mathscr{U}_m(n)}$ is a model for $G_n = H \wr S_n$.

Proof. By the transitivity of induction we have

$$\operatorname{Ind}_{G_x^\tau}^{G_n}(\phi_x^\tau) = \operatorname{Ind}_{G_{2x_0} \times G_{x_1} \times \cdots \times G_{x_m}}^{G_n} \left(\operatorname{Ind}_{V_{x_0}^\tau}^{G_{2x_0}}(\sigma_{x_0}^\tau) \odot \bigodot_{i=1}^m \operatorname{Ind}_{H_i \wr S_{x_i}}^{G_{x_i}} (\lambda_i \wr (x_i)) \right). \tag{1.3.4}$$

Note that if $\theta \in \mathscr{P}_H(n)$, then $\chi_\theta \otimes \widetilde{\operatorname{sgn}} = \chi_{\theta'}$ where $\theta' \in \mathscr{P}_H(n)$ is defined by setting $\theta'(\psi)$ equal to the transpose of $\theta(\psi)$. Therefore, since $\epsilon_\tau(\psi) = 1$ for all $\psi \in \operatorname{Irr}(H)$, we have by Proposition 1.3.5 that $\operatorname{Ind}_{V_k^\tau}^{G_{2k}}(\sigma_k^\tau) = \operatorname{Ind}_{V_k^\tau}^{G_{2k}}(\mathbb{1}) \otimes \widetilde{\operatorname{sgn}} = \sum_\theta \rho_\theta$ where the sum ranges over all $\theta \in \mathscr{P}_H(n)$ such that $\theta(\psi)$ has all even columns for all $\psi \in \operatorname{Irr}(H)$. Also, [10, Proposition 1] states that $\operatorname{Ind}_{H_i \wr S_{x_i}}^{G_{x_i}} (\lambda_i \wr (x_i)) = \sum_\theta \chi_\theta$ where the sum is over all $\theta \in \mathscr{P}_H(x_i)$ such that $\theta(\psi)$ is the zero partition if ψ is not a constituent of $\operatorname{Ind}_{H_i}^H(\lambda_i)$ and a trivial partition otherwise.

Given these facts, we can completely decompose $\operatorname{Ind}_{G_x^\tau}^{G_n}(\phi_x^\tau)$ by using [10, Lemma 1], which shows that if ψ is a representation of H and $\alpha \in \mathscr{P}(a)$ and $\beta \in \mathscr{P}(b)$, then $\operatorname{Ind}_{G_a \times G_b}^{G_{a+b}} ((\psi \wr \alpha) \odot (\psi \wr \beta)) = \sum_{\gamma \in \mathscr{P}(a+b)} c_{\alpha,\beta}^\gamma (\psi \wr \gamma)$ where the coefficients $c_{\alpha,\beta}^\gamma$ are the non-negative integers afforded by the Littlewood-Richardson rule. Thus, after applying our substitutions to (1.3.4) we can invoke Young's rule to obtain the desired decomposition. $\square$

The automorphism $\tau \in \operatorname{Aut}(H)$ extends to an automorphism of H^n and of G_n via the definitions

$$\begin{aligned} {}^\tau(h_1, \ldots, h_n) &\overset{\text{def}}{=} ({}^\tau h_1, \ldots, {}^\tau h_n) && \text{for } (h_1, \ldots, h_n) \in H^n, \\ {}^\tau(h, \pi) &\overset{\text{def}}{=} ({}^\tau h, \pi) && \text{for } \pi \in S_n,\ h \in H^n. \end{aligned} \tag{1.3.5}$$

As in (1.3.2), let $\omega_k = (1,2)(3,4)\cdots(2k-1,2k) \in S_{2k}$, where by convention $\omega_0 = 1$. We now have the following generalization of [10, Theorem 2].

Theorem 1.3.7. Suppose $\{\lambda_i : H_i \to \mathbb{C}\}_{i=1}^m$ is a generalized involution model for H with respect to $\tau \in \mathrm{Aut}(H)$, so that there exists a set $\{\varepsilon_i\}_{i=1}^m$ of orbit representatives in $\mathcal{I}_{H,\tau}$ with $H_i = C_{H,\tau}(\varepsilon_i)$. For each $x \in \mathscr{U}_m(n)$, define

$$\varepsilon_x = \left((\underbrace{1,\dots,1}_{2x_0 \text{ times}}, \underbrace{\varepsilon_1,\dots,\varepsilon_1}_{x_1 \text{ times}}, \underbrace{\varepsilon_2,\dots,\varepsilon_2}_{x_2 \text{ times}}, \dots, \underbrace{\varepsilon_m,\dots,\varepsilon_m}_{x_m \text{ times}}), \omega_{x_0} \right) \in G_n.$$

If we extend τ to an automorphism of G_n by (1.3.5), then the linear characters $\{\phi_x^\tau : G_x^\tau \to \mathbb{C}\}_{x\in\mathscr{U}_m(x)}$ form a generalized involution model for G_n with respect to τ.

Proof. By Theorem 1.2.1, we have $\epsilon_\tau(\psi) = 1$ for all $\psi \in \mathrm{Irr}(H)$. Since $\{\lambda_i\}_{i=1}^m$ is a model for H, it follows from Theorem 1.3.6 that $\{\phi_x^\tau\}_{x\in\mathscr{U}_m(x)}$ is a model for G_n. To show that this model is a generalized involution model, we must prove both of the following:

(1) For each $x \in \mathscr{U}_m(n)$, the group G_x^τ is the τ-twisted centralizer in G_n of $\varepsilon_x \in \mathcal{I}_{G_n,\tau}$.

(2) The set $\{\varepsilon_x\}_{x\in\mathscr{U}_m(n)}$ contains exactly one element from each orbit in $\mathcal{I}_{G_n,\tau}$.

To this end, fix $x \in \mathscr{U}_m(n)$ and let $\varepsilon_x' \in H^n$ be the element with $\varepsilon_x = (\varepsilon_x', \omega_{x_0})$. Since $\varepsilon_x' \cdot {}^\tau\varepsilon_x' = 1 \in H^n$ and $\omega_{x_0}(\varepsilon_x') = \varepsilon_x'$ by assumption, we have $\varepsilon_x \in \mathcal{I}_{G_n,\tau}$.

Next, let $\pi \in S_n$ and $h \in H^n$ and consider the twisted conjugation of ε_x by the arbitrary element $g = (\pi^{-1}(h), \pi) \in G_n$. This gives

$$(k,\sigma) \overset{\text{def}}{=} g \cdot \varepsilon_x \cdot {}^\tau g^{-1} = \left(\pi\omega_{x_0}\pi^{-1}(h) \cdot \pi(\varepsilon_x') \cdot {}^\tau h^{-1}, \pi\omega_{x_0}\pi^{-1}\right). \tag{1.3.6}$$

Hence $g \in C_{G_n,\tau}(\omega_x)$ only if $\pi \in C_{S_n}(\omega_{x_0})$. Assume this, and define $J_0 = \{1,\dots,2x_0\}$ and $J_i = \{2x_0 + (x_1+\cdots+x_{i-1}) + j : 1 \le j \le x_i\}$ for $i = 1,\dots,m$. Then π permutes the sets J_0 and $J_1 \cup \cdots \cup J_m$, so

$$k_j = \begin{cases} h_{j'} \cdot {}^\tau h_j^{-1}, & \text{if } j \in J_0, \text{ where } j' = \omega_{x_0}(j); \\ h_j \cdot \varepsilon_i \cdot {}^\tau h_j^{-1}, & \text{otherwise, where } i \text{ is the unique index with } \pi^{-1}(j) \in J_i. \end{cases} \tag{1.3.7}$$

It follows from the first case in this identity that $k = \varepsilon_x'$ only if $h_{j'} = {}^\tau h_j$ for all $j \in J_0$. It follows from the second case that if $j \in J_1 \cup \cdots \cup J_m$ then k_j lies in the H-orbit of ε_i, where i is the unique index with $\pi^{-1}(j) \in J_i$. Thus, $k = \varepsilon_x'$ only if π also permutes each of the sets J_i and $h_j \in C_{H,\tau}(\varepsilon_i) = H_i$ for all $j \in J_i$ and $i = 1,\dots,m$. Combining these observations, we see that $g \in C_{G_n,\tau}(\varepsilon_x)$ only if $g \in G_x^\tau$. The reverse implication follows easily, and so we have $C_{G_n,\tau}(\varepsilon_x) = G_x^\tau$.

It remains to show that the elements ε_x for $x \in \mathscr{U}_m(n)$ represent the distinct τ-twisted conjugacy classes in $\mathcal{I}_{G_n,\tau}$. This requires a straightforward but tedious calculation, similar to the one in the previous paragraph, which we omit. $\square$

We conclude this section with an observation on how to construct a Gelfand model for G_n from a generalized involution model for H. To make our notation more concise, we adopt the following convention: given $g = (h, \pi) \in G_n$, define $|g| \in S_n$ and $z_g : [n] \to H$ by

$$|g| = \pi \in S_n \qquad \text{and} \qquad z_g(i) = h_i \in H. \tag{1.3.8}$$

We can identify G_n with the set of $n \times n$ matrices which have exactly one nonzero entry in each row and column, and whose nonzero entries are elements of H. Viewing $g \in G_n$ as a matrix of this form, $|g|$ is the matrix given by replacing each nonzero entry of g with 1, and $z_g(i)$ is the value of the nonzero entry of the matrix g in the ith column.

In the following statement, it helps to recall the definition of sign_{S_n} from (1.2.3). The symbol τ continues to denote a fixed automorphism of H with $\tau^2 = 1$, which we have extended to an automorphism of G_n by (1.3.5). Also, $\mathbb{K}$ here denotes a fixed subfield of $\mathbb{C}$ and $\mathcal{V}_{H,\tau}$, $\mathcal{V}_{G,\tau}$ are the vector spaces over $\mathbb{K}$ defined by (1.2.1). Finally, we write $\mathrm{Fix}(\pi)$ for the set of fixed points of a permutation $\pi \in S_n$.

Proposition 1.3.8. Suppose $\mathrm{sign}_H : H \times \mathcal{I}_{H,\tau} \to \mathbb{K}$ is a function such that the map $\rho : H \to \mathrm{GL}(\mathcal{V}_{H,\tau})$ defined by

$$\rho(h)C_\omega = \mathrm{sign}_H(h, \omega) \cdot C_{h\cdot\omega\cdot{}^\tau h^{-1}} \qquad \text{for } h \in H,\ \omega \in \mathcal{I}_{H,\tau}$$

is a Gelfand model for H. Then the map $\rho_{n,H} : G_n \to \mathrm{GL}(\mathcal{V}_{G,\tau})$ defined by

$$\rho_{n,H}(g)C_\omega = \mathrm{sign}_{G_n}(g, \omega) \cdot C_{g\cdot\omega\cdot{}^\tau g^{-1}} \qquad \text{for } g \in G_n,\ \omega \in \mathcal{I}_{G_n,\tau},$$

where

$$\mathrm{sign}_{G_n}(g, \omega) = \mathrm{sign}_{S_n}(|g|, |\omega|) \prod_{i \in \mathrm{Fix}(|\omega|)} \mathrm{sign}_H(z_g(i), z_\omega(i))$$

is a Gelfand model for $G_n = H \wr S_n$.

Proof. By Lemma 1.2.2, H possesses a generalized involution model $\{\lambda_i : H_i \to \mathbb{K}\}_{i=1}^m$ with respect to τ. Retaining the notation of Theorem 1.3.7, we may assume without loss of generality that $\lambda_i(h) = \mathrm{sign}_H(h, \varepsilon_i)$ for all $h \in H_i$ and $H_i = C_{H,\tau}(\varepsilon_i)$ for each $i = 1, \dots, m$. To prove that $\rho_{n,H}$ is a Gelfand model, it suffices by Lemma 1.2.2 to show only two things: that $\phi_x^\tau(g) = \mathrm{sign}_{G_n}(g, \varepsilon_x)$ for all $g \in G_x^\tau$ and each $x \in \mathscr{U}_m(n)$, and that $\rho_{n,H}$ is a representation.

To this end, fix $x \in \mathscr{U}_m(n)$ and consider $g \in H_i \wr S_{x_i}$. Since λ_i is a linear character, we have by Lemma 1.3.1 that

$$(\lambda_i \wr (x_i))(g) = \prod_{j=1}^{x_i} \lambda_i(z_g(j)) = \prod_{j=1}^{x_i} \mathrm{sign}_H(z_g(j), \varepsilon_i).$$

Thus if $g = (g_0, g_1, \ldots, g_m) \in G_x^\tau$, where $g_0 \in V_{x_0}^\tau$ and $g_i \in H_i \wr S_{x_i}$ for $i = 1, \ldots, m$, then

$$\begin{aligned}\phi_x^\tau(g) &= \sigma_{x_0}^\tau(g_0) \prod_{i=1}^{m} (\lambda_i \wr x_i)(g_i) \\ &= \mathrm{sign}_{S_n}(|g|, |\varepsilon_x|) \prod_{i \in \mathrm{Fix}(\omega_x)} \mathrm{sign}_H(z_g(i), z_{\omega_x}(i)) = \mathrm{sign}_{G_n}(g, \varepsilon_x).\end{aligned}$$

It remains to show that $\rho_{n,A}$ is a representation. Let $g, h \in G_n$ and $\omega \in \mathcal{I}_{G_n,\tau}$ and write $\omega' = h \cdot \omega \cdot {}^\tau h^{-1}$. First, by Lemma 1.2.4 we have

$$\mathrm{sign}_{S_n}(|g|, |\omega'|) \cdot \mathrm{sign}_{S_n}(|h|, |\omega|) = \mathrm{sign}_{S_n}(|gh|, |\omega|). \tag{1.3.9}$$

Now let $\pi = |h|$. Choose $i \in \mathrm{Fix}(|\omega|)$ and observe that $\pi(i) \in \mathrm{Fix}(|\omega'|)$. It follows from the fact that $\omega \cdot {}^\tau\omega = \omega' \cdot {}^\tau\omega' = 1$ that both $z_\omega(i)$ and $z_{\omega'} \circ \pi(i)$ belong to $\mathcal{I}_{H,\tau}$. Furthermore, one can check that

$$z_g \circ \pi(i) \cdot z_h(i) = z_{gh}(i) \qquad \text{and} \qquad z_{\omega'} \circ \pi(i) = z_h(i) \cdot z_\omega(i) \cdot {}^\tau z_h(i)^{-1}.$$

Since $\mathrm{sign}_H(a, b \cdot x \cdot {}^\tau b^{-1}) \cdot \mathrm{sign}_H(b, x) = \mathrm{sign}_H(ab, x)$ for $a, b \in H$ and $x \in \mathcal{I}_{H,\tau}$, it follows that

$$\mathrm{sign}_H(z_g \circ \pi(i), z_{\omega'} \circ \pi(i)) \cdot \mathrm{sign}_H(z_h(i), z_\omega(i)) = \mathrm{sign}_H(z_{gh}(i), z_\omega(i)). \tag{1.3.10}$$

Since $\mathrm{Fix}(|\omega'|) = \{\pi(i) : i \in \mathrm{Fix}(|\omega|)\}$, combining the identities (1.3.9) and (1.3.10) shows that $\mathrm{sign}_{G_n}(g, \omega') \cdot \mathrm{sign}_{G_n}(h, \omega) = \mathrm{sign}_{G_n}(gh, \omega)$, which suffices to show that $\rho_{n,H}$ is a representation, and therefore a Gelfand model. $\square$

1.3.4 Applications

We now construct a generalized involution model and a Gelfand model for $G_n = H \wr S_n$ when H is abelian. This gives a simple proof of [2, Theorem 1.2], which asserts that the representation $\rho_{r,n}$ from the introduction is a Gelfand model for G_n in the special case that H is the cyclic group of order r. Using Theorem 1.3.6, we prove some facts concerning the decomposition of this representation into irreducible constituents, and in so doing prove a conjecture of Adin, Postnikov, and Roichman from [2].

Throughout this section, let A be a finite abelian group and let $\tau \in \mathrm{Aut}(A)$ be the automorphism defined by ${}^\tau a = a^{-1}$. For this particular case, we note that

$$\begin{aligned} &\mathcal{I}_A = \{a \in A : a^2 = 1\}, && C_A(a) = \{b \in A : bab^{-1} = a\} = A, \\ &\mathcal{I}_{A,\tau} = \{a \in A : a \cdot {}^\tau a = 1\} = A, && C_{A,\tau}(a) = \{b \in A : b \cdot a \cdot {}^\tau b^{-1} = a\} = \mathcal{I}_A. \end{aligned}$$

The automorphism τ gives rise to the following generalized involution model for A.

Lemma 1.3.9. If A is abelian, then the set $\mathrm{Irr}(\mathcal{I}_A)$ of all irreducible characters of the subgroup $\mathcal{I}_A = \{a \in A : a^2 = 1\}$ forms a generalized involution model for A with respect to the automorphism $\tau : a \mapsto a^{-1}$. In particular, for each $\lambda \in \mathrm{Irr}(\mathcal{I}_A)$, the induced character $\mathrm{Ind}_{\mathcal{I}_A}^A(\lambda)$ is the sum of all $\psi \in \mathrm{Irr}(A)$ with $\mathrm{Res}_{\mathcal{I}_A}^A(\psi) = \lambda$.

Remark. This generalized involution model is clearly unique, up to the arbitrary assignment of irreducible representations of $\mathcal{I}_A$ to orbits in $\mathcal{I}_{A,\tau}$, since we must have $\mathcal{I}_{A,\tau} = A$ as the degree of any Gelfand model for A is $|A|$.

Proof. Since $\mathcal{I}_{A,\tau} = A$ and $\mathcal{I}_A = C_{A,\tau}(a)$ for every $a \in A$, there are $|\mathcal{I}_A|$ distinct twisted conjugacy classes in $\mathcal{I}_{A,\tau}$ and so each irreducible character of $\mathcal{I}_A$ can be viewed as a linear character of the τ-twisted centralizer of a representative of a distinct orbit in $\mathcal{I}_{A,\tau}$. The claimed decomposition of $\mathrm{Ind}_{\mathcal{I}_A}^A(\lambda)$ is immediate by Frobenius reciprocity, and since each element of $\mathrm{Irr}(A)$ restricts to an element of $\mathrm{Irr}(\mathcal{I}_A)$, our assertion follows. $\square$

Seeing this result, we naturally want to use Proposition 1.3.8 to obtain a Gelfand model for the wreath product $A \wr S_n$. In order to do this, we must first define a function $\mathrm{sign}_A : A \times A \to \mathbb{C}$ which corresponds to the generalized involution model for A just described. We will define this function in two different ways: first from a completely abstract standpoint which does depend on the structure of A, and then with an explicit construction which relies on a given decomposition of A as a direct product of cyclic groups.

For our first definition, we must introduce a few pieces of notation to keep track of our arbitrary but unspecified sets of orbit representatives. Let $B = \{a^2 : a \in A\}$ and observe that the cosets of this subgroup in A are precisely the orbits in $\mathcal{I}_{A,\tau}$ under the twisted conjugacy action $a : x \mapsto a \cdot x \cdot {}^{\tau}a^{-1} = a^2 x$. Fix a bijection between A/B and $\mathrm{Irr}(\mathcal{I}_A)$, and for each $x \in A$, let $\lambda_x : \mathcal{I}_A \to \mathbb{C}$ denote the linear character corresponding to the orbit xB. Now choose two maps

$$\widetilde{s}_{\mathrm{orb}} : A/B \to A \qquad \text{and} \qquad \widetilde{s} : A/\mathcal{I}_A \to A$$

assigning representatives to the cosets of B and $\mathcal{I}_A$ in A, and let

$$s_{\mathrm{orb}}(a) = \widetilde{s}_{\mathrm{orb}}(aB) \qquad \text{and} \qquad s(a) = \widetilde{s}(a\mathcal{I}_A), \qquad \text{for } a \in A.$$

The image of s_{orb} is then a set of orbit representatives in A, which explains our notation. Our next definition is our most complicated: let $q : A \to A$ be the map

$$q(a) = \widetilde{s}\left(\left\{b \in A : s_{\mathrm{orb}}(a) \cdot b^2 = a\right\}\right), \qquad \text{for } a \in A,$$

The set $\{b \in A : s_{\mathrm{orb}}(a) \cdot b^2 = a\}$ is a coset of $\mathcal{I}_A$ in A and so the map q is well-defined. We can think of the value of $q(a)$ as the square root of a modulo B. In the case that A is cyclic, q has a much more direct formula which we will compute.

We now define $\mathrm{sign}_A : A \times A \to \mathbb{C}$ as the function

$$\mathrm{sign}_A(a, x) = \lambda_x\left(a \cdot q(x) \cdot s\left(a \cdot q(x)\right)^{-1}\right) \tag{1.3.11}$$

and let $\rho_A : A \to \mathrm{GL}(\mathcal{V}_{A,\tau})$ be the map given by

$$\rho_A(a)C_x = \mathrm{sign}_A(a,x) \cdot C_{a^2x}, \qquad \text{for } a, x \in A. \tag{1.3.12}$$

These definitions come with the following result.

Proposition 1.3.10. The map ρ_A defines a Gelfand model for the abelian group A.

Proof. If $a \in \mathcal{I}_A$, then $s(a \cdot q(x)) = s(q(x)) = q(x)$ and so $\mathrm{sign}_A(a,x) = \lambda_x(a)$. Therefore, by Lemma 1.2.2 and the preceding lemma, it suffices to show that ρ_A is a representation. For this, fix $a, b, x \in A$ and observe that $q(b^2x) = s(b \cdot q(x))$ since

$$s_{\mathrm{orb}}(x) \cdot (b \cdot q(x))^2 = b^2 \cdot s_{\mathrm{orb}}(x) \cdot q(x)^2 = b^2x.$$

In addition, since $s(c)\mathcal{I}_A = c\mathcal{I}_A$ for all $c \in A$, we have $s\left(a \cdot s(b \cdot q(x))\right) = s\left(ab \cdot q(x)\right)$. Thus, since $\lambda_x = \lambda_{b^2x}$ by construction, $\mathrm{sign}_A(a, b^2x) = \lambda_x\left(a \cdot s(b \cdot q(x)) \cdot s\left(ab \cdot q(x)\right)^{-1}\right)$ and so $\mathrm{sign}_A(b,x) \cdot \mathrm{sign}_A(a, b^2x) = \mathrm{sign}_A(ab, x)$, which suffices to show that ρ_A is a representation. □

Using this abstract formulation, we can provide a concrete definition of sign_A using the structure of A as a finite abelian group. For any two integers $a \leq b$, let

$$[a,b] = \{i \in \mathbb{Z} : a \leq i \leq b\} \qquad \text{and define} \qquad [n] = [1,n].$$

Identify the cyclic group $\mathbb{Z}_r$ with the set $[0, r-1]$ so that the group operation is addition modulo r, and define a function $\mathrm{sign}_r : \mathbb{Z}_r \times \mathbb{Z}_r \to \{\pm 1\}$ by

$$\mathrm{sign}_r(a,x) = \begin{cases} -1 & \text{if } r \text{ is even and there exists } k \in [0, r/2-1] \\ & \text{with } x = 2k+1 \text{ and } a + k \in [r/2, r-1] \\ 1 & \text{otherwise} \end{cases} \qquad \text{for } a, x \in \mathbb{Z}_r.$$

If $A = \prod_{i=1}^{k} \mathbb{Z}_{r_i}$ where each r_i is a prime power, then we define $\mathrm{sign}_A : A \times A \to \{\pm 1\}$ by

$$\mathrm{sign}_A(a,x) = \prod_{i=1}^{k} \mathrm{sign}_{r_i}(a_i, x_i), \quad \text{for } a = (a_1, \ldots, a_k) \in A,\ x = (x_1, \ldots, x_k) \in A. \tag{1.3.13}$$

Every finite abelian group is isomorphic to a direct product of this form which is unique up to rearrangement of factors, so the formula (1.3.13) is well-defined for all abelian groups. The definition (1.3.13) is just a special case of (1.3.11), which explains the following corollary.

Corollary 1.3.11. If A is abelian then the map ρ_A with sign_A defined by (1.3.13) is a Gelfand model.

Proof. It suffices to prove this when $A = \mathbb{Z}_r$ is cyclic, for this we only need to show that $\mathrm{sign}_A = \mathrm{sign}_r$ for some choice of the sections s_{orb} and s and of the arbitrary correspondence between orbits in $\mathcal{I}_{A,\tau}$ and irreducible representations of $\mathcal{I}_A$. If r is odd then this always happens since $\mathcal{I}_A = \{1\}$ so $\mathrm{sign}_A(a,x) = \mathrm{sign}_\tau(a,x) = 1$ for all $a, x \in A$. Suppose r is even. Then $\mathcal{I}_A = \{0, r/2\}$; the cosets $A/\mathcal{I}_A$ are $[0, r/2-1]$ and $[r/2, r-1]$; and the two orbits in $\mathcal{I}_{A,\tau} = A$ are given by the sets of odd and even integers in $[0, r-1]$. Assign the trivial representation of $\mathcal{I}_A$ to the even orbit and the nontrivial representation to the odd orbit, so that the notation $\lambda_x : \mathcal{I}_A \to \mathbb{C}$ becomes

$$\lambda_x(0) = 1 \quad \text{and} \quad \lambda_x(r/2) = \begin{cases} 1 & \text{if } x \text{ is even;} \\ -1 & \text{if } x \text{ is odd;} \end{cases} \qquad \text{for } x \in A.$$

If we define the sections s_{orb} and s by

$$s_{\mathrm{orb}}(a) = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \text{ is odd} \end{cases} \qquad \text{and} \qquad s(a) = \begin{cases} a & \text{if } a \in [0, r/2-1] \\ a - r/2 & \text{if } a \in [r/2, r-1] \end{cases}$$

then the function $q : A \to A$ is given by the simple formula $q(a) = \lfloor a/2 \rfloor$ for $a \in A$, where the floor function takes its usual meaning for integers. It now follows by inspection that with respect to these choices, the definition (1.3.11) of sign_A matches sign_τ as required. □

We are now in a position to apply Proposition 1.3.8 to obtain a Gelfand model for the wreath product $G_n = A \wr S_n$. In particular, extend τ to an automorphism $\tau \in \mathrm{Aut}(G_n)$ by ${}^\tau(a,\pi) = (a^{-1}, \pi)$, and define a map $\rho_{n,A} : G_n \to \mathrm{GL}(\mathcal{V}_{G_n,\tau})$ by

$$\rho_{n,A}(g) C_\omega = \mathrm{sign}_{G_n}(g,\omega) \cdot C_{g \cdot \omega \cdot {}^\tau g^{-1}}, \qquad \text{for } g \in G_n,\ \omega \in \mathcal{I}_{G_n,\tau},$$

where

$$\mathrm{sign}_{G_n}(g,\omega) = \mathrm{sign}_{S_n}(|g|, |\omega|) \prod_{i \in \mathrm{Fix}(|\omega|)} \mathrm{sign}_A(z_g(i), z_\omega(i)).$$

Here sign_{S_n} is given by (1.2.3) and sign_A is given by either (1.3.11) or (1.3.13). The following theorem is now immediate from Proposition 1.3.8 and the preceding two results.

Theorem 1.3.12. The map $\rho_{n,A}$ defines a Gelfand model for $G_n = A \wr S_n$ when A is abelian.

By restating this theorem in slightly greater detail in the special case that A is cyclic, we can provide an alternate proof of [2, Theorem 1.2]. For this, we view $\mathbb{Z}_r$ as the additive group of integers $[0, r-1]$, so that

$$(a,\pi)(b,\sigma) = (\sigma^{-1}(a) + b, \pi\sigma), \qquad \text{for } (a,\pi), (b,\sigma) \in \mathbb{Z}_r \wr S_n. \tag{1.3.14}$$

We let $(a,\pi)^T = (-a,\pi)^{-1} = (\pi(a), \pi^{-1})$ for $(a,\pi) \in \mathbb{Z}_r \wr S_n$ and define

$$\mathcal{V}_{r,n} = \mathbb{Q}\text{-span}\left\{ C_\omega : \omega \in \mathbb{Z}_r \wr S_n,\ \omega^T = \omega \right\}.$$

Observe that $g^T = {}^\tau g^{-1}$ for $g \in \mathbb{Z}_r \wr S_n$, where τ is the automorphism ${}^\tau(a,\pi) = (-a,\pi)$. Therefore $\mathcal{V}_{r,n} = \mathcal{V}_{G,\tau}$ with $G = \mathbb{Z}_r \wr S_n$ in our earlier notation. Also, if we view elements of the wreath product $\mathbb{Z}_r \wr S_n$ as generalized permutation matrices, then g^T is to the usual matrix transpose of g. An element $g \in \mathbb{Z}_r \wr S_n$ is *symmetric* or an *absolute involution* if $g^T = g$.

Recall the definition of $|g|$ and z_g for $g \in \mathbb{Z}_r \wr S_n$ from (1.3.8). The following notation comes from [2, Definitions 6.1 and 6.3]. For $g, \omega \in \mathbb{Z}_r \wr S_n$, let $B(g,\omega)$ denote the subset of $[n]$ given by

$$B(g,\omega) = \begin{cases} \varnothing & \text{if } r \text{ is odd;} \\ \left\{ i \in \mathrm{Fix}(|\omega|) : \begin{array}{l} z_\omega(i) \text{ is odd and } z_g(i) + k \in [r/2, r-1] \\ \text{for the } k \in [0, r/2-1] \text{ with } 2k+1 = z_\omega(i) \end{array} \right\} & \text{if } r \text{ is even.} \end{cases}$$

Next define

$$\mathrm{sign}_{r,n}(g,\omega) = (-1)^{|B(g,\omega)|} \cdot (-1)^{|\mathrm{Inv}(|g|) \cap \mathrm{Pair}(|\omega|)|}$$

with $\mathrm{Inv}(\cdot)$ and $\mathrm{Pair}(\cdot)$ given as in Section 1.2.2, and let $\rho_{r,n} : \mathbb{Z}_r \wr S_n \to \mathrm{GL}(\mathcal{V}_{r,n})$ be the map given by

$$\rho_{r,n}(g) C_\omega = \mathrm{sign}_{r,n}(g,\omega) \cdot C_{g\omega g^T}, \qquad \text{for } g, \omega \in \mathbb{Z}_r \wr S_n \text{ with } \omega^T = \omega. \tag{1.3.15}$$

The map $\rho_{r,n}$ is precisely the representation $\rho_{n,A}$ above with $A = \mathbb{Z}_r$ and $\mathrm{sign}_A = \mathrm{sign}_r$, and one can check that our definition of $\mathrm{sign}_{r,n}$ agrees with the one given on generators in the introduction. We thus obtain the following corollary, which appears as [2, Theorem 1.2].

Corollary 1.3.13 (Adin, Postnikov, Roichman [2]). The map $\rho_{r,n}$ defines a Gelfand model for the wreath product $\mathbb{Z}_r \wr S_n$.

By directly applying Theorem 1.3.7 to Lemma 1.3.9, we can explicitly describe the generalized involution model for $\mathbb{Z}_r \wr S_n$ whose existence is implicit in our construction of $\rho_{r,n}$. In this situation, it is convenient to identify $\mathbb{Z}_r$ with the multiplicative subgroup of $\mathbb{C}^\times$ given by all rth roots of unity; thus $\mathbb{Z}_2 = \{\pm 1\}$. Let $\zeta_r = e^{2\pi i/r}$ be a primitive rth root of unity. We view $\mathbb{Z}_r \wr S_n$ as the multiplicative group of $n \times n$ generalized permutation matrices whose nonzero entries are taken from $\mathbb{Z}_r$. Given $g \in \mathbb{Z}_r \wr S_n$, let $|g|$ denote the permutation matrix given by replacing each entry of g with its absolute value, and let $z_g(i)$ for $i = 1, \dots, n$ denote the nonzero entry of g in its ith column. Under our previous conventions, the matrix g can then be identified with the abstract pair (x, π) where $\pi = |g| \in S_n$ and $x_i = z_g(i) \in \mathbb{Z}_r$ for $i = 1, \dots, n$. The matrix transpose g^T then coincides with our previous definition of the transpose.

For each $i \in [0, r-1]$, let $\psi_i : \mathbb{Z}_r \to \mathbb{C}$ denote the irreducible character

$$\psi_i(x) = x^i, \qquad \text{for } x \in \mathbb{Z}_r \text{ viewed as an element of } \mathbb{C}^\times,$$

so that $\mathrm{Irr}(\mathbb{Z}_r) = \{\psi_i : i \in [0, r-1]\}$. Additionally let $\mathscr{P}_r(n)$ denote the set of r-tuples $\theta = (\theta_0, \theta_1, \dots, \theta_{r-1})$ of partitions with $|\theta_0| + |\theta_1| + \cdots + |\theta_{r-1}| = n$. We refer to elements of $\mathscr{P}_r(n)$ as *r-partite partitions of n*. Define $\psi_i \wr \lambda$ for $i \in [0, r-1]$ and $\lambda \in \mathscr{P}$ as the character of $\mathbb{Z}_r \wr S_{|\lambda|}$ given by

$$(\psi_i \wr \lambda)(g) = \chi^\lambda(|g|) \left(\prod_{j=1}^{n} z_g(j) \right)^i \qquad \text{for } g \in \mathbb{Z}_r \wr S_{|\lambda|}.$$

One checks via Lemma 1.3.1 that this coincides with our constructions in Section 1.3.2 since $\mathbb{Z}_r$ is abelian and since $\det(g)/\det(|g|)$ is the product of the nonzero entries of generalized permutation matrix g. Now, following Theorem 1.3.2, each irreducible character of $\mathbb{Z}_r \wr S_n$ is of the form

$$\chi_\theta \overset{\text{def}}{=} \mathrm{Ind}_{S_\theta}^{\mathbb{Z}_r \wr S_n} \left(\bigodot_{i=0}^{r-1} \psi_i \wr \theta_i \right) \qquad \text{where } S_\theta = \prod_{i=0}^{r-1} \mathbb{Z}_r \wr S_{|\theta_i|} \tag{1.3.16}$$

for a unique $\theta \in \mathscr{P}_r(n)$. We refer to the r-partite partition θ of n as the *shape* of the irreducible character χ_θ. The shape of an irreducible $\mathbb{Z}_r \wr S_n$-representation is then the shape of its character.

We recall also the following additional definitions from Section 1.3.2:

$$\begin{aligned} \omega_k &= (1,2)(3,4)\cdots(2k-1,2k) \in S_{2k}, \\ V_k^\tau &= \left\{ g \in \mathbb{Z}_r \wr S_{2k} : |g| \in C_{S_n}(\omega_k),\ z_g(2i-1)\cdot z_g(2i) = 1 \text{ for all } i \right\}. \end{aligned}$$

The next pair of theorems says precisely how to construct $\rho_{r,n}$ by inducing linear representations.

Theorem 1.3.14. Assume r is odd. Then the wreath product $G(r,n) = \mathbb{Z}_r \wr S_n$ has a generalized involution model with respect to the automorphism $g \mapsto (g^{-1})^T$, given by the $1 + \lfloor n/2 \rfloor$ linear characters $\lambda_k : C_{G_n,\tau}(\varepsilon_k) \to \mathbb{Q}$ with $0 \le 2k \le n$, where

$$\begin{aligned} \varepsilon_k &= \begin{pmatrix} \omega_k & 0 \\ 0 & I_{n-2k} \end{pmatrix} \text{ for } 0 \le 2k \le n \text{ are orbit representatives in } \mathcal{I}_{G_n,\tau}, \\ C_{G_n,\tau}(\varepsilon_k) &= \left\{ g \overset{\text{def}}{=} \begin{pmatrix} \nu & 0 \\ 0 & \pi \end{pmatrix} : \nu \in V_k^\tau,\ \pi \in S_{r-2k} \right\}, \\ \lambda_k(g) &= \det(\nu) \text{ for } g \in C_{G_n,\tau}(\varepsilon_k). \end{aligned}$$

If $\theta \in \mathscr{P}_r(n)$ then the irreducible character χ_θ is a constituent of $\mathrm{Ind}_{C_{G_n,\tau}(\varepsilon_k)}^{G_n}(\lambda_k)$ if and only if the partitions $\theta_0, \theta_1, \ldots, \theta_{r-1}$ have $n - 2k$ odd columns in total.

Theorem 1.3.15. Assume r is even. Then the wreath product $G(r,n) = \mathbb{Z}_r \wr S_n$ has a generalized involution model with respect to the automorphism $g \mapsto (g^{-1})^T$, given by the

$\lceil \frac{n+1}{2} \rceil \cdot \lfloor \frac{n+3}{2} \rfloor$ linear characters $\lambda_{k,\ell} : C_{G_n,\tau}(\varepsilon_{k,\ell}) \to \mathbb{Q}$ with $0 \leq 2k + \ell \leq n$, where

$$\varepsilon_{k,\ell} = \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & I_{n-2k-\ell} & 0 \\ 0 & 0 & \zeta_r I_\ell \end{pmatrix} \text{ for } 0 \leq 2k+\ell \leq n \text{ are orbit representatives in } \mathcal{I}_{G_n,\tau},$$

$$C_{G_n,\tau}(\varepsilon_{k,\ell}) = \left\{ g \stackrel{\text{def}}{=} \begin{pmatrix} \nu & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} : \nu \in V_k^\tau,\ x \in \mathbb{Z}_2 \wr S_{n-2k-\ell},\ y \in \mathbb{Z}_2 \wr S_\ell \right\},$$

$$\lambda_{k,\ell}(g) = \det(\nu) \tfrac{\det(y)}{\det(|y|)} \text{ for } g \in C_{G_n,\tau}(\varepsilon_{k,\ell}).$$

If $\theta \in \mathscr{P}_r(n)$ then the irreducible character χ_θ is a constituent of $\mathrm{Ind}_{C_{G_n,\tau}(\varepsilon_k)}^{G_n}(\lambda_k)$ if and only if the partitions $\theta_0, \theta_2, \ldots, \theta_{r-2}$ have $n - 2k - \ell$ odd columns in total and the partitions $\theta_1, \theta_3, \ldots, \theta_{r-1}$ have ℓ odd columns in total.

The proofs of these results are straightforward exercises in translating the notations of Theorem 1.3.7 and Lemma 1.3.9. We prove only the second theorem, since the proof of the odd case is the same but less complicated.

Proof of Theorem 1.3.15. Let $\mathcal{I}_r = \mathbb{Z}_2 = \{\pm 1\}$ denote the subgroup of involutions in $\mathbb{Z}_r$, and define $\mathbb{1}, \chi : \mathcal{I}_r \to \mathbb{C}$ to be the trivial and nontrivial characters of $\mathcal{I}_r$, respectively. By Lemma 1.3.10,

$$\mathrm{Ind}_{\mathcal{I}_r}^{\mathbb{Z}_r}(\mathbb{1}) = \psi_0 + \psi_2 + \cdots + \psi_{r-2} \qquad \text{and} \qquad \mathrm{Ind}_{\mathcal{I}_r}^{\mathbb{Z}_r}(\chi) = \psi_1 + \psi_3 + \cdots + \psi_{r-1}.$$

As in Section 1.3.3, let $\mathscr{U}_2(n)$ denote the set of triples of nonnegative integers $x = (x_0, x_1, x_2)$ with $2x_0 + x_1 + x_2 = n$. For each $x \in \mathscr{U}_2(n)$ define $\phi_x^\tau : G_x^\tau \to \mathbb{C}$ by (1.3.3) and $\varepsilon_x \in G$ as in Theorem 1.3.7, where we take $H_1 = H_2 = \mathcal{I}_r$, define τ by ${}^\tau g = (g^{-1})^T$, set $\varepsilon_1 = 0 \in \mathbb{Z}_r$ and $\varepsilon_2 = 1 \in \mathbb{Z}_r$. By Theorems 1.3.6 and 1.3.7, the linear characters $\{\phi_x^\tau : x \in \mathscr{U}_2(n)\}$ form a generalized involution model for G_n, and χ_θ is a constituent of $\mathrm{Ind}_{G_x^\tau}^{G_n}(\phi_x^\tau)$ if and only if the partitions $\theta_0, \theta_2, \ldots, \theta_{r-2}$ have x_1 odd columns in total and the partitions $\theta_1, \theta_3, \ldots, \theta_{r-1}$ have x_2 odd columns in total. The theorem is immediate after noting that $\varepsilon_x = \varepsilon_{x_0,x_2}$ and $\phi_x^\tau = \lambda_{x_0,x_2}$ in the notation of the current theorem, which follows easily from the fact that the product of the nonzero entries of an invertible generalized permutation matrix g is precisely $\det(g)/\det(|g|)$. □

In the following corollary, let $2\mathbb{Z}_r = \langle \zeta_r^2 \rangle$, where $\zeta_r = e^{2\pi i/r}$ generates $\mathbb{Z}_r$. If r is odd then of course $2\mathbb{Z}_r = \mathbb{Z}_r$, while if r is even then $2\mathbb{Z}_r = \mathbb{Z}_{r/2} = \{1 = \zeta_r^0, \zeta_r^2, \ldots, \zeta_r^{r-2}\}$.

Corollary 1.3.16. Fix $\omega \in \mathbb{Z}_r \wr S_n$ such that $\omega = \omega^T$. Let

$$k = \text{the number of 2-cycles in } |\omega|,$$
$$\ell = \text{the number of } i \in \mathrm{Fix}(|\omega|) \text{ with } z_\omega(i) \notin 2\mathbb{Z}_r.$$

The character of the subrepresentation of $\rho_{r,n}$ generated by vector $C_\omega \in \mathcal{V}_{r,n}$ is then the sum $\sum_\theta \chi_\theta$ over all $\theta \in \mathscr{P}_r(n)$ such that

(i) When r is odd, the partitions $\theta_0, \theta_1, \ldots, \theta_{r-1}$ have $n-2k$ odd columns in total.

(ii) When r is even, the partitions $\theta_0, \theta_2, \ldots, \theta_{r-2}$ have $n-2k-\ell$ odd columns in total and the partitions $\theta_1, \theta_3, \ldots, \theta_{r-1}$ have ℓ odd columns in total.

Proof. This follows from the preceding theorem after checking that the orbit of ω under the twisted conjugacy action $g : \omega \mapsto g\omega g^T$ contains ε_k when r is odd and $\varepsilon_{k,\ell}$ when r is even. □

This corollary allows us to prove [2, Conjecture 7.1]. Recall the definition given above of an r-partite partition of n. One obtains an *r-partite standard Young tableau* of shape $\theta \in \mathscr{P}_r(n)$ by inserting the integers $1, 2, \ldots, n$ bijectively into the cells of the Ferrers diagrams of the partitions $\theta_0, \theta_1, \ldots, \theta_{r-1}$ so that entries increase along each row and column of each partition.

The natural subrepresentations considered in the preceding corollary have the following connection with the generalized Robinson-Schensted correspondence for wreath products due to Stanton and White [100]. Recall, for example from [97], that the usual Robinson-Schensted-Knuth (RSK) correspondence is a bijective map

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \xrightarrow{\text{RSK}} (P, Q)$$

from two-line arrays of lexicographically ordered positive integers to pairs of semistandard Young tableaux (P, Q) with the same shape. Viewing $\sigma \in S_n$ as the two-line array with $a_i = i$ and $b_i = \sigma(i)$, this map restricts to a bijection from permutations to pairs of standard Young tableaux with the same shape. Schützenberger proves in [99] that the RSK correspondence associates to each involution $\omega \in \mathcal{I}_{S_n}$ with f fixed points a pair of standard Young tableaux (P, Q) with $P = Q$ whose common shape has f odd columns.

To define Stanton and White's colored RSK correspondence for wreath products, fix an element $g \in \mathbb{Z}_r \wr S_n$ and for each $j \in [0, r-1]$, let (P_j, Q_j) be the pair of tableaux obtained by RSK correspondence applied to the array

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_\ell \\ \sigma(i_1) & \sigma(i_2) & \cdots & \sigma(i_\ell) \end{pmatrix} \tag{1.3.17}$$

where $\{i_1 < i_2 < \cdots < i_\ell\}$ is the set of $i \in [n]$ with $z_g(i) = \zeta_r^j$. The colored RSK correspondence is then the bijection from elements of $\mathbb{Z}_r \wr S_n$ to pairs of r-partite standard Young tableaux of the same shape defined by

$$g \longrightarrow (\boldsymbol{P}, \boldsymbol{Q}) = \Big((P_0, P_1, \ldots, P_{r-1}), (Q_0, Q_1, \ldots, Q_{r-1}) \Big).$$

To begin, we have the following easy corollary of Schützenberger's result.

Lemma 1.3.17. Fix $\omega \in \mathbb{Z}_r \wr S_n$ such that $\omega = \omega^T$ and suppose $\omega \mapsto (\boldsymbol{P}, \boldsymbol{Q})$ under the colored RSK correspondence. Then $\boldsymbol{P} = \boldsymbol{Q}$ and for each $j \in [0, r-1]$, the number of odd columns in the shape of P_j is equal to the cardinality of $\{i \in \mathrm{Fix}(|\omega|) : z_\omega(i) = \zeta_r^j\}$.

Proof. Since ω is a symmetric element, we have $z_\omega(i) = z_\omega(j)$ whenever i and j are in the same cycle of the involution $|\omega| \in S_n$. Therefore each array (1.3.17) corresponds to an involution in the group of permutations of the set $\{i_1, \dots, i_\ell\}$, and it follows by Schützenberger's result that $\boldsymbol{P} = \boldsymbol{Q}$ and the number of odd columns in the shape of P_j is as claimed. $\square$

We can now prove the theorem promised in the introduction.

Theorem 1.3.18. Let $\mathcal{X}$ be a set of symmetric elements in $\mathbb{Z}_r \wr S_n$. If the elements of $\mathcal{X}$ span a $\rho_{r,n}$-invariant subspace of $\mathcal{V}_{r,n}$, then the subrepresentation of $\rho_{r,n}$ on this space is equivalent to the multiplicity free sum of all irreducible $\mathbb{Z}_r \wr S_n$-representations whose shapes are obtained from the elements of $\mathcal{X}$ by the colored RSK correspondence.

Remark. Caselli and Fulci prove a similar result concerning the decomposition of a different Gelfand model for $\mathbb{Z}_r \wr S_n$ in [28]. Comparing the preceding theorem with [28, Theorem 1.2] shows that there exist abstract isomorphisms between various natural subrepresentations of these two Gelfand models.

The symmetric elements $\omega \in \mathbb{Z}_r \wr S_n$ whose underlying permutations $|\omega| \in S_n$ have a fixed number of 2-cycles form a union of twisted conjugacy classes with respect to the inverse transpose automorphsim, and so they span an invariant subspace of $\mathcal{V}_{r,n}$. Hence, this result implies [2, Conjecture 7.1].

Proof. It suffices to prove the theorem when $\mathcal{X} = \{g\omega g^T : g \in \mathbb{Z}_r \wr S_n\}$ is the orbit of some $\omega \in \mathbb{Z}_r \wr S_n$ with $\omega^T = \omega$. In this case, it follows by comparing Corollary 1.3.16 and Lemma 1.3.17 that the colored RSK correspondence defines an injective map from $\mathcal{X}$ to the set of r-partite standard Young tableaux whose shapes index irreducible constituents of the subrepresentation generated by $\mathcal{X}$. Since the number of such tableaux is equal to the cardinality of $\mathcal{X}$ due to the well-known fact that the number of r-partite standard Young tableaux of shape θ is equal to $\chi_\theta(1)$, this map is in fact a bijection, which proves the theorem. $\square$

We conclude by deriving two additional results which will be useful in the subsequent work [79]. Assume r is even. We then have two $\rho_{r,n}$-invariant subspaces of $\mathcal{V}_{r,n}$ given by

$$\begin{aligned} \mathcal{V}^+_{r,n} &= \mathbb{Q}\text{-span}\left\{C_\omega : \omega \in \mathbb{Z}_r \wr S_n,\ \omega^T = \omega,\ \det(\omega)/\det(|\omega|) \in 2\mathbb{Z}_r\right\}, \\ \mathcal{V}^-_{r,n} &= \mathbb{Q}\text{-span}\left\{C_\omega : \omega \in \mathbb{Z}_r \wr S_n,\ \omega^T = \omega,\ \det(\omega)/\det(|\omega|) \notin 2\mathbb{Z}_r\right\}. \end{aligned} \tag{1.3.18}$$

Let $\chi^+_{r,n}$ and $\chi^-_{r,n}$ denote the characters of $\mathbb{Z}_r \wr S_n$ corresponding to the subrepresentations of $\rho_{r,n}$ on $\mathcal{V}^+_{r,n}$ and $\mathcal{V}^-_{r,n}$ respectively.

Corollary 1.3.19. Let r, n be positive integers with r even. Given $\theta \in \mathscr{P}_r(n)$, define $\Omega(\theta)$ as the sum of the numbers of odd columns in the partitions $\theta_1, \theta_3, \ldots, \theta_{r-1}$. Then

$$\chi_{r,n}^{+} = \sum_{\substack{\theta \in \mathscr{P}_r(n), \\ \Omega(\theta) \text{ is even}}} \chi_\theta \qquad \text{and} \qquad \chi_{r,n}^{-} = \sum_{\substack{\theta \in \mathscr{P}_r(n), \\ \Omega(\theta) \text{ is odd}}} \chi_\theta.$$

Proof. Since $\det(\omega)/\det(|\omega|) \in 2\mathbb{Z}_r$ for a symmetric element $\omega \in \mathbb{Z}_r \wr S_n$ if and only if the union of the disjoint sets $\{i \in \mathrm{Fix}(|\omega|) : z_\omega(i) = \zeta_r^j\}$ over all odd $j \in [0, r-1]$ has even cardinality, this is immediate from Lemma 1.3.17 and Theorem 1.3.18. $\square$

Suppose p is a positive integer dividing r. Let $\gamma : \mathbb{Z}_r \wr S_n \to \mathbb{C}$ denote the linear character defined by

$$\gamma(g) = \left(\psi_{r/p} \wr (n)\right)(g) = \left(\prod_{i=1}^{n} z_g(i)\right)^{r/p} \qquad \text{for } g \in \mathbb{Z}_r \wr S_n. \tag{1.3.19}$$

Here (n) denotes the trivial partition of n. A straightforward calculation shows that for all $\theta \in \mathscr{P}_r(n)$ we have

$$\gamma \otimes \chi_\theta = \chi_{\theta'}, \qquad \text{where} \qquad \theta'_i = \theta_{i-r/p} \text{ for } i \in [0, r-1] \tag{1.3.20}$$

where with slight abuse of notation we define $\theta_{i-r} = \theta_i$ for $i \in [0, r-1]$. This observation leads to the following lemma.

Lemma 1.3.20. Let r, p, n be positive integers with r even and p dividing r. Then

$$\gamma \otimes \chi_{r,n}^{+} = \begin{cases} \chi_{r,n}^{-}, & \text{if } n \text{ and } r/p \text{ are odd,} \\ \chi_{r,n}^{+}, & \text{otherwise;} \end{cases} \qquad \gamma \otimes \chi_{r,n}^{-} = \begin{cases} \chi_{r,n}^{+}, & \text{if } n \text{ and } r/p \text{ are odd;} \\ \chi_{r,n}^{-}, & \text{otherwise.} \end{cases}$$

Proof. Recall the definition of Ω from Corollary 1.3.19 and let $\Omega'(\theta)$ for $\theta \in \mathscr{P}_r(n)$ be the sum of the numbers of odd columns in the partitions $\theta_0, \theta_2, \ldots, \theta_{r-2}$. Suppose r/p is odd; then (1.3.20) implies that the map $\chi \mapsto \gamma \otimes \chi$ exchanges the two sets

$$\{\chi_\theta : \theta \in \mathscr{P}_r(n),\ \Omega(\theta) \text{ is odd}\} \qquad \text{and} \qquad \{\chi_\theta : \theta \in \mathscr{P}_r(n),\ \Omega'(\theta) \text{ is odd}\}. \tag{1.3.21}$$

If n is odd, then $\theta \in \mathscr{P}_r(n)$ has $\Omega'(\theta)$ odd if and only if $\Omega(\theta)$ is even, and it follows immediately from Corollary 1.3.19 that $\gamma \otimes \chi_{r,n}^{\pm} = \chi_{r,n}^{\mp}$. If n is even, then $\theta \in \mathscr{P}_r(n)$ has $\Omega'(\theta)$ odd if and only if $\Omega(\theta)$ is odd, so the two sets in (1.3.21) are the same, and necessarily $\gamma \otimes \chi_{r,n}^{+} = \chi_{r,n}^{+}$. Alternatively, if r/p is even, then by (1.3.20) the map $\chi \mapsto \gamma \otimes \chi$ defines a permutation of the set of irreducible constituents of $\chi_{r,n}^{+}$ so $\gamma \otimes \chi_{r,n}^{+} = \chi_{r,n}^{+}$. Similar arguments show that $\gamma \otimes \chi_{r,n}^{-} = \chi_{r,n}^{-}$ if n or r/p is even. $\square$

1.4 Complex reflection groups

We devote the rest of this chapter to the problem of showing how and when the Gelfand model $\rho_{r,n}$ for $G(r,1,n) = \mathbb{Z}_r \wr S_n$ can be extended to the complex reflection groups $G(r,p,n)$. More generally, we will classify all finite complex reflection groups which have generalized involution models.

1.4.1 Definitions and notation

Before proceeding we recall a few general facts about complex reflection groups. As mentioned at the beginning of this chapter, every finite complex reflection group decomposes as a direct product of irreducible complex reflection groups. The finite irreducible complex reflection groups were identified through the work of a number of mathematicians in the nineteenth and first half of the twentieth century. Shephard and Todd completed this classification in their seminal paper [92]; a useful modern treatment of this material appears in [60].

The finite irreducible groups include one infinite family $G(r,p,n)$ and thirty-four exceptional groups labeled $G_4, \dots G_{37}$. Presentations for the exceptions as abstract groups appear in [21]. We can describe the infinite series of groups $G(r,p,n)$ more concretely. Let r,p,n be positive integers with p dividing r. As a subgroup of $\mathrm{GL}_n(\mathbb{C})$, the group $G(r,p,n)$ can be realized as the set of generalized permutation matrices whose nonzero entries are complex rth roots of unity, such that the product of the nonzero entries in any matrix is an (r/p)th root of unity. This group acts irreducibly on $\mathbb{C}^n$ when $r > 1$ and $(r,p,n) \neq (2,2,2)$, and on the codimension 1 subspace of $\mathbb{C}^n$ consisting of vectors whose coordinates sum to zero when $r = 1$ and $n > 1$.

The wreath product $\mathbb{Z}_r \wr S_n$ (viewed a group of matrices following the conventions set down after Corollary 1.3.13) is just $G(r,1,n)$, and henceforth we refer to elements of $G(r,1,n)$ and its subgroups $G(r,p,n)$ by the notation (x,π) exactly as for elements of $\mathbb{Z}_r \wr S_n$ in Section 1.3.4, with multiplication again given by (1.3.14). Likewise, every finite Coxeter group is a finite complex reflection group. The Coxeter groups of type A_n, B_n, D_n, G_2, and $I_2(n)$ appear within the infinite series as $G(1,1,n+1)$, $G(2,1,n)$, $G(2,2,n)$, $G(6,6,2)$, and $G(n,n,2)$ respectively. The remaining finite Coxeter groups of type H_3, F_4, H_4, E_6, E_7, and E_8 appear as the exceptional groups G_{23}, G_{28}, G_{30}, G_{35}, G_{36}, and G_{37}, respectively.

Given an element $g = (x,\pi) \in G(r,1,n)$, we define $|g| \in S_n$ and $z_g : [n] \to \mathbb{Z}_r$ and $\overline{g}, g^T \in G(r,1,n)$ in the same way as for $\mathbb{Z}_r \wr S_n$ above, by setting

$$|g| \overset{\text{def}}{=} \pi, \qquad z_g(i) \overset{\text{def}}{=} x_i, \qquad \overline{g} \overset{\text{def}}{=} (-x,\pi), \qquad \text{and} \qquad g^T \overset{\text{def}}{=} \overline{g^{-1}} = \left(\pi(x), \pi^{-1}\right). \quad (1.4.1)$$

We call g^T the *transpose* of g, and say that g is *symmetric* if $g = g^T$. After fixing a primitive rth root of unity ζ_r, it makes sense to view each element $(x,\pi) \in \mathbb{Z}_r \wr S_n$ as the $n \times n$ matrix with ζ^{x_i} in the position $(\pi(i), i)$ for $i = 1, \dots, n$ and zeros in all other positions. If we identify g with a generalized permutation matrix in this way, then g^T corresponds to the usual matrix transpose of g, and $\overline{g}$ is the complex conjugate of g. In particular, the map

$g \mapsto g^T$ is an anti-automorphism and $g \mapsto \overline{g} = (g^{-1})^T$ is an automorphism.

There is a homomorphism $\Delta : (\mathbb{Z}_r)^n \to \mathbb{Z}_r$ given by

$$\Delta(x) = x_1 + x_2 + \cdots + x_n, \qquad \text{for } x \in (\mathbb{Z}_r)^n.$$

This map extends to a homomorphism $\Delta : G(r,1,n) \to \mathbb{Z}_r$ by the formula $\Delta(x,\pi) = \Delta(x)$ for $x \in (\mathbb{Z}_r)^n$ and $\pi \in S_n$. Observe that $\Delta(\overline{g}) = \Delta(g^{-1}) = -\Delta(g)$ and $\Delta(g) = \Delta(g^T)$ for $g \in \mathbb{Z}_r \wr S_n$. Given positive integers r, p, n, the complex reflection group $G(r,p,n)$ is then the normal subgroup of $G(r,p,n)$ given by

$$G(r,p,n) = \{g \in \mathbb{Z}_r \wr S_n : \Delta(g) \in p\mathbb{Z}_r\},$$

where $p\mathbb{Z}_r$ denotes the subgroup $\{0, d, 2d, \dots, r-d\} \subset \mathbb{Z}_r$ generated by the greatest common divisor $d = \gcd(r,p)$. In particular, if $d = r$ then $p\mathbb{Z}_r = \{0\}$. To avoid redundancy in this definition, we from now on require that p divide r.

1.4.2 Irreducible characters

To understand the irreducible characters of $G(r,p,n)$ we begin from a more general standpoint. Throughout, let $\mathrm{Irr}(G)$ denote the set of irreducible characters of a finite group G. Now consider a finite group G with a normal subgroup H such that G/H is cyclic. Let $C \cong G/H$ denote the cyclic group of linear characters γ of G with $\ker\gamma \supset H$. Then the tensor product $\otimes$ defines an action of C on the irreducible characters of G, and we can say the following:

(i) Each irreducible character χ of G restricts to a multiplicity free sum of k irreducible characters of H, where k is the order of the stabilizer subgroup $\{\gamma \in C : \gamma \otimes \chi = \chi\}$. Furthermore, each irreducible character of H is a constituent of some such restriction.

(ii) If χ, ψ are irreducible characters of G, then the following are equivalent:

  (a) $\mathrm{Res}^G_H(\chi)$ and $\mathrm{Res}^G_H(\psi)$ share a common irreducible constituent.

  (b) $\mathrm{Res}^G_H(\chi) = \mathrm{Res}^G_H(\psi)$.

  (c) $\chi = \gamma \otimes \psi$ for some $\gamma \in C$.

These statements follow from Clifford theory; see [98, Section 6] for proofs.

Specializing to the case at hand, we fix r, p, n with p dividing r and let $G = G(r,1,n)$ and $H = G(r,p,n)$. Observe that H is a normal subgroup of G whose corresponding quotient group is cyclic and of order p. The irreducible characters of G are the functions χ_θ for $\theta \in \mathscr{P}_r(n)$ defined by (1.3.16). If $\gamma : G \to \mathbb{C}$ denotes the linear character given by (1.3.19), then $\ker\gamma \supset H$ and, since γ has order p in the group of linear characters of G, it follows that $C = \langle\gamma\rangle = \{\mathbb{1}, \gamma, \gamma^2, \dots, \gamma^{p-1}\}$ in the notation above. In light of (1.3.20), it follows for $i \in [0, p-1]$ and $\theta \in \mathscr{P}_r(n)$ that $\gamma^i \otimes \chi_\theta = \chi_\theta$ if and only if

$$\theta_j = \theta_{j-ir/p} = \theta_{j-2ir/p} = \cdots = \theta_{j-(p-1)ir/p}$$

for all j. If this holds then $\frac{p}{\gcd(p,i)}$ is a nontrivial divisor of both p and n since $\sum_{j=0}^{r-1} |\theta_j| = n$. Hence, if $\gcd(p,n) = 1$, then $\gamma^i \otimes \chi_\theta \neq \chi_\theta$ for all $0 < i < p$, so by the observations above we arrive at the following fact.

Observation 1.4.1. If $\gcd(p,n) = 1$, then each irreducible character of $G(r,p,n)$ is equal to the restriction of exactly p distinct irreducible characters of $G(r,1,n)$.

Concerning the irreducible characters of $G(r,p,n)$, we will make use of one additional result due to Caselli. The next theorem derives from the combination of [26, Proposition 4.4, Theorem 4.5, and Proposition 4.6].

Theorem 1.4.2 (Caselli [26]). Let r,p,n be positive integers with p dividing r. Then

$$\left|\left\{\omega \in G(r,p,n) : \omega^T = \omega\right\}\right| \leq \sum_{\psi \in \mathrm{Irr}(G(r,p,n))} \psi(1)$$

and we have equality if and only if $\gcd(p,n) \leq 2$.

1.5 Constructions for the infinite series

In this section we describe how one can obtain a generalized involution model for $G(r,p,n)$ in the cases where this is possible. We have two methods for doing this: by explicitly identifying the set of linear characters comprising our model, or by giving a Gelfand model of the particular form appearing in Lemma 1.2.2. We apply the second method when $\gcd(p,n) = 1$ and the first when $\gcd(p,n) = n = 2$ and r/p is odd. In Section 1.7 we will discover that $G(r,p,n)$ does not have a generalized involution model in any other cases.

1.5.1 Gelfand models with p and n coprime

Some work has been done on this topic. In [26], Caselli describes a representation for $G(r,p,n)$ in the complex vector space

$$\mathcal{V}^{\mathbb{C}}_{r,p,n} \stackrel{\text{def}}{=} \mathbb{C}\text{-span}\left\{C_\omega : \omega \in G(r,p,n),\ \omega^T = \omega\right\}$$

which defines a Gelfand model whenever equality obtains in Theorem 1.4.2, i.e., when $\gcd(p,n) \leq 2$. He calls such complex reflection groups *involutory*. Caselli's constructions do not arise from generalized involution models and require the field of complex numbers for their definition. The Gelfand models we present coincide with Caselli's only when $r = 1$, and are by contrast rational representations.

To give these, we begin by noting that the Gelfand model $\rho_{r,n}$ for $G(r,1,n)$ restricts to a representation of $G(r,p,n)$ for any p dividing r, and that one obvious subrepresentation of this restriction poses a natural candidate for a Gelfand model. Specifically, if we define $\mathcal{V}_{r,p,n} \subset \mathcal{V}_{r,n}$ as the subspace

$$\mathcal{V}_{r,p,n} = \mathbb{Q}\text{-span}\left\{C_\omega : \omega \in G(r,p,n),\ \omega^T = \omega\right\}$$

then since $G(r, p, n)$ is closed under taking transposes—as defined by (1.4.1)—the map $\rho_{r,p,n} : G(r, p, n) \to \mathrm{GL}(\mathcal{V}_{r,p,n})$ given by

$$\rho_{r,p,n}(g)C_\omega = \mathrm{sign}_{r,n}(g, \omega) \cdot C_{g\omega g^T}, \qquad \text{for } g, \omega \in G(r, p, n) \text{ with } \omega^T = \omega$$

is automatically a well-defined $G(r, p, n)$-representation. The following theorem says exactly when this representation is a Gelfand model.

Theorem 1.5.1. Let r, p, n be positive integers with p dividing r. Then the representation $\rho_{r,p,n}$ is a Gelfand model for $G(r, p, n)$ if and only if $\gcd(p, n) = 1$ and p or r/p is odd.

Proof. Let $G = G(r, 1, n)$ and $H = G(r, p, n)$. By Theorem 1.4.2, $\rho_{r,p,n}$ can only be a Gelfand model for H if $\gcd(p, n) = 1$ or 2, so we only consider those cases. View $\mathcal{V}_{r,n}$ as a G-module by defining $gC_\omega = \rho_{r,n}(g)C_\omega$ for $g \in G$, and for any $i \in \mathbb{Z}_r$, let $\mathcal{V}_{r,n}(i)$ denote the H-submodule

$$\mathcal{V}_{r,n}(i) = \mathbb{Q}\text{-span}\left\{C_\omega : \omega \in G,\ \omega^T = \omega,\ \Delta(\omega) - i \in p\mathbb{Z}_r\right\}.$$

Observe that $\mathcal{V}_{r,p,n} = \mathcal{V}_{r,n}(0)$ and that $\mathcal{V}_{r,n} = \mathcal{V}_{r,n}(0) \oplus \mathcal{V}_{r,n}(1) \oplus \cdots \oplus \mathcal{V}_{r,n}(p-1)$.

Suppose $\gcd(p, n) = 1$, and let $c \in G$ denote the central element

$$c = ((1, 1, \ldots, 1), 1) \in G \qquad \text{so that} \qquad c^i = ((i, i, \ldots, i), 1). \tag{1.5.1}$$

Observe that $\mathcal{V}_{r,n}(2ni + j) = c^i\mathcal{V}_{r,n}(j)$ and so $\mathcal{V}_{r,n}(j) \cong \mathcal{V}_{r,n}(2ni + j)$ as H-modules since c^i is central. Consequently, if p is odd then $\gcd(p, 2n) = 1$ and the H-modules $\mathcal{V}_{r,n}(i)$ are all isomorphic, since as i ranges over $0, 1, \ldots, p-1 \in \mathbb{Z}_r$, the elements $2ni$ represent every coset of $p\mathbb{Z}_r$ in $\mathbb{Z}_r$. In this case, it follows that an irreducible H-module $\mathcal{U}$ is a constituent of $\mathcal{V}_{r,p,n} = \mathcal{V}_{r,n}(0)$ with multiplicity m if and only if $\mathcal{U}$ is a constituent of $\mathcal{V}_{r,n}$ with multiplicity pm. Therefore if p is odd then $\rho_{r,p,n}$ is a Gelfand model for H since $\gcd(p, n) = 1$ implies that each irreducible H-module appears as a constituent of $\mathcal{V}_{r,n}$ with multiplicity p by Observation 1.4.1.

Suppose alternatively that $\gcd(p, n) = 1$ but p is even, so that n is odd. Then by the same considerations the H-modules $\mathcal{V}^+_{r,n}$ and $\mathcal{V}^-_{r,n}$ defined by (1.3.18) are isomorphic to $p/2$ copies of $\mathcal{V}_{r,n}(0)$ and $\mathcal{V}_{r,n}(1)$, respectively. Since every irreducible H-module is isomorphic to a constituent of $\mathcal{V}_{r,n}$ with multiplicity p as $\gcd(p, n) = 1$, it follows that every irreducible H-module is isomorphic to a constituent of $\mathcal{V}_{r,p,n} = \mathcal{V}_{r,n}(0)$ with multiplicity one if and only if $\mathcal{V}^+_{r,n} \cong \mathcal{V}^-_{r,n}$ as H-modules. We claim that this holds if and only if r/p is odd.

To show this, observe that $\mathcal{V}^+_{r,n} \cong \mathcal{V}^-_{r,n}$ as H-modules if and only if $\gamma \otimes \chi^+_{r,n} = \chi^-_{r,n}$, where γ is the character defined by (1.3.19). The "if" direction of this statement is immediate since γ restricts to the trivial character of H, and the other direction follows from Lemma 1.3.20, since if $\gamma \otimes \chi^+_{r,n} \neq \chi^-_{r,n}$ then $\gamma \otimes \chi^-_{r,n} = \chi^-_{r,n}$ which implies that no irreducible constituent of the nonzero H-module $\mathcal{V}^-_{r,n}$ appears as a constituent of $\mathcal{V}^+_{r,n}$. Since n is odd, Lemma 1.3.20 implies that $\gamma \otimes \chi^+_{r,n} = \chi^-_{r,n}$ if and only if r/p is odd, which proves our claim. Thus if $\gcd(p, n) = 1$ then $\rho_{r,p,n}$ is a Gelfand model for H if and only if p or r/p is odd.

To complete the proof, suppose $\gcd(p, n) = 2$ so that n and p are both even. Then r is even and it follows from Lemma 1.3.20 that $\gamma \otimes \chi^-_{r,n} = \chi^-_{r,n}$. Hence any irreducible constituent of the nonzero H-module $\mathcal{V}^-_{r,n}$ does not appear as a constituent of $\mathcal{V}^+_{r,n}$, or in the submodule $\mathcal{V}_{r,p,n} = \mathcal{V}_{r,n}(0)$, so $\rho_{r,p,n}$ cannot be a Gelfand model for H. $\square$

Suppose $\gcd(p, n) = 1$ but both p and r/p are even. Then while Theorem 1.5.1 does not hold, by modifying our construction slightly we can still produce a Gelfand model for $G(r, p, n)$ in $\mathcal{V}_{r,p,n}$. In this case, for every $\omega \in G(r, p, n)$ exactly one of the containments $\Delta(\omega) \in 2p\mathbb{Z}_r$ or $\Delta(\omega) - p \in 2p\mathbb{Z}_r$ holds. Thus, we may define $\widetilde{B}(g, \omega)$ for two elements $g, \omega \in G(r, p, n)$ as the subset of $[n]$ given by

$$\widetilde{B}(g,\omega) = \left\{ i \in \mathrm{Fix}(|\omega|) : \begin{array}{l} z_\omega(i) \text{ is odd and } z_g(i) + k \in [r/2, r-1] \\ \text{for the } k \in [0, r/2-1] \text{ with } 2k+1 = z_\omega(i) \end{array} \right\}$$

when $\Delta(\omega) \in 2p\mathbb{Z}_r$, and by

$$\widetilde{B}(g,\omega) = \left\{ i \in \mathrm{Fix}(|\omega|) : \begin{array}{l} z_\omega(i) \text{ is even and } z_g(i) + k \in [r/2, r-1] \\ \text{for the } k \in [0, r/2-1] \text{ with } 2k = z_\omega(i) \end{array} \right\}$$

when $\Delta(\omega) \notin 2p\mathbb{Z}_r$. Let $\widetilde{\rho}_{r,p,n} : G(r,p,n) \to \mathrm{GL}(\mathcal{V}_{r,p,n})$ be the map given by

$$\widetilde{\rho}_{r,p,n}(g) C_\omega = \widetilde{\mathrm{sign}}_{r,p,n}(g,\omega) \cdot C_{g\omega g^T}, \qquad \text{for } g, \omega \in G(r,p,n) \text{ with } \omega^T = \omega$$

where

$$\widetilde{\mathrm{sign}}_{r,p,n}(g,\omega) = (-1)^{|\widetilde{B}(g,\omega)|} \cdot (-1)^{|\mathrm{Inv}(|g|) \cap \mathrm{Pair}(|\omega|)|}$$

for $g, \omega \in G(r, 1, n)$. We then have the following result.

Theorem 1.5.2. Let r, p, n be positive integers with p dividing r. If $\gcd(p, n) = 1$ but r/p and p are both even, then $\widetilde{\rho}_{r,p,n}$ is a Gelfand model for $G(r, p, n)$.

In the following proof, it is helpful to note that if $c \in G(r, 1, n)$ is the central element defined by (1.5.1), then

$$\widetilde{\mathrm{sign}}_{r,p,n}(g,\omega) = \begin{cases} \mathrm{sign}_{r,n}(g,\omega) \cdot C_{g\omega g^T} & \text{if } \Delta(\omega) \in 2p\mathbb{Z}_r, \\ \mathrm{sign}_{r,n}(g,\omega c) \cdot C_{g\omega g^T} & \text{if } \Delta(\omega) - p \in 2p\mathbb{Z}_r. \end{cases}$$

Given this observation, one can check without difficulty that $\widetilde{\rho}_{r,p,n}$ is a well-defined representation when p and r/p are both even, using the fact that $\rho_{r,p,n}$ is a representation and that $\Delta(g\omega g^T) - \Delta(\omega) = 2\Delta(g) \in 2p\mathbb{Z}_r$ for all $g, \omega \in G(r, p, n)$.

Proof. Again let $G = G(r, 1, n)$ and $H = G(r, p, n)$, and view $\mathcal{V}_{r,p}$ as a G-module as in the proof of Theorem 1.5.1. Since $2p$ divides r, $\mathcal{V}_{r,p}$ decomposes into a direct sum of $2p$ distinct H-submodules as $\mathcal{V}_{r,n} = \widetilde{\mathcal{V}}_{r,n}(0) \oplus \widetilde{\mathcal{V}}_{r,n}(1) \oplus \cdots \oplus \widetilde{\mathcal{V}}_{r,n}(2p-1)$ where

$$\widetilde{\mathcal{V}}_{r,n}(i) = \mathbb{Q}\text{-span}\left\{ C_\omega : \omega \in G,\ \omega^T = \omega,\ \Delta(\omega) - i \in 2p\mathbb{Z}_r \right\}.$$

Defining $c \in G$ by (1.5.1), we again have $\widetilde{\mathcal{V}}_{r,n}(2ni+j) = c^i \widetilde{\mathcal{V}}_{r,n}(j)$ for all $i, j \in \mathbb{Z}_r$, so since $\gcd(2p,n) = 1$ as p is even, the H-modules $\mathcal{V}_{r,n}^+$ and $\mathcal{V}_{r,n}^-$ defined by (1.3.18) are isomorphic to p copies of $\widetilde{\mathcal{V}}_{r,n}(0)$ and $\widetilde{\mathcal{V}}_{r,n}(1)$, respectively. Since $\delta \otimes \mathcal{V}_{r,n}^{\pm} = \mathcal{V}_{r,n}^{\pm}$ by Lemma 1.3.20, the H-modules $\mathcal{V}_{r,n}^+$ and $\mathcal{V}_{r,n}^-$ do not share any irreducible constituents. Therefore, since each irreducible H-module appears as a constituent of $\mathcal{V}_{r,n}$ with multiplicity p by Observation 1.4.1, it follows that each irreducible H-module appears as a constituent of $\widetilde{\mathcal{V}}_{r,n}(0) \oplus \widetilde{\mathcal{V}}_{r,n}(1)$ with multiplicity one.

If we view $\mathcal{V}_{r,p,n}$ as an H-module by defining $gC_\omega = \widetilde{\rho}_{r,p,n}(g)C_\omega$ for $g \in H$, then $\mathcal{V}_{r,p,n}$ decomposes into H-submodules as $\mathcal{V}_{r,p,n} = \mathcal{V}_0 \oplus \mathcal{V}_1$ where

$$\mathcal{V}_0 = \mathbb{Q}\text{-span}\{C_\omega : \Delta(\omega) \in 2p\mathbb{Z}_r\} \qquad \text{and} \qquad \mathcal{V}_1 = \mathbb{Q}\text{-span}\{C_\omega : \Delta(\omega) - p \in 2p\mathbb{Z}_r\}.$$

By definition $\mathcal{V}_0 = \widetilde{\mathcal{V}}_{r,n}(0)$, and the linear map $\mathcal{V}_1 \to \widetilde{\mathcal{V}}_{r,n}(p+n)$ defined on basis elements by $C_\omega \mapsto C_{\omega c}$ is an isomorphism of H-modules. Since n is odd, $\widetilde{\mathcal{V}}_{r,n}(p+n) \cong \widetilde{\mathcal{V}}_{r,n}(1)$ as H-modules, and so we conclude that $\widetilde{\rho}_{r,p,n}$ is a Gelfand model. □

Since in the notation of the previous section $\mathcal{V}_{r,p,n}$ is precisely the vector space $\mathcal{V}_{G,\tau}$ with $\mathbb{K} = \mathbb{Q}$, $G = G(r,p,n)$, and $\tau \in \mathrm{Aut}(G)$ the inverse transpose automorphism $\tau : g \mapsto \overline{g}$, we are afforded the following corollary by Lemma 1.2.2.

Corollary 1.5.3. Let r, p, n be positive integers such that p divides r. Then $G(r,p,n)$ has a generalized involution model with respect to the inverse transpose automorphism $g \mapsto \overline{g}$ if $\gcd(p,n) = 1$.

One can form a generalized involution model for $G = G(r,p,n)$ by choosing a set of representatives $\{\omega\}$ for the τ-twisted conjugacy classes in $\mathcal{I}_{G,\tau}$, and then taking the linear characters $\lambda : C_{G,\tau}(\omega) \to \mathbb{Q}$ defined as the coefficients in $\mathbb{Q}$ such that $\rho(g)C_\omega = \lambda(g)C_\omega$ for those $g \in G$ with $g\omega g^T = \omega$, where ρ is our Gelfand model.

1.5.2 Models in rank two

We can only expect to be able to construct a generalized involution model for $G(r,p,n)$ when $\gcd(p,n) \leq 2$, and we will in fact be unable to do so when $\gcd(p,n) = 2$ in most cases. Here we deal with the one exception to this rule, occurring when $n = 2$ and r/p is odd. In contrast to the previous section, here we produce the generalized involution model directly.

Throughout this section, fix positive even integers r, p with p dividing r such that r/p is odd. We write $G = G(r,p,2)$ and let $\tau \in \mathrm{Aut}(G)$ denote the inverse transpose automorphism $\tau : g \mapsto \overline{g}$. Of immediate relevance is the following consequence of Theorem 1.2.1.

Lemma 1.5.4. Let $g = ((a,b),\pi) \in G$, so that $a, b \in \mathbb{Z}_r$ such that $a + b \in p\mathbb{Z}_r$ and $\pi \in S_2$. Then

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g) = \begin{cases} (r^2+2r)/p & \text{if } a = b = 0 \text{ and } \pi = 1, \\ 2r/p & \text{if } a = -b \in 2\mathbb{Z}_r \setminus \{0\} \text{ and } \pi = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.2}$$

Proof. By Theorems 1.2.1 and 1.4.2, it suffices to show that the right-hand side is equal to the number of $\omega \in G$ with $\omega \cdot {}^{\tau}\omega = g$. Let $\omega = ((x,y),\sigma) \in G$; then $(x,y) \in (\mathbb{Z}_r)^2$ can assume r^2/p distinct values. If $\sigma = 1$ then $\omega \cdot {}^{\tau}\omega = ((0,0),1)$ while if $\sigma \neq 1$ then $\omega \cdot {}^{\tau}\omega = ((y-x, x-y),1)$. As there are $2r/p$ choices of $(x,y) \in (\mathbb{Z}_r)^2$ such that $x+y \in p\mathbb{Z}_r$ and $x - y = b$ if $a, b \in 2\mathbb{Z}_r$ and zero choices if $a, b \notin 2\mathbb{Z}_r$, the lemma follows. $\square$

Let $s_1 \in S_2$ denote the simple reflection $s_1 = (1,2)$. One checks that the elements

$$\omega_1 = ((0,0),1), \qquad \omega_2 = ((1,-1),1), \qquad \omega_3 = ((0,0),s_1) \qquad \omega_4 = ((p/2,p/2),s_1)$$

represent the distinct τ-twisted conjugacy classes in $\mathcal{I}_{G,\tau}$, and that

$$C_{G,\tau}(\omega_1) = \left\{(0,1), \left((\tfrac{r}{2},\tfrac{r}{2}),1\right), (0,s_1), \left((\tfrac{r}{2},\tfrac{r}{2}),s_1\right)\right\} \cong S_2 \times S_2$$

and

$$C_{G,\tau}(\omega_2) = \left\{(0,1), \left((\tfrac{r}{2},\tfrac{r}{2}),1\right), ((-1,1),s_1), \left((\tfrac{r}{2}-1,\tfrac{r}{2}+1),s_1\right)\right\} \cong S_2 \times S_2$$

and

$$C_{G,\tau}(\omega_3) = C_{G,\tau}(\omega_4) = G(r,r,2).$$

Define linear characters $\lambda_i : C_{G,\tau}(\omega_i) \to \mathbb{Q}$ by

	$((0,0),1)$	$\left((\frac{r}{2},\frac{r}{2}),1\right)$	$((-1,1),s_1)$	$\left((\frac{r}{2}-1,\frac{r}{2}+1),s_1\right)$
λ_2	1	-1	-1	1

and

$$\lambda_1(g) = 1, \qquad \lambda_3(g) = \operatorname{sgn}(|g|), \qquad \lambda_4(g) = \operatorname{sgn}(|g|) \cdot (-1)^{z_g(1)}.$$

In the definition of λ_4 we are of course viewing $z_g(1) \in \mathbb{Z}_r$ as an integer in $[0, r-1]$; the given formula only makes sense because $n = 2$.

We now have the following result.

Proposition 1.5.5. Let r,p be even positive integers with p dividing r, such that r/p is odd. Then the linear characters $\lambda_i : C_{G,\tau}(\omega_i) \to \mathbb{Q}$ for $1 \leq i \leq 4$ form a generalized involution model for $G = G(r,p,2)$.

Proof. If we define $h_{ij} = ((ip+j, -j),1) \in G$ for $i,j \in \mathbb{Z}_r$ and let

$$\begin{aligned} \mathcal{C}_1 = \mathcal{C}_2 &= \{h_{ij} : i \in [0, r/p-1],\ j \in [0, r/2-1]\}, && \text{so that } |\mathcal{C}_1| = |\mathcal{C}_2| = r^2/(2p) \\ \mathcal{C}_3 = \mathcal{C}_4 &= \{h_{i0} : i \in [0, r/p-1]\}, && \text{so that } |\mathcal{C}_3| = |\mathcal{C}_4| = r/p \end{aligned}$$

then each $\mathcal{C}_i$ forms a set of left coset representatives of $C_{G,\tau}(\omega_i)$ in G. Let $g = ((a,b),\pi) \in G$ denote an arbitrary element of G with $a,b \in \mathbb{Z}_r$, $\pi \in S_2$ and $a + b \in p\mathbb{Z}_r$. Observe that

$$h_{ij} \cdot g \cdot (h_{ij})^{-1} = \begin{cases} ((a,b),1), & \text{if } \pi = 1, \\ ((a-ip-2j, b+ip+2j),s_1), & \text{if } \pi \neq 1. \end{cases}$$

Write $\Lambda_i = \mathrm{Ind}_{C_{G,\tau}(\omega_i)}^G(\lambda_i)$. Using the preceding observation with the Frobenius formula for induced characters, it is not difficult to check that:

(i) If $a+b \neq 0$ then $\Lambda_i(g) = 0$ for $1 \leq i \leq 4$.

(ii) If $a+b=0$ and $\pi = 1$ then

$$\Lambda_1(g) = \begin{cases} r^2/(2p), & \text{if } a = b \in \{0, r/2\}, \\ 0, & \text{otherwise,} \end{cases} \qquad \Lambda_3(g) = r/p,$$

$$\Lambda_2(g) = \begin{cases} r^2/(2p), & \text{if } a = b = 0, \\ -r^2/(2p), & \text{if } a = b = r/2, \\ 0, & \text{otherwise,} \end{cases} \qquad \Lambda_4(g) = \begin{cases} r/p, & \text{if } a \in 2\mathbb{Z}_r, \\ -r/p, & \text{if } a \notin 2\mathbb{Z}_r. \end{cases}$$

(iii) If $a+b=0$ and $\pi \neq 1$ then

$$\Lambda_1(g) = \begin{cases} 2r/p, & \text{if } a \in 2\mathbb{Z}_r \text{ and } r/2 \text{ is even,} \\ 0, & \text{if } a \notin 2\mathbb{Z}_r \text{ and } r/2 \text{ is even,} \\ r/p, & \text{if } r/2 \text{ is odd,} \end{cases} \qquad \Lambda_3(g) = -r/p,$$

$$\Lambda_2(g) = \begin{cases} 0, & \text{if } r/2 \text{ is even,} \\ r/p, & \text{if } a \in 2\mathbb{Z}_r \text{ and } r/2 \text{ is odd,} \\ -r/p, & \text{if } a \notin 2\mathbb{Z}_r \text{ and } r/2 \text{ is odd,} \end{cases} \qquad \Lambda_4(g) = \begin{cases} -r/p, & \text{if } a \in 2\mathbb{Z}_r, \\ r/p, & \text{if } a \notin 2\mathbb{Z}_r. \end{cases}$$

In turn, these formulas imply that $(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)(g)$ is precisely equal to the right-hand side of (1.5.2), which completes our proof. $\square$

1.6 Automorphisms for the infinite series

To prove that the groups $G(r,p,n)$ do not have generalized involution models other than in the situations addressed by the previous section, we require some understanding of these groups' automorphisms. In particular, we require a sufficiently explicit description of the elements of $\mathrm{Aut}\,(G(r,p,n))$ to be able to deduce precisely which automorphisms can satisfy the conditions of Theorem 1.2.1.

Marin and Michel provide in [84] several useful general results concerning the structure of $\mathrm{Aut}(G)$ when G is any finite complex reflection group. In particular, they prove that when G is an irreducible complex reflection group not equal to the symmetric group S_6, each automorphism of G is the composition of an automorphism which preserves the pseudo-reflections in G and a "central automorphism," by which we mean a map τ such that ${}^\tau g \cdot g^{-1}$ is always central. Letting V denote the vector space on which G acts irreducibly, Marin and Michel describe how each reflection-preserving automorphism can be interpreted as the composition of an automorphism induced from the normalizer of G in $\mathrm{GL}(V)$ and an automorphism induced from the Galois group of $\mathbb{K}$ over $\mathbb{Q}$, where $\mathbb{K}$ is the field of definition, i.e., the extension of $\mathbb{Q}$ generated by the traces of elements of G. They further discuss how to construct central automorphisms from the linear characters of G.

Marin and Michel's paper does not go as far as to actually write down the definitions of all the automorphisms in a very accessible fashion. Shi and Wang in the article [93] do write down explicit formulas, but only for the subgroup of reflection-preserving automorphisms of $G(r,p,n)$. From elementary considerations and without too much difficulty, one can give a complete and explicit description of $\mathrm{Aut}\,(G(r,p,n))$, and we provide this here for completeness. It is possible to glean many of these results from Shi and Wang's classification [93] and Marin and Michel's work [84]. The content of this section is, as such, to produce from a short, self-contained argument actual formulas for all the automorphisms of $G(r,p,n)$ which we can use to classify their generalized involution models.

In what follows, we denote by $\mathrm{Inn}(G)$ the group of inner automorphisms of a group G; by $\mathrm{Out}(G)$ the quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$; and by $Z(G)$ the center of G. Fix positive integers r,p,n with p dividing r. Let $e_i = (0,\dots,0,1,0,\dots,0) \in (\mathbb{Z}_r)^n$ denote the standard vector in the obvious free basis of $(\mathbb{Z}_r)^n$ over $\mathbb{Z}_r$, and define elements $s_i, s_i', s \in G(r,r,n)$ and $t, c \in G(r,1,n)$ by

$$\begin{aligned}
s_i &= (0,(i,i+1)) \ \text{ for } i=1,\dots,n-1,\\
s_i' &= (e_i - e_{i+1},(i,i+1)) \ \text{ for } i=1,\dots,n-1,\\
s &= (e_1-e_2,1)\,,\\
t &= (e_1,1)\,,\\
c &= (e_1+e_2+\cdots+e_n,1)\,.
\end{aligned} \tag{1.6.1}$$

Note that the elements s_1', s, are only defined for $r \geq 2$ and $n \geq 2$, and that $s_1' = s_1 s$. Also, observe that each s_i and s_i' has order 2 while s, t, c all have order r. In particular when $r=1$ we have $s=t=c=1$.

The group $G(r,p,n)$ is generated by s_1', s_1, ..., s_{n-1}, t^p or by s_1, ..., s_{n-1}, s, t^p; we can omit from these lists s_1' and s if $p=1$ and t^p if $p=r$. Broué, Malle, and Rouquier [21] give presentations for $G(r,p,n)$, as well as for the exceptional groups G_i, using the former set of generators; however, the latter set will be more convenient in many of our definitions.

For each integer j we note that

$$c^j = t^j \cdot \left(s_1 t^j s_1\right) \cdot \left(s_2 s_1 t^j s_1 s_2\right) \cdots \left(s_{n-1} \cdots s_2 s_1 t^j s_1 s_2 \cdots s_{n-1}\right) \in Z\left(G(r,p,n)\right). \tag{1.6.2}$$

The center of $G(r,p,n)$ almost always lies in the subgroup of $G(r,1,n)$ generated by c, as we note for later use in the following basic lemma.

Lemma 1.6.1. Let r,p,n be integers with p dividing r. If $d = \gcd(p,n)$ then

$$(\mathbb{Z}_r)^n \cap Z\left(G(r,p,n)\right) = \{c^{jp/d} : j \in [0, dr/p-1]\}.$$

This subgroup is equal to the center of $G(r,p,n)$ unless (r,p,n) is $(1,1,2)$ or $(2,2,2)$, in which case $G(r,p,n)$ is abelian.

Proof. We leave this easy exercise to the reader. □

To make our notation less cumbersome, we set

$$C(r,p,n) = (\mathbb{Z}_r)^n \cap Z\left(G(r,p,n)\right).$$

The following definition in some sense names all nontrivial outer automorphisms of $G(r,p,n)$. Given $j,k \in \mathbb{Z}$ and $z \in C(r,1,n)$, let $\alpha_{j,k,z} : G(r,1,n) \to G(r,1,n)$ be the map

$$\alpha_{j,k,z} : (x,\pi) \mapsto z^{\ell(\pi)} c^{\Delta(x)\cdot k}(jx,\pi), \qquad \text{for } x \in (\mathbb{Z}_r)^n,\ \pi \in S_n. \tag{1.6.3}$$

We recall that $\Delta : (\mathbb{Z}_r)^n \to \mathbb{Z}_r$ is the homomorphism $\Delta(x) = x_1 + x_2 + \dots x_n$. In our superscripts we naturally identify $\mathbb{Z}_r$ with the integers $[0,r-1]$ and view $\mathbb{Z}_r$ as a $\mathbb{Z}$-module. Since c has order r, this is well defined. Also, $\ell : S_n \to \mathbb{Z}_{\geq 0}$ denotes the usual length function, defined as the minimum number of factors needed to write a permutation as a product of the simple transpositions s_i, or equivalently the cardinality of a permutation's inversion set.

The map $\alpha_{j,k,z}$ has the following effect on our generators:

$$s_i \mapsto zs_i, \qquad s_i' \mapsto zs_i'(s_is_i')^{j-1}, \qquad s \mapsto s^j, \qquad t \mapsto c^k t^j, \qquad c \mapsto c^{j+nk}.$$

Observe that $\alpha_{1,0,1}$ is the identity and $\alpha_{-1,0,1}$ is the inverse transpose automorphism $g \mapsto \overline{g}$. The map $\alpha_{j,k,z}$ is often but not always an automorphism, as we see in the following lemma.

Lemma 1.6.2. Let r,p,n be positive integers with p dividing r. If $j,k \in \mathbb{Z}$ and $z \in C(r,1,n)$, then the map $\alpha_{j,k,z}$ restricts to an automorphism of $G(r,p,n)$ if and only if

$$\gcd(j,r) = \gcd(j+nk, r/p) = 1 \qquad \text{and} \qquad z \in C(r,p,n) \text{ and } z^2 = 1. \tag{1.6.4}$$

Proof. Assume $\alpha_{j,k,z}$ restricts to an automorphism of $G(r,p,n)$. The image zs_i of s_i then has order two and belongs to $G(r,p,n)$, so $z^2 = 1$ and $z = zs_i \cdot s_i \in G(r,p,n)$, which implies $z \in C(r,p,n)$ since $z \in C(r,1,n)$. Likewise, the image s^j of s has order r so $\gcd(j,r) = 1$, and the image $c^{p(j+nk)}$ of c^p has order r/p so $\gcd(j+nk, r/p) = 1$.

Conversely, suppose (1.6.4) holds. Since $c^p \in G(r,p,n)$ and $\Delta(jx) = j \cdot \Delta(x)$, it follows that $\alpha_{j,k,z}$ maps $G(r,p,n)$ into itself. One easily checks that $\alpha_{j,k,z}$ is a homomorphism using the following observations: c,z are central; if $\pi,\sigma \in S_n$ then $\ell(\pi) + \ell(\sigma) - \ell(\pi\sigma)$ is even; and $\Delta(\sigma(x)) = \Delta(x)$ for all $x \in (\mathbb{Z}_r)^n$ and $\sigma \in S_n$. It remains only to show that $\alpha_{j,k,z} : G(r,p,n) \to G(r,p,n)$ is bijective, and for this it suffices to show that $\alpha_{j,k,z}$ is surjective.

To prove this, we first observe that z is either the identity or the element $c^{r/2}$ when r is even and $nr/2$ is a multiple of p. Assume this latter case occurs; since $\Delta\left(\frac{r}{2}e_1 + \dots + \frac{r}{2}e_n\right) = nr/2$ we then have $\alpha_{j,k,z}(z) = c^{(j+nk)r/2}$. If r/p is even then $\gcd(j+nk, r/p) = 1$ implies that $j+nk$ is odd. If r/p is odd then p is even and $r/2$ is an odd multiple of $p/2$, so n must be even in order for $nr/2$ to be a multiple of p. Since r is even and $\gcd(j,r) = 1$, j is odd, so again $j+nk$ is odd. Hence $c^{(j+nk)r/2} = c^{r/2} = z$, and we conclude that in either case $\alpha_{j,k,z}(z) = z$.

Given this observation, it follows that $\alpha_{j,k,z}(zs_i) = s_i$ for all i. Furthermore, if j' is an integer such that $jj' \equiv 1 \pmod{r}$, then $\alpha_{j,k,z}(s^{j'}) = s$. Finally, if k' is an integer such that $(j+nk)k' \equiv -j'k \pmod{r/p}$, then

$$\alpha_{j,k,z}\left(c^{pk'}t^{pj'}\right) = c^{p(j+nk)k'} \cdot c^{pj'k}t^{pjj'} = t^p.$$

Since there exist such integers j', k' by assumption and since $s_1, \ldots s_{n-1}, s, t^p$ generate $G(r,p,n)$, it follows that our map is surjective and hence an automorphism. $\square$

Given $g \in G(r,1,n)$, let $\mathrm{Ad}(g) : x \mapsto gxg^{-1}$ denote the corresponding inner automorphism. Each such $\mathrm{Ad}(g)$ of course restricts to an automorphism of the normal subgroup $G(r,p,n)$, and with slight abuse of notation we regard $\mathrm{Ad}(g)$ for $g \in G(r,1,n)$ as an element of $\mathrm{Aut}\,(G(r,p,n))$. The following lemma gives a useful characterization of which maps $\mathrm{Ad}(g)$ restrict to elements of $\mathrm{Inn}\,(G(r,p,n))$.

Lemma 1.6.3. Let r,p,n be positive integers with p dividing r. If $g \in G(r,1,n)$, then the following are equivalent:

(i) $\mathrm{Ad}(g)$ restricts to an inner automorphism of $G(r,p,n)$.

(ii) $\mathrm{Ad}(g)(\pi)$ is conjugate to π in $G(r,p,n)$ for all $\pi \in S_n$.

(iii) $\Delta(g) \in d\mathbb{Z}_r$ where $d = \gcd(p,n)$.

Proof. The lemma is trivially true if $n = 1$ so assume $n \geq 2$. Clearly (i) implies (ii), so assume (ii) holds. Choose $a \in [0, p-1]$ such that $g = g't^a$ for some $g' \in G(r,p,n)$ and let $\pi = (1, 2, \cdots\ n)^{-1} \in S_n$. Then (ii) implies

$$\mathrm{Ad}(t^a)(\pi) = (-ae_1 + ae_2, \pi) = (x,\sigma)\pi(x,\sigma)^{-1}$$

for some $(x,\sigma) \in G(r,p,n)$. Conjugating both sides of this equation by σ^{-1} gives

$$\left(-ae_{\sigma^{-1}(1)} + ae_{\sigma^{-1}(2)}, \sigma^{-1}\pi\sigma\right) = \left((x_n - x_1)e_1 + (x_1 - x_2)e_2 + \cdots + (x_{n-1} - x_n)e_n, \pi\right). \tag{1.6.5}$$

Since $C_{S_n}(\pi) = \langle \pi \rangle$, we must have $\sigma = \pi^{i-2}$ for some $i \in [n]$. In this case $\sigma^{-1}(2) = i$ and $\sigma^{-1}(1) = i-1$ or n, and so if we make the abusive definition $x_{j-n} = x_j$ for $j \in [n]$, then equation (1.6.5) implies

$$x_{i-1} - x_i = a, \qquad x_{i-2} - x_{i-1} = -a, \qquad \text{and} \qquad x_{i-2} = x_{i-3} = \cdots = x_{i-n} \overset{\text{def}}{=} b \in \mathbb{Z}_r.$$

From these identities, one computes $a + bn = x_1 + x_2 + \cdots + x_n \in p\mathbb{Z}_r$, so we have $\Delta(g) + bn = \Delta(g') + a + bn \in p\mathbb{Z}_r \subset d\mathbb{Z}_r$. Since $bn \in d\mathbb{Z}_r$, (iii) follows.

Finally assume (iii) holds. Then since $d = ip + jn$ for some $i, j \in \mathbb{Z}$, it follows that (viewing $\mathbb{Z}_r$ as a $\mathbb{Z}$-module) $\Delta(g) - kn \in p\mathbb{Z}_r$ for some $k \in \mathbb{Z}$ in which case $gc^{-k} \in G(r,p,n)$. Since c is central, we have $\mathrm{Ad}(g) = \mathrm{Ad}(gc^{-k})$ which gives (i). $\square$

In almost all cases every automorphism of $G(r,p,n)$ arises by composing $\mathrm{Ad}(g)$ for some $g \in G(r,1,n)$ with some $\alpha_{j,k,z}$. To make this precise, we first require the next lemma.

Lemma 1.6.4. Let r, p, n be positive integers with p dividing r. Then every automorphism of $G(r,p,n)$ which preserves the normal subgroup $(\mathbb{Z}_r)^n \cap G(r,p,n) = \{g \in G(r,p,n) : |g| = 1\}$ is of the form

$$\mathrm{Ad}(g) \circ \alpha_{j,k,z}, \qquad \text{for some } g \in G(r,1,n) \text{ and } j,k,z \text{ as in (1.6.4)},$$

unless (r,p,n) is $(1,1,6)$.

The following elementary proof uses many of the same arguments as the proof of [84, Proposition 4.1], which describes a related but less specific result.

Proof. Let $G = G(r,p,n)$ and $N = (\mathbb{Z}_r)^n \cap G$, so that $G = N \rtimes S_n$. For the present we assume $n \neq 6$. Fix $\upsilon \in \mathrm{Aut}(G)$ and suppose $\upsilon(N) = N$. If $\mathrm{proj} : G \to S_n$ denotes the homomorphism $(x,\pi) \mapsto \pi$, then it follows that map $\pi \mapsto \mathrm{proj} \circ \upsilon(0,\pi)$ is an inner automorphism of S_n, and so $\upsilon = \mathrm{Ad}(\omega) \circ \upsilon'$ for some $\omega \in S_n$ and some $\upsilon' \in \mathrm{Aut}(G)$ with

$$\upsilon'(s_i) = (x_{i,1}e_1 + \cdots + x_{i,n}e_n, (i,i+1)) \qquad \text{for some choice of } x_{i,j} \in \mathbb{Z}_r \tag{1.6.6}$$

for $i = 1,\ldots,n-1$. Since $s_i^2 = 1$ we must have $x_{i,i} = -x_{i,i+1}$ and $2x_{i,j} = 0$ for $j \notin \{i,i+1\}$. Since s_1 and s_j commute for $j > 2$, inspection of the equal expressions $\upsilon'(s_1)$ and $\upsilon'(s_j)\upsilon'(s_1)\upsilon'(s_j)$ shows that $x_{1,3} = x_{1,4} = \cdots = x_{1,n}$. Therefore

$$\upsilon'(s_1) = zs^{a_1}s_1, \qquad \text{for some } a_1 \in \mathbb{Z}_r \text{ and some } z \in C(r,p,n) \text{ with } z^2 = 1.$$

The conjugacy class of $zs^{a_1}s_1$ consists of elements of the form $z\,(ae_i - ae_j, (i,j))$ for $a \in \mathbb{Z}_r$ and $1 \leq i < j \leq n$. The element $\upsilon'(s_i)$ must be of this form, as well of the form (1.6.6), so we conclude that $\upsilon'(s_i) = z\,(a_ie_i - a_ie_{i+1}, (i,i+1))$ for some $a_i \in \mathbb{Z}_r$ for each $i = 1,\ldots,n-1$. Once can check that if

$$y = \left(\sum_{i=1}^n \sum_{j=i}^n a_je_i, 1\right) \in G(r,1,n) \qquad \text{then} \qquad y^{-1} \cdot \upsilon'(s_i) \cdot y = zs_i \text{ for all } i,$$

and so $\upsilon = \mathrm{Ad}(\omega y) \circ \upsilon''$ where $\upsilon'' \in \mathrm{Aut}(G)$ has $\upsilon''(s_i) = zs_i$ for all i. Since N is normal in $G(r,1,n)$, it follows that $\upsilon''(N) = N$.

Since s and t commute with s_j for $j > 2$ and $j > 1$ respectively, and since $s_1ss_1 = s^{-1}$, it follows that we can write

$$\upsilon''(s) = z's^j \quad \text{and} \quad \upsilon''(t) = z''t^{pj'}, \qquad \text{for some } j,j' \in \mathbb{Z}_r \text{ and } z',z'' \in C(r,p,n) \text{ with } (z')^2 = 1.$$

If $n = 2$ then it follows that either $z' = 1$ or $z' = s^{r/2}$, and so in this case we lose no generality by assuming $z' = 1$. If $n > 2$, then since $s_1s^j = (s_2s^js_2)^{-1}s_1(s_2s^js_2)$, we have

$$zz's_1s^j = \upsilon''(s_1s) = \upsilon''((s_2ss_2)^{-1}s_1(s_2ss_2)) = zs_1s^j$$

so $z' = 1$ automatically. Hence $v''(s) = s^j$ for some $j \in \mathbb{Z}_r$. Since $s^p = t^p s_1 t^{-p} s_1$, we obtain

$$(pje_1 - pje_2, 1) = v''(s^p) = v''(t^p) \cdot zs_1 \cdot v''(t^p)^{-1} \cdot zs_1 = (pj'e_1 - pj'e_2, 1)\,.$$

Therefore $t^{pj} = t^{pj'}$ so we can assume $j' = j$. Since t^p has order r/p, the central element z'' must be of the form c^{pk} for some integer k, and so $v''(t^p) = c^{pk}t^{pj}$. But now v'' agrees with the map $\alpha_{j,k,z}$ on the generators $s_1, \ldots, s_{n-1}, s, t^p$, and so we conclude by Proposition 1.6.2 that $v'' = \alpha_{j,k,z}$. Thus $v = \mathrm{Ad}(\omega y) \circ \alpha_{j,k,z}$ as desired.

To finish our proof we must treat the case $n = 6$ and $r > 1$. In this situation, N is a characteristic subgroup by [84, Lemma 4.2] and so the map $\pi \mapsto \mathrm{proj} \circ v(0, \pi)$ again induces an automorphism of S_n. Our desired conclusion will follow if we can show that this automorphism is inner, since then we can invoke all of the preceding arguments. This is shown in the last paragraph of the proof of [84, Proposition 4.1]. $\square$

Barring a finite number of cases, the subgroup $(\mathbb{Z}_r)^n \cap G(r,p,n)$ is typically characteristic and so every automorphism of $G(r,p,n)$ is of the form given in the lemma. To account for the possible exceptions, we define additional automorphisms $\eta_{r,p,n}, \eta'_{r,p,n} \in \mathrm{Aut}\,(G(r,p,n))$ on generators by

$$\eta_{2,1,2} : \begin{aligned} s_1 &\mapsto t \\ t &\mapsto s_1 \end{aligned} \qquad \eta_{2,2,2} : \begin{aligned} s_1 &\mapsto s \\ s_1' &\mapsto s_1 \\ s &\mapsto s_1' \end{aligned} \qquad \eta_{4,2,2} : \begin{aligned} s_1 &\mapsto t^2 \\ s_1' &\mapsto s_1 \\ t^2 &\mapsto s_1' \end{aligned} \qquad \eta_{3,3,3} : \begin{aligned} s_1 &\mapsto s_2 \\ s_2 &\mapsto s_1' \\ s_1' &\mapsto s_1 \end{aligned}$$

$$\eta'_{3,3,3} : \begin{aligned} s_1 &\mapsto s_1 \\ s_2 &\mapsto s_2 \\ s_1' &\mapsto s_2' \end{aligned} \qquad \eta_{2,2,4} : \begin{aligned} s_1 &\mapsto s_1' \\ s_2 &\mapsto s_2 \\ s_3 &\mapsto s_1 \\ s_1' &\mapsto s_3 \end{aligned} \qquad \eta_{1,1,6} : \begin{aligned} s_1 &\mapsto (1,2)(3,4)(5,6) \\ s_2 &\mapsto (1,5)(2,3)(4,6) \\ s_3 &\mapsto (1,2)(3,6)(4,5) \\ s_4 &\mapsto (1,5)(2,6)(3,4) \\ s_5 &\mapsto (1,2)(3,5)(4,6) \end{aligned}$$

and in all other cases $\eta_{r,p,n} = 1$ and $\eta'_{r,p,n} = 1$. Thus $\eta'_{r,p,n}$ is the identity unless $(r,p,n) = (3,3,3)$.

Many of these automorphisms are well known: for example, $\eta_{2,1,2}$ and $\eta_{2,2,4}$ are the graph automorphisms ${}^2B_2$ and ${}^3D_4$, and $\eta_{1,1,6}$ is the outer automorphism of S_6. The automorphisms $\eta_{4,2,2}$ and $\eta_{3,3,3}$, $\eta'_{3,3,3}$ and $\eta_{2,2,4}$ derive from normal embeddings of $G(4,2,2) \lhd G_6$ and $G(3,3,3) \lhd G_{26}$ and $G(2,2,4) \lhd G_{28}$ in exceptional groups. For more information on the structure of this embedding, see [20, Proposition 3.13] or [84, Section 3].

Theorem 1.6.5. Let r, p, n be positive integers with p dividing r. Then every automorphism of $G(r,p,n)$ is of the form

$$(\eta_{r,p,n})^{i_1} \circ (\eta'_{r,p,n})^{i_2} \circ \mathrm{Ad}(g) \circ \alpha_{j,k,z}$$

for some $g \in G(r,1,n)$, $i_1, i_2, j, k \in \mathbb{Z}$, and $z \in C(r,p,n)$.

Remark. Of course, by Lemma 1.6.2 the given expression is an automorphism of $G(r,p,n)$ if and only if $\gcd(j,r) = \gcd(j+nk, r/p) = 1$ and $z^2 = 1$. Note, furthermore, that $\eta_{r,p,n} = \eta'_{r,p,n} = 1$ if (r,p,n) is not $(2,1,2)$, $(2,2,2)$, $(4,2,2)$, $(3,3,3)$, $(2,2,4)$, or $(1,1,6)$. In these cases, it is easy to see that the reflection-preserving automorphisms of $G(r,p,n)$ are precisely the maps of the form $\mathrm{Ad}(g) \circ \alpha_{j,0,1}$, as predicted by [93, Theorem 7.1].

Proof. One can check directly that the theorem holds if (r,p,n) is $(2,1,2)$, $(2,2,2)$, $(4,2,2)$, $(3,3,3)$, $(2,2,4)$, or $(1,1,6)$; we have done so using the computer algebra system GAP. If (r,p,n) is not one of these exceptions, then by [84, Lemma 4.2] the subgroup $(\mathbb{Z}_r)^n \cap G(r,p,n)$ is characteristic, and hence preserved by $\mathrm{Aut}(G)$, in which case the theorem follows immediately from Lemma 1.6.4. □

We can be a little more specific about the uniqueness of the decomposition given in the theorem, and this will allow us to give a formula for the order of $\mathrm{Aut}\,(G(r,p,n))$. Given integers j,k and $z \in C(r,1,n)$, we adopt the shorthand

$$\beta_j \stackrel{\text{def}}{=} \alpha_{j,0,1} : (x,\pi) \mapsto (jx,\pi) \qquad \text{and} \qquad \gamma_{k,z} \stackrel{\text{def}}{=} \alpha_{1,k,z} : (x,\pi) \mapsto z^{\ell(\pi)} c^{\Delta(x)\cdot k}(x,\pi).$$

One checks that

$$\beta_j \circ \beta_{j'} = \beta_{jj'}, \qquad \gamma_{k,z} \circ \gamma_{k',z'} = \gamma_{k'',zz'}, \qquad \text{and} \qquad \beta_j \circ \gamma_{k,z} = \gamma_{k,z} \circ \beta_j = \alpha_{j,jk,z},$$

where $k'' = k + k' + nkk'$. Since in this notation $\alpha_{0,1,1} = \beta_1 = \gamma_{0,1}$ all equal the identity automorphism of $G(r,p,n)$, it follows that the sets

$$B \stackrel{\text{def}}{=} \{\beta_j : \gcd(j,r) = 1\} \qquad \text{and} \qquad C \stackrel{\text{def}}{=} \{\gamma_{k,z} : \gcd(1{+}nk, r/p) = 1,\ z \in C(r,p,n),\ z^2 = 1\}$$

are subgroups of $\mathrm{Aut}\,(G(r,p,n))$. Define additionally the subgroups

$$X \stackrel{\text{def}}{=} \langle \eta_{r,p,n}, \eta'_{r,p,n} \rangle \qquad \text{and} \qquad A \stackrel{\text{def}}{=} \{\mathrm{Ad}(g) : g \in G(r,1,n)\}$$

of $\mathrm{Aut}\,(G(r,p,n))$. Using the previous result, we have the following.

Proposition 1.6.6. Let r,p,n be positive integers with p dividing r. Then

(i) $\mathrm{Aut}\,(G(r,p,n)) = XABC$ and XA is a normal subgroup of $\mathrm{Aut}\,(G(r,p,n))$.

(ii) a. If $n = 1$ then $B = C$, and if $n > 1$ then $B \cap C = \{1\}$.

b. We have $A \cap BC = \begin{cases} \{\alpha_{j,k,z} : (j,k) \in \{(-1,1),(1,0)\}\} & \text{if } n = 2 \\ \{1\} & \text{if } n \neq 2. \end{cases}$

c. We have $X \cap ABC = \begin{cases} A & \text{if } (r,p,n) = (3,3,3) \\ \{1\} & \text{if } (r,p,n) \neq (3,3,3). \end{cases}$

(iii) If $n > 2$ then $\mathrm{Aut}\,(G(r,p,n)) \cong (XA \rtimes B) \times C$.

This result is closely related to [84, Proposition 4.1], which asserts that any automorphism of $G(r,p,n)$ is the composition of an automorphism which preserves the pseudo-reflections in $G(r,p,n)$ and a central automorphism. The subgroup $C \subset \mathrm{Aut}\,(G(r,p,n))$ is the set of central automorphisms of $G(r,p,n)$ and $XAB \subset \mathrm{Aut}\,(G(r,p,n))$ is the subgroup of automorphisms which preserve the reflections. Furthermore, if we view $G(r,p,n)$ as a subgroup of $\mathrm{GL}(\mathbb{C}^n)$ whose elements are generalized permutation matrices, then the subgroup XA consists of all automorphisms induced from elements of the normalizer of $G(r,p,n)$ in $\mathrm{GL}(\mathbb{C}^n)$. The subgroup B likewise consists of all automorphisms induced from elements of the Galois group of $\mathbb{Q}(\zeta_r)$ over $\mathbb{Q}$.

Proof. To prove the first half of (i), one checks that BC is precisely the set of all automorphisms of $G(r,p,n)$ of the form $\alpha_{j,k,z}$; hence, by the preceding theorem $\mathrm{Aut}\,(G(r,p,n)) = XABC$. One can check the remaining assertions in the finite number of cases when $X \neq \{1\}$ directly, by hand or with a computer algebra system. Therefore assume $X = \{1\}$.

To show $A \lhd \mathrm{Aut}\,(G(r,p,n))$ we observe that if $g = (y,\omega) \in G(r,1,n)$ then $\alpha_{j,k,z} \circ \mathrm{Ad}(g) \circ \alpha_{j,k,z}^{-1} = \mathrm{Ad}(g')$ where $g' = (jy, \omega)$. Our description of $B \cap C$ is trivially verified. Suppose $n = 2$ and $z \in C(r,p,n)$ has $z^2 = 1$. We have two cases: either $z = 1$ or $z = \left(\frac{r}{2}e_1 + \frac{r}{2}e_2, 1\right)$, the latter of which can only occur if r is even. One checks that $\alpha_{-1,1,z} = \mathrm{Ad}(g)$ for $g = s_1$ if $z = 1$ and for $g = \left(\frac{r}{2}e_1, (1,2)\right)$ if $z \neq 1$. Likewise, $\gamma_{0,z}$ is the identity if $z = 1$ and $\mathrm{Ad}(g)$ for $g = \left(\frac{r}{2}e_1, 1\right)$ if $z \neq 1$. Since $n = 2$ we have

$$\mathrm{Ad}(g)(s_1) = s^a s_1 \text{ for some } a \in \mathbb{Z} \qquad \text{and} \qquad \begin{pmatrix} \mathrm{Ad}(g)(s) = s & \text{or} & \mathrm{Ad}(g)(s) = s^{-1} \\ \mathrm{Ad}(g)(t^p) = t^p & & \mathrm{Ad}(g)(t^p) = c^p t^{-p} \end{pmatrix}$$

and it follows that only elements of the form $\alpha_{j,k,z}$ with $(j,k) = (1,0)$ or $(-1,1)$ can be contained in $A \cap BC$, which proves (ii).

If $n = 1$ then $A = \{1\}$. Suppose $n > 2$ and $\mathrm{Ad}(g) = \alpha_{j,k,z}$ for some $g = (y,\pi) \in G(r,1,n)$ and j,k,z. This implies $\pi\sigma\pi^{-1} = \sigma$ for all $\sigma \in S_n$ so $\pi = 1$ since $Z(S_n)$ is trivial. Consequently $\mathrm{Ad}(g)$ fixes both s and t so we must have $j = 1$ and $k = 0$. Furthermore, $\mathrm{Ad}(g)(s_1) = s^{y_1 - y_2} s_1$ so we must have $y_1 = y_2$ and $z = 1$ since $n > 2$. Therefore $\alpha_{j,k,z} = 1$ so $A \cap BC = \{1\}$. Since each element in C commutes with all elements of A and B, part (iii) now follows from (i) and (ii). $\square$

From this proposition we are now able to derive a formula for the order of the automorphism group of $G(r,p,n)$. Here $\phi(x)$ denotes Euler's totient function, which we recall is defined as the number of positive integers $y \leq x$ with $\gcd(x,y) = 1$.

Corollary 1.6.7. Let r,p,n be positive integers with p dividing r. Assume $n > 1$ and write e for the greatest divisor of r/p with $\gcd(e,n) = 1$. Then

- $|\mathrm{Aut}\,(G(r,p,n))\,| = \dfrac{c_{r,p,n}}{c'_{r,p,n}} \cdot \phi(r) \cdot \phi(e)/e \cdot n! \cdot r^n/p$
- $|\mathrm{Out}\,(G(r,p,n))\,| = c_{r,p,n} \cdot \phi(r) \cdot \phi(e)/e \cdot r/p \cdot \gcd(p,n)$

- $|Z\,(G(r,p,n))\,| = c'_{r,p,n} \cdot r/p \cdot \gcd(p,n)$

where

$$\begin{aligned}
&c_{1,1,2} = 1, && c_{2,2,2} = 3, && c_{2,1,2} = 1, && c_{4,2,2} = 3/2,\\
&c_{3,3,3} = 4, && c_{2,2,4} = 6, && c_{1,1,6} = 2, &&\\
&c'_{1,1,2} = 2, && c'_{2,2,2} = 2, && &&
\end{aligned}$$

and in all other cases

$$c_{r,p,n} = \begin{cases} 1/2 & \text{if } n = 2,\\ 1 & \text{if } r \text{ is odd and } n > 2,\\ 1 & \text{if } p \text{ is even but } r/p \text{ and } n > 2 \text{ are odd},\\ 2 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad c'_{r,p,n} = 1.$$

Remark. If $n = 1$ then $G(r,p,n) \cong \mathbb{Z}_{r/p}$ so $|\mathrm{Aut}\,(G(r,p,n))\,| = |\mathrm{Out}\,(G(r,p,n))\,| = \phi(r/p)$.

Proof. The formula for the order of the center of $G = G(r,p,n)$ follows immediately from Lemma 1.6.1, so it suffices to prove our formula for $|\mathrm{Aut}(G)|$. If (r,p,n) is one of the exceptional cases $(1,1,2)$, $(2,2,2)$, $(2,1,2)$, $(4,2,2)$, $(3,3,3)$, $(2,2,4)$, or $(1,1,6)$, then our formula asserts that $|\mathrm{Aut}(G)|$ is 1, 6, 8, 48, 432, 1152, or 1440, respectively. One easily checks that these orders are correct: all but two of the exceptions are Weyl groups whose automorphisms are well known (e.g., see [9]), and one can compute the outer automorphisms of $G(4,2,2)$ and $G(3,3,3)$ by hand or with a computer.

Assuming (r,p,n) is not one of these exceptions, we have $|A| = |G(r,1,n)|/|C(r,1,n)| = n! \cdot r^{n-1}$ by Lemma 1.6.1 and $|B| = \phi(r)$. To compute $|C|$, we note that $\gamma_{k,z} = \gamma_{k',z'}$ if and only if $k \equiv k' \pmod{r/p}$ and $z = z'$. Thus the elements of C are in bijection with all choices of $k \in [0, r/p-1]$ and $z \in C(r,p,n)$ such that $\gcd(1+nk, r/p) = 1$ and $z^2 = 1$. To satisfy these conditions, the central element z must be the identity if r is odd, or if r is even but $nr/2$ is not a multiple of p, which occurs if and only if p is even but r/p and n are odd. In all other cases, z can be either 1 or $c^{r/2}$. Additionally, since $1+nk$ is coprime to r/p if and only if $1+nk$ is coprime to e, it follows that there are $\phi(e)\cdot(r/p)/e$ possible choices of $k \in [0, r/p-1]$ with $\gcd(1+nk, r/p) = 1$. Thus $|C| = \widetilde{c}_{r,p,n} \cdot \phi(e)/e \cdot r/p$ where

$$\widetilde{c}_{r,p,n} = \begin{cases} 1, & \text{if } r \text{ is odd, or if } p \text{ is even but } r/p \text{ and } n \text{ are odd,}\\ 2, & \text{otherwise.}\end{cases}$$

As $n > 1$ and $X = \{1\}$, by the preceding proposition we thus have

$$|\mathrm{Aut}(G)| = \frac{|A||B||C|}{|A \cap BC||B \cap C|} = \frac{\widetilde{c}_{r,p,n}}{|A \cap BC|} \cdot \phi(r) \cdot \phi(e)/e \cdot n! \cdot r^n/p.$$

If $n > 2$, then $|A \cap BC| = 1$ and if $n = 2$ then it follows from part (ii) of Proposition 1.6.6 that $|A \cap BC| = 2\widetilde{c}_{r,p,n}$, so in both cases we obtain $\widetilde{c}_{r,p,n}/|A \cap BC| = c_{r,p,n}/c'_{r,p,n}$ as desired. $\square$

1.7 First main theorem

In this section we complement the results of Section 1.5 by proving that $G(r,p,n)$ does not have a generalized involution model if $\gcd(p,n) \geq 2$, unless $\gcd(p,n) = n = 2$ and r/p is odd. Combining this with a computer assisted investigation of the exceptional groups, we will be able to completely determine which finite complex reflection groups have generalized involution models.

1.7.1 Reductions

We begin by observing that finding all generalized involution models of a group often is equivalent to classifying the generalized involution models defined with respect to a single, fixed automorphism. Say that an automorphism $\tau \in \mathrm{Aut}(G)$ of a group G is *class-preserving* if ${}^\tau g$ is conjugate to g for all $g \in G$, or equivalently if $\psi \circ \tau = \psi$ for all $\psi \in \mathrm{Irr}(G)$. Clearly all inner automorphisms are class-preserving, but a finite group can possess outer automorphisms which are class-preserving, as was first shown by Burnside [23]. The non-existence of class-preserving outer automorphisms can greatly reduce the problem of finding all generalized involution models of a group by the following lemma.

Lemma 1.7.1. Let G be a finite group with an automorphism $\tau \in \mathrm{Aut}(G)$ such that $\tau^2 = 1$ and

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(1) = |\{g \in G : g \cdot {}^\tau g = 1\}|. \tag{1.7.1}$$

If G has no class-preserving outer automorphisms, then the following hold:

(i) The image of τ in $\mathrm{Out}(G)$ is central.

(ii) Any generalized involution model for G can be defined with respect to τ.

Remark. The conclusion of the lemma can fail if G has class-preserving outer automorphisms. Wall [102] showed that the semidirect product $G = \mathbb{Z}_m \rtimes \mathbb{Z}_m^\times$ consisting of all pairs $(a,x) \in \mathbb{Z}_m \times \mathbb{Z}_m^\times$ with the multiplication

$$(a,x)(b,y) = (a+xb, xy), \qquad \text{for } a,b \in \mathbb{Z}_m,\ x,y \in \mathbb{Z}_m^\times$$

has a class-preserving outer automorphism τ of order two if m is divisible by 8. Taking $m = 8$ gives a group G of order 32, the smallest group with a class-preserving outer automorphism. One can check using a computer algebra system (we used `GAP`) that this G has a generalized involution model with respect to $1 \in \mathrm{Aut}(G)$ but not with respect to the class-preserving outer automorphism τ, even though (1.7.1) holds.

Proof. Assume G has no class-preserving outer automorphisms. If $\alpha \in \mathrm{Aut}(G)$ and $\tau' = \alpha \circ \tau \circ \alpha^{-1}$, then by Theorem 1.2.1, we have $\epsilon_\tau(\psi) = 1$ and $\epsilon_{\tau'}(\psi) = \epsilon_\tau(\psi \circ \alpha) = 1$ for all $\psi \in \mathrm{Irr}(G)$. Therefore, by [24, Proposition 2], ${}^\tau g$ is conjugate to ${}^{\tau'} g$ for all $g \in G$, so $\alpha \circ \tau \circ \alpha^{-1} \circ \tau^{-1}$ is class-preserving, and therefore an inner automorphism. This proves (i).

Now suppose G has a generalized involution model with respect to $\upsilon \in \mathrm{Aut}(G)$ with $\upsilon^2 = 1$. By Theorem 1.2.1, it follows that $\epsilon_\tau(\psi) = \epsilon_\upsilon(\psi) = 1$ for all $\psi \in \mathrm{Irr}(G)$, so by [24, Proposition 2] each $g \in G$ is conjugate to both ${}^\tau g^{-1}$ and ${}^\upsilon g^{-1}$. Replacing g with g^{-1}, one sees that ${}^\tau g$ is therefore conjugate to ${}^\upsilon g$ for all $g \in G$, which suffices to show that $\tau \circ \upsilon^{-1}$ is a class-preserving automorphism. Hence $\tau = \mathrm{Ad}(x) \circ \upsilon$ for some $x \in G$. Since $\tau^2 = \upsilon^2 = 1$, the element $z \stackrel{\text{def}}{=} x \cdot {}^\upsilon x \in G$ is central. Fix $\psi \in \mathrm{Irr}(G)$ and let $\omega_\psi(z) \stackrel{\text{def}}{=} \frac{\psi(z)}{\psi(1)}$ denote the value of its central character at z; then $\psi(zg) = \omega_\psi(z)\psi(g)$ for all $g \in G$, and it follows that $\omega_\psi(z)\epsilon_\tau(\psi) = \epsilon_\upsilon(\psi) = \epsilon_\tau(\psi) = 1$ so $\psi(z) = \psi(1)$. Since this holds for all irreducible characters of G, we have $z = 1$. This means that $g \cdot {}^\tau g = gx \cdot {}^\upsilon g \cdot x^{-1} = (gx) \cdot {}^\upsilon(gx)$, and it follows that the map $\mathcal{I}_{G,\tau} \to \mathcal{I}_{G,\upsilon}$ given by $g \mapsto gx$ is an isomorphism of G-sets. In particular, the twisted conjugacy classes with respect to τ and υ are in bijection and have the same twisted centralizers. Therefore the generalized involution model with respect to υ can also be defined with respect to τ, which proves (ii). $\square$

As a consequence of this result, to determine whether a group G with no class-preserving outer automorphisms has a generalized involution model, one only needs to check (1) if there exists $\tau \in \mathrm{Aut}(G)$ with $\tau^2 = 1$ such that (1.7.1) holds, and (2) if G has a generalized involution model with respect to τ. This strategy is especially apposite for irreducible complex reflection groups in light of the following result given in a slightly different form as [84, Proposition 3.1].

Lemma 1.7.2 (Marin, Michel [84]). A finite complex reflection group has no class-preserving outer automorphisms.

Remark. We remark that it is a tedious but not overly difficult exercise to prove the lemma directly for the irreducible groups $G(r,p,n)$, and via computer calculations for the exceptional groups. The lemma then holds for all finite complex reflection groups because a class-preserving automorphism of a direct product must restrict to a class-preserving automorphism of each factor.

These results become especially useful when doing calculations. To determine which of the exceptional irreducible complex reflection groups $G_4, \dots, G_{37}$ have generalized involution models, we will rely on a computer-assisted brute force search. The preceding lemmas greatly diminish the size of this calculation, because they show that one needs to examine at most one automorphism for each group to determine if a generalized involution model exists. In Table A.1 we provide a list of automorphisms $\tau \in \mathrm{Aut}(G_i)$ for which (1.7.1) holds, if this is possible. These automorphisms are defined on the generators s, t, u, v, w which appear in the presentations for $G_4, \dots, G_{37}$ in the appendix of [21]. These generators coincide with the generators for the exceptional groups in the `GAP` package `CHEVIE`, which allows one easily to compute things with this data.

Vinroot, elaborating upon the work of Baddeley [11], describes in [101] all finite Coxeter groups with involution models in the classical sense; in particular, the only irreducible finite Coxeter groups which fail to have involution models are those of type D_{2n} $(n > 1)$, E_6, E_7, E_8, F_4, and H_4. If G is a finite Coxeter group then all of its representations are equivalent to real representations, and so by the Frobenius-Schur involution counting theorem, (1.7.1) holds with $\tau = 1$. Hence, by Lemmas 1.7.1 and 1.7.2, a finite Coxeter group has a generalized involution model if and only if it has an involution model, and we are left with the following corollary of [101, Theorem 1].

Corollary 1.7.3. A finite Coxeter group has a generalized involution model if and only if it has an involution model, which occurs if and only if all of its irreducible factors are of type A_n, B_n, D_{2n+1}, H_3, or $I_2(n)$.

Remark. The Coxeter group of type G_2 is omitted from this list only because it is isomorphic to the one of type $I_2(6)$. We note that the Coxeter group of type $I_2(n)$ is the involutory complex reflection group $G(n,n,2)$, and that restricted to this group the map $\tau : g \mapsto \overline{g}$ is a nontrivial inner automorphism. Thus, while the group has an involution model in the classical sense, it also has a generalized involution model with respect to τ, which is consistent with Lemmas 1.7.1 and 1.7.2. The same is true for groups of types A_n, B_n, and D_{2n+1}, but vacuously since in these cases the inverse transpose τ acts as the identity map.

In order to reduce our investigation of finite complex reflection groups to irreducible groups, we require one additional lemma. This next statement generalizes [101, Lemma 1] which considers only involution models.

Lemma 1.7.4. If $H_1, \dots, H_n$ are finite groups then $H = \prod_{i=1}^n H_i$ has a generalized involution model if and only if each H_i has a generalized involution model.

Proof. If H has a generalized involution model with respect to $\tau \in \mathrm{Aut}(H)$, then each $h \in H$ is conjugate to ${}^\tau h^{-1}$ by [24, Theorem 1.2.1 and Proposition 2], and so τ restricts to an automorphism of each factor H_i. Given this fact, it follows that any generalized involution model for H decomposes in an obvious way as a "product" of generalized involution models of the factor groups H_i, and the proof of the lemma becomes a simple exercise. ☐

1.7.2 Addressing the case with p and n not coprime

We now demonstrate that $G(r,p,n)$ does not have a generalized involution model if $\gcd(p,n) = 2$, unless $n = 2$ and r/p is odd. Our proof of this proceeds in two steps, and will use somewhat different methods. We begin in the case when $\gcd(p,n) = 2$ and r/p is even.

Lemma 1.7.5. Let r, p, n be positive integers with p dividing r. If $\gcd(p,n) = 2$ and r/p is even, then $G(r,p,n)$ does not have a generalized involution model.

Proof. We can tackle this case by much more direct methods than when r/p is odd. Let $G = G(r,p,n)$ and define τ as the usual inverse transpose automorphism $g \mapsto \overline{g}$. Since $\gcd(p,n) = 2$, it follows from Theorem 1.4.2 that equation (1.7.1) is satisfied, so by Lemmas 1.7.1 and 1.7.2, we need only show that G has no generalized involution models with respect to τ. Towards this goal, our strategy is simple. Since $r/2$ is a multiple of p, the central element $z \stackrel{\text{def}}{=} c^{r/2} \in G(r,1,n)$ is contained in G; here c is defined as in (1.6.1). We claim that

z lies in the commutator subgroup of the twisted centralizer $C_{G,\tau}(\omega)$ for every generalized involution $\omega \in \mathcal{I}_{G,\tau}$.

If this holds, then z lies in the kernel of every linear character λ of $C_{G,\tau}(\omega)$ and therefore also in the kernel of the induced character $\mathrm{Ind}_{C_{G,\tau}(\omega)}^G(\lambda)$ since z is central. In this case, if G has a generalized involution model $\{\lambda_i : H_i \to \mathbb{C}\}$ with respect to τ, then z lies in the kernel of $\sum_i \mathrm{Ind}_{H_i}^G(\lambda_i)$, implying the contradiction

$$1 \neq z \in \bigcap_{\psi \in \mathrm{Irr}(G)} \ker(\psi) = \{1\}.$$

To prove our claim, suppose $\omega = (a, \pi) \in G$ has $\omega \cdot {}^\tau\omega = (\pi^{-1}(a) - a, \omega^2) = (0,1) = 1$. Then $\pi \in S_n$ must be an involution with $\pi(a) = a$. We lose no generality by conjugating ω by an element of $S_n \subset G$ since this has the effect of conjugating the twisted centralizer $C_{G,\tau}(\omega)$ and fixing z. Therefore, we can assume that $\omega = (1,2)(3,4)\cdots(2k-1,2k)$ for some $k \leq n/2$, in which case $\pi(a) = a$ implies $a_{2i-1} = a_{2i}$ for all $i = 1, \ldots, k$. Since r and p are even and $\Delta(a) \in p\mathbb{Z}_r$, the number of $a_i \notin 2\mathbb{Z}_r$ is even; therefore, letting $\ell = n/2 - k$, there are distinct indices $\{i_1, j_1, \ldots, i_\ell, j_\ell\} = [2k+1, n]$ such that $a_{i_t} - a_{j_t} \in 2\mathbb{Z}_r$ for all $t = 1, \ldots, \ell$. For each t, let $b_t \in \mathbb{Z}_r$ such that $2b_t = a_{i_t} - a_{j_t}$. Now define $g = (x, \sigma) \in G$ by

$$\sigma = (1,2)\cdots(2k-1,2k)(i_1,j_1)\cdots(i_\ell,j_\ell) \in S_n \qquad \text{and} \qquad x_i = \begin{cases} 0, & \text{if } i \in [1,k], \\ b_t, & \text{if } i = j_t, \\ -b_t, & \text{if } i = i_t. \end{cases}$$

One can check that we then have $\sigma \in C_{S_n}(\pi)$, $\pi(x) = x$, and $a + 2x = \sigma(a)$, and so

$$g \cdot \omega \cdot {}^\tau g^{-1} = (\sigma^{-1}\pi^{-1}(x) + \sigma^{-1}(a) + \sigma^{-1}(x), \sigma\pi\sigma^{-1}) = \big(\sigma^{-1}(a + 2x), \pi\big) = \omega.$$

Thus $g \in C_{G,\tau}(\omega)$. Since r is divisible by 4 and $r/2$ is divisible by p, we can define $h = (y, 1) \in G$ by setting $y \in (\mathbb{Z}_r)^n$ to have

$$y_i = \begin{cases} r/4, & \text{if } i \in \{1,3,\ldots,2k-1\}, \\ -r/4, & \text{if } i \in \{2,4,\ldots,2k\}, \\ r/2, & \text{if } i = i_t, \\ 0, & \text{if } i = j_t. \end{cases}$$

Observe that $\pi^{-1}(y) = -y$ since $r/2 = -r/2$, so $h \cdot \omega \cdot {}^\tau h^{-1} = \omega h^{-1} h = \omega$ and $h \in C_{G,\tau}(\omega)$. Our claim now follows by calculating $ghg^{-1}h^{-1} = (\sigma^{-1}(y) - y, 1) = z$, which completes the proof. $\square$

If $\gcd(p,n) = 2$ but r/p is odd, then the crucial step in the preceding proof does not hold. However, in this case the group $G(r,p,n)$ still fails to have a generalized involution model, provided $n > 2$. To show this, we will use two results from Baddeley's thesis [11].

First, recall that a model for a group G is a set $\{\lambda_i : H_i \to \mathbb{C}\}$ of linear characters of subgroups of G such that $\sum_i \mathrm{Ind}_{H_i}^G(\lambda_i) = \sum_{\psi \in \mathrm{Irr}(G)} \psi$. Following Baddeley, we say that a model $\{\lambda_i : H_i \to \mathbb{C}\}$ is *based* on a set $\mathcal{S}$ of subgroups of G if for each i there exists a subgroup $H_i' \in \mathcal{S}$ with a linear character $\lambda_i' : H_i' \to \mathbb{C}$ such that $\mathrm{Ind}_{H_i}^G(\lambda_i) = \mathrm{Ind}_{H_i'}^G(\lambda_i')$. Thus $\{\lambda_i : H_i \to \mathbb{C}\}$ is based on the set of subgroups $\{H_i\}$, as well as on any set of subgroups which are conjugate in G to the subgroups H_i, as well as on the set of all subgroups of G. For each $n \geq 1$, let $\mathcal{G}(n)$ denote the set of subgroups of S_n of the form

$$W_k \times S_i \times S_j, \qquad \text{where } i, j, k \text{ are nonnegative integers with } i + j + 2k = n$$

and $W_k \subset S_{2k}$ is the centralizer of the permutation $(1,2)(3,4)\cdots(2k-1,2k) \in S_{2k}$. The centralizer of any involution in S_n is conjugate to a subgroup of the form $W_k \times S_j = W_k \times S_0 \times S_j \in \mathcal{G}(n)$ for some j, k with $j + 2k = n$, so any involution model for S_n is based on $\mathcal{G}(n)$. (This is not a vacuous statement; [49] constructs an involution model for the symmetric group.) Baddeley states the following result as [11, Corollary 4.3.16].

Lemma 1.7.6 (Baddeley [11]). Suppose $\mathcal{M}$ is a model for S_n based on $\mathcal{G}(n)$. If $\mathcal{M}$ contains both the trivial character $\mathbb{1} \in \mathrm{Irr}(S_n)$ and the sign character $\mathrm{sgn} \in \mathrm{Irr}(S_n)$ then $n = 2$.

To state our second needed result, let $\Phi : G \to G'$ be a surjective group homomorphism. Suppose $H \subset G$ is a subgroup and $\psi \in \mathrm{Irr}(H)$. If $\ker(\psi) \supset \ker(\Phi) \cap H$, then there exists a unique irreducible character $\psi' \in \mathrm{Irr}(\Phi(H))$ such that $\psi = \psi' \circ \Phi$, and we define $\mathcal{R}_\Phi(\psi) \in \{0\} \cup \mathrm{Irr}(\Phi(H))$ by

$$\mathcal{R}_\Phi(\psi) = \begin{cases} \psi', & \text{if } \ker(\psi) \supset \ker(\Phi) \cap H, \\ 0, & \text{otherwise.} \end{cases}$$

The following appears as [11, Theorem 4.2.3].

Theorem 1.7.7 (Baddeley [11]). Let $\Phi : G \to G'$ be a surjective group homomorphism. If $\mathcal{M}$ is a model for G and

$$\hat{\mathcal{M}} = \{\lambda \in \mathcal{M} : \mathcal{R}_\Phi(\lambda) \neq 0\},$$

then $\mathcal{M}_\Phi \overset{\text{def}}{=} \left\{\mathcal{R}_\Phi(\lambda) : \lambda \in \hat{\mathcal{M}}\right\}$ is a model for G'.

We now apply the preceding lemma and theorem to prove that most of the complex reflection groups $G(r,p,n)$ with $\gcd(p,n) = 2$ do not have generalized involution models. We proceed by an argument similar to one used by Baddeley to prove that the Weyl group of type D_{2n} does not have an involution model if $n > 1$ [11, Proposition 4.8.1].

Lemma 1.7.8. Let r, p, n be positive integers with p dividing r. If $\gcd(p,n) = 2$, then $G(r,p,n)$ has a generalized involution model if and only if $n = 2$ and r/p is odd.

Proof. Assume $\gcd(p,n) = 2$ so that n and r are both even. Given Lemma 1.7.5, we may assume that r/p is odd. Suppose $\mathcal{M} = \{\lambda_i : H_i \to \mathbb{C}\}$ is a generalized involution model for $G = G(r,p,n)$ with respect to some automorphism $\tau \in \mathrm{Aut}(G)$. Then each $H_i = C_{G,\tau}(\omega_i)$ for a set of orbit representatives $\omega_i \in \mathcal{I}_{G,\tau}$, and by Theorem 1.4.2 and Lemmas 1.7.1 and 1.7.2, we may assume that τ is the usual inverse transpose automorphism $g \mapsto \overline{g}$. Identify

$\mathbb{Z}_2 \subset \mathbb{Z}_r$ as the subgroup $\mathbb{Z}_2 = \{0, r/2\} = (r/2)\mathbb{Z}_r$, so that we can view $G(2,2,n)$ as a subgroup of G.

If we define $\Phi : G(r,1,n) \to S_n$ as the surjective homomorphism given by $\Phi : (x,\pi) \mapsto \pi$, then Φ restricts to a surjective homomorphism $G(r,p,n) \to S_n$, and the image under Φ of each τ-twisted centralizer H_i is conjugate to some subgroup $W_k \times S_i \times S_j$ in $\mathcal{G}(n)$. Thus the model $\mathcal{M}_\Phi$ for S_n defined by Theorem 1.7.7 is based on $\mathcal{G}(n)$.

We now observe that the generalized involutions

$$e = ((0,\dots,0),1) = 1 \in \mathcal{I}_{G,\tau} \qquad \text{and} \qquad \omega = ((1,-1,1,-1\dots,1,-1),1) \in \mathcal{I}_{G,\tau}$$

belong to disjoint twisted conjugacy classes, since every element of the orbit of e is of the form $(x,1) \in G$ with $x_i \in 2\mathbb{Z}_r$ for all i. Since the stabilizers of elements in a given orbit are all conjugate, we may therefore assume without loss of generality that one linear character of $C_{G,\tau}(e)$ appears in $\mathcal{M}$ and one linear character of $C_{G,\tau}(\omega)$ appears in $\mathcal{M}$.

Because r/p is odd, so that $r/2 \in \mathbb{Z}_2 \subset \mathbb{Z}_r$ is an odd multiple of $p/2$, we have

$$C_{G,\tau}(e) = (\mathbb{Z}_2 \wr S_n) \cap G = G(2,2,n).$$

To calculate $C_{G,\tau}(\omega)$, we observe that if $z = ((1,0,1,0,\dots,1,0),1) \in G(r,1,n)$ and $c \in G(r,1,n)$ is the central element defined by (1.6.1),then $z \cdot e \cdot {}^\tau z^{-1} = \omega c$. Consequently, if $g \in G$ then $g \cdot \omega \cdot {}^\tau g^{-1} = \omega$ if and only if $g \cdot \omega c \cdot {}^\tau g^{-1} = \omega c$, and so $C_{G,\tau}(\omega) = \mathrm{Ad}(z)\,(G(2,2,n))$.

The group $C_{G,\tau}(e) = G(2,2,n)$ has only two linear characters λ_1 and λ_2, given by restricting the linear characters $\mathbb{1}_{\mathbb{Z}_r} \wr (n)$ and $\mathbb{1}_{\mathbb{Z}_r} \wr (1^n)$ of $G(2,1,n) = \mathbb{Z}_2 \wr S_n$, respectively. It is evident from the definition of these characters that $\ker(\Phi) \cap G(2,2,n) = (\mathbb{Z}_r)^n \cap G(2,2,n) \subset \ker(\lambda_i)$ for $i = 1,2$ and that

$$\mathcal{R}_\Phi(\lambda_1) = \chi^{(n)} = \mathbb{1} \in \mathrm{Irr}(S_n) \qquad \text{and} \qquad \mathcal{R}_\Phi(\lambda_2) = \chi^{(1^n)} = \mathrm{sgn} \in \mathrm{Irr}(S_n).$$

Let $\lambda_i' = \lambda_i \circ \mathrm{Ad}(z)^{-1}$; then λ_1', λ_2' are the only linear characters of $C_{G,\tau}(\omega)$, and since $\Phi \circ \mathrm{Ad}(z)^{-1} = \Phi$ as $z \in \ker(\Phi)$, we have $\mathcal{R}_\Phi(\lambda_i') = \mathcal{R}_\Phi(\lambda_i) \neq 0$. Thus either $\mathcal{R}_\Phi(\lambda_1), \mathcal{R}_\Phi(\lambda_2') \in \mathcal{M}_\Phi$ or $\mathcal{R}_\Phi(\lambda_2), \mathcal{R}_\Phi(\lambda_1') \in \mathcal{M}_\Phi$. In particular $\mathbb{1}$ and sgn must both appear in $\mathcal{M}_\Phi$, so by Lemma 1.7.6 we have $n = 2$. In this case we know by Corollary 1.5.3 that G indeed has a generalized involution model, which completes the proof. $\square$

1.7.3 Summary

We may now prove the theorem promised in the introduction.

Theorem 1.7.9. A finite complex reflection group has a generalized involution model if and only if each of its irreducible factors is one of the following:

(i) $G(r,p,n)$ with $\gcd(p,n) = 1$.

(ii) $G(r,p,2)$ with r/p odd.

(iii) G_{23}, the Coxeter group of type H_3.

Proof. Let G be a finite complex reflection group. Then G is a product of irreducible complex reflection groups, so by Lemma 1.7.4 it suffices to prove that the only irreducible complex reflection groups are those of types (i), (ii), and (iii).

To this end, first suppose $G = G_i$ for some $4 \leq i \leq 37$ is an exceptional irreducible complex reflection group. If $G = G_{23}$ is the Coxeter group of type H_3, then G has a generalized involution model by Corollary 1.7.3. To prove no other exceptional groups have generalized involution models, we resort to an exhaustive computer search using the GAP package CHEVIE. Several fortunate circumstances make this computation tractable. First, by Corollary 1.7.3, we do not need to examine the Coxeter groups G_{28}, G_{30}, G_{35}, G_{36}, and G_{37}. Second, it follows from Theorem 1.2.1 that the exceptional groups G_{27}, G_{29}, and G_{34} do not have generalized involution models because, upon examination of their character tables, one finds that if G is one of these groups then $\sum_{\psi\in\mathrm{Irr}(G)} \psi$ assumes negative values. Checking that each of the remaining exceptional groups does not have a generalized involution model by a brute force search is a feasible and not very time consuming calculation. In particular, by Lemmas 1.7.1 and 1.7.2 one only needs to examine at most one automorphism for each group; we list candidates for this automorphism in Table A.1. The remaining exceptional groups neither are prohibitively large nor have an excessive number of twisted conjugacy classes.

To deal with the infinite series, suppose $G = G(r,p,n)$ for some positive integers r, p, n with p dividing r. If $\gcd(p,n) \leq 2$ then it follows from Corollary 1.5.3 and Lemma 1.7.8 that G has a generalized involution model if and only if G is of the form (i) or (ii). We may therefore assume $\gcd(p,n) > 2$, so that $n > 2$ and $r > 2$.

Suppose G has a generalized involution model with respect to some $\upsilon \in \mathrm{Aut}(G)$ with $\upsilon^2 = 1$. By Theorem 1.2.1 we then have $\sum_{\psi\in\mathrm{Irr}(G)} \psi(1) = |\mathcal{I}_{G,\upsilon}|$ and $\epsilon_\upsilon(\psi) = 1$ for all $\psi \in \mathrm{Irr}(G)$, so by [24, Proposition 2] the elements g^{-1} and ${}^{\upsilon}g$ are conjugate for all $g \in G$. It follows that υ preserves the normal subgroup $N = (\mathbb{Z}_r)^n \cap G$, and so by Lemma 1.6.4 we can write

$$\upsilon = \mathrm{Ad}(g) \circ \alpha_{j,k,z}, \qquad \text{for some } g \in G(r,1,n) \text{ and } j,k,z \text{ as in (1.6.4).}$$

For some $a \in \mathbb{Z}_r$ we have $gt^{-a} \in G$, and if we let $\upsilon' = \mathrm{Ad}(t^a) \circ \alpha_{j,k,z}$ then g^{-1} and ${}^{\upsilon'}g$ are conjugate for all $g \in G$. This fact implies that υ is the composition of an inner automorphism with the inverse transpose automorphism.

To see this, observe that $\mathrm{Ad}(t^a)$ fixes all element of N. Therefore, if $x = (e_1 - 2e_2 + e_3, 1) \in N$ then ${}^{\upsilon'}x = (j(e_1 - 2e_2 + e_3), 1)$, while all conjugates of x^{-1} in G are of the form

$$(-e_{i_1} + 2e_{i_2} - e_{i_3}, 1) \text{ for distinct } i_1, i_2, i_3 \in [n].$$

Since $r > 2$, we must have $j \equiv -1 \pmod{r}$, and we may assume $j = -1$. If $p = r$ then $\alpha_{j,k,z} = \alpha_{j,0,z}$ for all k. If $p < r$, then

$${}^{\upsilon'}t^p = c^{pk}t^{-p} = (p(k-1)e_1 + pk(e_2 + \cdots + e_n), 1)$$

while all conjugates of t^{-p} are of the form $(-pe_i, 1)$ for $i \in [n]$. Since $n > 2$, it follows that t^{-p} and ${}^{v'}t^p$ are conjugate only if $pk = 0$ in $\mathbb{Z}_r$, in which case $\alpha_{j,k,z} = \alpha_{-1,0,z}$. As $\mathrm{Ad}(t^a)(s_2) = s_2$, we have ${}^{v'}s_2 = zs_2$ while all conjugates of $s_2^{-1} = s_2$ in G are of the form $(be_{i_1} - be_{i_2}, (i_1, i_2))$ for $b \in \mathbb{Z}_r$ and $1 \leq i_1 < i_2 \leq n$. Again since $n > 2$, it follows that ${}^{v'}s_2$ and $s_2^{-1} = s_2$ are conjugate only if $z = 1$. Thus $\alpha_{j,k,z} = \alpha_{-1,0,1}$ is precisely the inverse transpose map. Furthermore, since $\alpha_{-1,0,1}$ fixes all elements of S_n and since each $\pi \in S_n$ is conjugate in S_n to π^{-1}, it follows from Lemma 1.6.3 that $\mathrm{Ad}(t^a)$ defines an inner automorphism of G.

We therefore may assume that $v = \mathrm{Ad}(g) \circ \tau$ where $g \in G$ and $\tau : g \mapsto \overline{g}$ is the inverse transpose automorphism. We now observe that $\omega \in G$ has $\omega \cdot {}^v\omega = 1$ if and only if $(\omega g) \cdot {}^\tau(\omega g) = g \cdot {}^\tau g$, so $|\mathcal{I}_{G,v}| = |\{\omega \in G : \omega \cdot {}^\tau\omega = g \cdot {}^\tau g\}|$. Since $\tau = \tau^{-1}$ and $v^2 = 1$, the element $g \cdot {}^\tau g$ is central, and as $n > 2$, this implies that $|g \cdot {}^\tau g| = 1$. Define $\mathcal{X}_\pi(h) \subset (\mathbb{Z}_r)^n$ for each fixed $\pi \in S_n$ and $h \in G$ as the set of $x \in (\mathbb{Z}_r)^n$ with

$$x_1 + \cdots + x_n \in p\mathbb{Z}_r \qquad \text{and} \qquad (x, \pi) \cdot {}^\tau(x, \pi) = \left(\pi^{-1}(x) - x, \pi^2\right) = h \cdot {}^\tau h.$$

If $x, y \in \mathcal{X}_\pi(g)$ then $x - y \in \mathcal{X}_\pi(1)$ since if $\mathcal{X}_\pi(g)$ is nonempty then $\pi^2 = |g \cdot {}^\tau g| = 1$. Hence $|\mathcal{X}_\pi(g)| \leq |\mathcal{X}_\pi(1)|$ for all $\pi \in S_n$. Since $\{\omega \in G : \omega \cdot {}^\tau\omega = h \cdot {}^\tau h\} = \{(x, \pi) : \pi \in S_n,\ x \in \mathcal{X}_\pi(h)\}$, it follows that

$$|\mathcal{I}_{G,v}| = |\{\omega \in G : \omega \cdot {}^\tau\omega = g \cdot {}^\tau g\}| = \sum_{\pi \in S_n} |\mathcal{X}_\pi(g)| \leq \sum_{\pi \in S_n} |\mathcal{X}_\pi(1)| = |\{\omega \in G : \omega \cdot {}^\tau\omega = 1\}|.$$

We thus have $\sum_{\psi \in \mathrm{Irr}(G)} \psi(1) = |\mathcal{I}_{G,v}| \leq |\mathcal{I}_{G,\tau}|$. By Theorem 1.4.2 this inequality must become equality, which contradicts the assumption that $\gcd(p, n) > 2$. We conclude that the only irreducible groups with generalized involution models are those of types (i)-(iii), which completes our proof. $\square$

1.8 Second main theorem

In this section we derive necessary and sufficient conditions for two projective reflection groups $G(r, p, q, n)$ and $G(r, p', q', n)$ to be isomorphic. From this will derive the second formulation of Theorem 1.7.9 given in the first section of this chapter. The results here are joint with Fabrizio Caselli and appear also in the preprint [29].

To begin, we recall the definition of the groups $G(r, p, q, n)$ and some notation for referring to their elements. Fix positive integers r, p, n with p dividing r. As we did in (1.5.1), we again write c to denote the central element

$$c = (e_1 + e_2 + \cdots + e_n, 1) \in G(r, n). \tag{1.8.1}$$

If q divides r and pq divides rn, then $G(r, p, n)$ contains the cyclic central subgroup $C_q = \langle c^{r/q} \rangle$ of order q, and $G(r, p, q, n)$ is the quotient group

$$G(r, p, q, n) \stackrel{\text{def}}{=} G(r, p, n)/C_q$$

of order $\frac{r^n}{pq} \cdot n!$. We continue to denote by c the image of the element (1.8.1) in $G(r,1,q,n)$.

1.8.1 Isomorphisms between projective reflection groups

Let r and n be positive integers and let p, p', q, q' be positive integer divisors of r. Throughout we assume $pq = p'q'$ and that this product divides rn, and we let

$$G = G(r,p,q,n) \qquad \text{and} \qquad G' = G(r,p',q',n).$$

In this subsection we determine a necessary and sufficient condition for G and G' to be isomorphic when $n \neq 2$. (Note thus that only the case $pq \neq p'q'$ is of interest, since otherwise $|G| \neq |G'|$.) We start with following result, which is equivalent to [25, Proposition 4.2].

Proposition 1.8.1 (Caselli [25]). If $\gcd(\frac{rn}{q}, p') = \gcd(\frac{rn}{q'}, p)$ then for every $g \in G$ there exists a unique $g' \in G'$ such that g and g' have common representatives in $G(r,n)$, and in this case the map $g \mapsto g'$ determines an isomorphism $G \cong G'$.

Results in [25, §4] completely characterize when $G \cong G^*$ if $n \neq 2$ (where we define $G^* = G(r,q,p,n)$). Our strategy is to generalize the ideas in that work to the present context.

Say that a prime integer P appears in a number k with multiplicity e if P^e divides k and P^{e+1} does not divide k. A prime is then *special* if it appears in p and p' with different multiplicities. Since $pq = p'q'$, a prime is special if and only if it also appears in q and q' with different multiplicities. We now have the following proposition.

Proposition 1.8.2. Assume that

$$\gcd(p,n) = \gcd(p',n) \qquad \text{and} \qquad \gcd(q,n) = \gcd(q',n),$$

and write $\frac{rn}{pq} = \eta\delta$ where η (respectively, δ) is a positive integer equal to a product of non-special (respectively, special) primes. Then $G(r,\delta p,q,n)$ is well-defined and

$$G(r,p,q,n) \cong G(r,\delta p,q,n) \times \mathbb{Z}_\delta.$$

Proof. Since $\gcd(q,n) = \gcd(q',n)$, the multiplicity of any special prime in n is not greater than the corresponding multiplicity in q. As n divides $\eta\delta q = \frac{rn}{p}$, it follows that n divides $\eta q = \frac{rn}{\delta p}$. Thus δp divides r, and since δpq likewise divides rn as $\frac{rn}{\delta pq} = \eta$, we conclude that $G(r,\delta p,q,n)$ is well-defined.

A symmetric argument using the assumption that $\gcd(p,n) = \gcd(p',n)$ shows that δq likewise divides r. Therefore $c^{\frac{r}{\delta q}}$ is a well-defined element of order δ in G; let $C_\delta \cong \mathbb{Z}_\delta$ be the cyclic subgroup it generates. Both $G(r,\delta p,q,n)$ and C_δ are normal subgroups of $G(r,p,q,n)$, so to complete the proof of the proposition, we have only to show that $G(r,\delta p,q,n)$ and C_δ intersect trivially. For this, it suffices to verify that

$$(c^{\frac{r}{\delta q}})^k \in G(r,\delta p,q,n) \quad \text{iff} \quad \tfrac{rnk}{\delta q} \equiv 0 \pmod{\delta p} \quad \text{iff} \quad k \equiv 0 \pmod{\delta}.$$

The first equivalence follows by definition, and the second equivalence follows from the fact that if $\frac{rnk}{\delta q} = \delta pk'$ for some integers k, k', then by dividing both sides by p one obtains $\eta k = \delta k'$, which can only hold if k is a multiple of δ as η and δ are necessarily coprime. $\square$

The next pair of results establish Theorem 1.8.10 in the case $n \neq 2$. This generalizes [25, Theorem 4.4].

Theorem 1.8.3. If $pq = p'q'$, then the groups $G(r,p,q,n)$ and $G(r,p',q',n)$ are isomorphic whenever $\gcd(p,n) = \gcd(p',n)$ and $\gcd(q,n) = \gcd(q',n)$.

Proof. Write $\frac{rn}{pq} = \eta\delta$ as in Proposition 1.8.2. The theorem will follow immediately from Proposition 1.8.2 once we show that $G(r,\delta p,q,n) \cong G(r,\delta p',q',n)$. Since $\frac{rn}{q} = \eta\delta p$ and $\frac{rn}{q'} = \eta\delta p'$, it suffices by Proposition 1.8.1 to verify that $\gcd(\eta\delta p, \delta p') = \gcd(\eta\delta p', \delta p)$, which is equivalent to the identity $\gcd(\eta p, p') = \gcd(\eta p', p)$. This holds because every prime dividing η appears in p and p' with equal multiplicity, and so we have in fact that $\gcd(\eta p, p') = \gcd(\eta p', p) = \gcd(p, p')$. $\square$

The next proposition implies the converse of Theorem 1.8.3, provided $n \neq 2$.

Proposition 1.8.4. Assume $n \neq 2$ and let $G = G(r,p,q,n)$.

(1) The center of G has order $\frac{r}{pq} \cdot \gcd(p,n)$.

(2) The abelianization $G/[G,G]$ of G has order $\frac{2r}{pq} \cdot \gcd(q,n)$.

Proof. One can easily check that, since $n \neq 2$, the center of G is given by the set of its scalar elements (i.e. of the form c^i). The number of scalar elements in G is $\frac{1}{q}$ times the number of scalar elements in $G(r,p,n)$, which is $\frac{r}{p} \cdot \gcd(p,n)$ by [79, Corollary 4.1].

To prove (2), it suffices to count the linear characters of G since these are equal in number to the order of $G/[G,G]$. By [25, §6], the linear characters of $G(r,n)$ are parametrized by r-tuples of partitions $(\lambda_0, \ldots, \lambda_{r-1})$ where all partitions λ_i are empty except one which can be either (n) or (1^n). The linear representations of $G(r,1,q,n)$ are parametrized by these r-tuples of partitions where, if the only non-empty partition appears in a position i, then $ni \equiv 0 \pmod{q}$ (i.e. $(\lambda_0, \ldots, \lambda_{r-1}) \in \mathrm{Fer}(r,q,1,n)$ in the notation of [25, §6]). Therefore the number of linear characters of $G(r,1,q,n)$ is $\frac{2r}{q} \cdot \gcd(q,n)$. One can likewise check that, since $n \neq 2$, each linear character of G is given by the common restriction of exactly p distinct linear characters of $G(r,1,q,n)$. Thus the number of linear characters of G is $\frac{1}{p}$ times the number of linear characters of $G(r,1,q,n)$. $\square$

Combining the preceding theorem and proposition gives this corollary.

Corollary 1.8.5. Assume $n \neq 2$ and $pq = p'q'$. Then $G(r,p,q,n) \cong G(r,p',q',n)$ if and only if $\gcd(p,n) = \gcd(p',n)$ and $\gcd(q,n) = \gcd(q',n)$.

1.8.2 Isomorphisms in rank two

In this section we fix $n = 2$, and assume that p, p', q, q' divide r and $pq = p'q'$ divides $2r$. We now determine when the two groups $G = G(r, p, q, 2)$ and $G' = G(r, p', q', 2)$ are isomorphic.

In referring to elements of these groups, it is convenient to abbreviate our notation by writing $(a, b; \pi)$ for the image of the element $((a, b), \pi) \in G(r, p, 1, 2)$ in $G(r, p, q, 2)$. We thus view $G(r, p, q, 2)$ as the set of triples $(a, b; \pi) \in S_2 \times \mathbb{Z}_r \times \mathbb{Z}_r$ with $a + b$ divisible by p, where $(a, b; \pi) = (a', b'; \pi')$ if and only if $\pi = \pi'$ and $a - a' \equiv b - b' \equiv k\frac{r}{q} \pmod{r}$ for some integer k. Multiplication is given by

$$(a, b; \pi)(a', b'; \pi') = \begin{cases} (a + a', b + b'; \pi\pi') & \text{if } \pi' = 1 \in S_2, \\ (b + a', a + b'; \pi\pi') & \text{if } \pi' \neq 1 \in S_2. \end{cases}$$

We now have this lemma:

Lemma 1.8.6. If $p + p'$ and $q + q'$ are both odd and $\frac{r}{pq}$ is even then $G \ncong G'$.

Proof. Since $pq = p'q'$ we may assume without loss of generality that p' and q are odd and that p and q' are even. By Theorem 1.8.3 we then have that $G \cong G(r, pq, 1, 2)$ and $G' \cong G(r, 1, p'q', 2)$, and so it is enough to show that if p and $\frac{r}{p}$ are both even then $G(r, p, 1, 2) \ncong G(r, 1, p, 2)$.

To this end, let $A = \{g^{r/p} : g \in G(r, p, 1, 2)\}$ and $B = \{g^{r/p} : g \in G(r, 1, p, 2)\}$. It suffices to show that $|A| = p$ and $|B| = p + 1$. It is easy to check that A consists of the distinct elements $(\frac{ir}{p}, -\frac{ir}{p}; 1) \in G(r, p, 1, 2)$ for $i \in [p]$. It is likewise a straightforward exercise to show that B consists of the distinct images in $G(r, 1, p, 2)$ of $(0, \frac{ir}{p}; 1) \in G(r, 2)$ for $i \in [p]$ together with $(\frac{r}{2p}, \frac{r}{2p}; 1) \in G(r, 2)$. □

The next lemma is similar.

Lemma 1.8.7. If exactly one of the four parameters p, p', q, q' is odd then $G \ncong G'$.

Proof. We may assume that the unique odd parameter is either q' or p'. By Theorem 1.8.3, if q' is the unique odd parameter then $G' \cong G(r, pq, 1, 2)$, and if p' is the unique odd parameter then $G' \cong G(r, 1, pq, 2)$, and in either case $G \cong G(r, \frac{pq}{2}, 2, 2)$. It thus suffices to show that if p and $\frac{r}{p}$ are even then $G(r, 2p, 1, 2) \ncong G(r, p, 2, 2)$ and $G(r, 1, 2p, 2) \ncong G(r, p, 2, 2)$. With these hypotheses on p and $\frac{r}{p}$, let

$$A = \{g^{r/p} : g \in G(r, 2p, 1, 2)\},$$
$$B = \{g^{r/p} : g \in G(r, 1, 2p, 2)\},$$
$$C = \{g^{r/p} : g \in G(r, p, 2, 2)\}.$$

As in the proof of Lemma 1.8.6, it is not difficult to check that A consists of the distinct elements $(\frac{ir}{p}, -\frac{ir}{p}; 1) \in G(r, 2p, 2)$ for $i \in [p]$. On the other hand, one finds similarly that B consists of the distinct images in $G(r, 1, 2p, 2)$ of the elements $(0, \frac{ir}{p}; 1) \in G(r, 2)$ for

$i \in [p]$. Finally, C consists of the distinct images in $G(r,p,2,2)$ of the elements $(\frac{ir}{p}, -\frac{ir}{p}; 1) \in G(r,p,2)$ for $i \in [\frac{p}{2}]$. Thus $|A| = |B| = p$ and $|C| = \frac{p}{2}$, which establishes the desired non-isomorphisms. $\square$

We now examine a particular class of groups $G = G(r,p,2)$ where we can explicitly describe an isomorphism $\phi : G \to G^*$.

Lemma 1.8.8. If p or $\frac{r}{p}$ is odd then $G(r,p,1,2) \cong G(r,1,p,2)$.

Proof. If p is odd then $G(r,p,1,2) \cong G(r,1,p,2)$ by Theorem 1.8.3, so assume that $\frac{r}{p}$ is odd. Let p' be the largest power of 2 dividing p (and hence also r), and let $q = 1$ and $q' = p/p'$. With respect to these choices of p, p', q, q', the special primes are precisely the odd primes dividing p. Write $\frac{2r}{p} = \frac{rn}{pq} = \eta\delta$ as in Proposition 1.8.2, so that η is a product of non-special primes and δ is a product of special primes, and we have

$$G(r,p,1,2) \cong G(r,\delta p,1,2) \times \mathbb{Z}_\delta$$

and

$$G(r,1,p,2) \cong G(r,\delta,p,2) \times \mathbb{Z}_\delta \cong G(r,1,\delta p,2) \times \mathbb{Z}_\delta,$$

the second congruence on the right following from Theorem 1.8.3 as δ is odd. Because $\frac{r}{p}$ is also odd, η is even and $\frac{\eta}{2} = \frac{r}{\delta p}$ is odd; thus $\frac{r}{\delta p}$ is a product of odd primes not dividing p, and so is coprime to both p and δ and in particular to δp.

Since $G(r,p,1,2) \cong G(r,1,p,2)$ if $G(r,\delta p,1,2) \cong G(r,1,\delta p,2)$, the preceding argument shows that we may assume without loss of generality that $\frac{r}{p}$ and p are coprime. One checks that for $d = r/p'$ the map

$$\begin{array}{rccc} \phi : & G(r,p,1,2) & \to & G(r,1,p,2) \\ & (i,j;\pi) & \mapsto & (i,j+di;\pi) \end{array}$$

is a well-defined group homomorphism. To show that ϕ is an isomorphism it is enough to demonstrate injectivity, so let $g \in G(r,p,1,2)$ such that $\phi(g) = 1$. Then g is necessarily of the form $(i,j;1)$ with

$$i + j \equiv 0 \pmod{p} \qquad \text{and} \qquad i \equiv j + di \equiv k\tfrac{r}{p} \pmod{r} \text{ for some } k \in [p],$$

the second congruence following from the assumption that $\phi(i,j;1) = (i,j+di;1)$ represents the identity in $G(r,1,p,2)$. These two congruences imply that $k\frac{r}{p}(2-d)$ is a multiple of p. Since d is odd, no number dividing $2-d$ divides either 2 or d, and as every odd prime dividing p also divides d, it follows that $\gcd(2-d,p) = 1$. Since $\frac{r}{p}$ is coprime to p by hypothesis, we conclude that k is a multiple of p, which implies that $i \equiv j \equiv 0 \pmod{r}$ and in turn that $g = 1$, as desired. $\square$

Gathering together the preceding results yields the following proposition.

Proposition 1.8.9. Assume $pq = p'q'$. Then $G(r, p, q, 2) \cong G(r, p', q', 2)$ if and only if one of the following mutually exclusive conditions holds:

(i) $p + p'$ and $q + q'$ are both even;

(ii) $p + p'$, $q + q'$, and $\frac{r}{pq}$ are all odd integers.

Proof. If the first condition holds then $G \cong G'$ by Theorem 1.8.3. If the second condition holds then since $pq = p'q'$, exactly one of p, q is odd and it follows that pq in fact divides r. In this case, we may assume that p and q' are even and that p' and q are odd. Theorem 1.8.3 then implies that $G \cong G(r, pq, 1, 2)$ and $G' \cong G(r, 1, pq, 2)$, while Lemma 1.8.8 implies that $G(r, pq, 1, 2) \cong G(r, 1, pq, 2)$.

If $p + p'$ and $q + q'$ are both odd but $\frac{r}{pq}$ is even then $G \not\cong G'$ by Lemma 1.8.6. If $p + p'$ and $q + q'$ have different parities then exactly one of the parameters p, p', q, q' is odd as $pq = p'q'$, so $G \not\cong G'$ by Lemma 1.8.7. □

1.8.3 Conclusions

As in the previous sections, we let r, n be positive integers and p, p', q, q' be positive divisors of r such that $pq = p'q'$ divides rn. Combining Corollary 1.8.5 and Proposition 1.8.9 gives this summary theorem.

Theorem 1.8.10. The projective reflection groups $G(r, p, q, n)$ and $G(r, p', q', n)$ are isomorphic if and only if either (i) $\gcd(p, n) = \gcd(p', n)$ and $\gcd(q, n) = \gcd(q', n)$ or (ii) $n = 2$ and the numbers $p + p'$ and $q + q'$ and $\frac{r}{pq}$ are all odd integers.

Finally, by combining this with Theorem 1.7.9 we get the result promised at the beginning of this chapter.

Theorem 1.8.11. The complex reflection group $G(r, p, n) = G(r, p, 1, n)$ has a GIM if and only if $G(r, p, 1, n) \cong G(r, 1, p, n)$.

Proof. Replacing (p, q, p', q') by $(p, 1, 1, p)$ in Theorem 1.8.10 implies that $G(r, p, 1, n) \cong G(r, 1, p, n)$ if and only if $\gcd(p, n) = \gcd(1, n) = 1$, or $n = 2$ and $p+1$ is odd and r/p is odd. Since if $p + 1$ then $\gcd(p, 2) = 1$ automatically, it follows that $G(r, p, 1, n) \cong G(r, 1, p, n)$ if and only if $\gcd(p, n) = 1$, or $n = 2$ and r/p is odd. These conditions coincide precisely with the ones given in Theorem 1.7.9 for $G(r, p, n)$ to have a generalized involution model. □

On seeing this theorem one naturally asks whether for arbitrary projective reflection groups the property of having a GIM is equivalent to self-duality; i.e., the existence of an isomorphism $G(r, p, q, n) \cong G(r, q, p, n)$. This turns out to be false, and much of the preprint [29] (only a small part of which is incorporated into this thesis) is devoted to clarifying which groups $G(r, p, q, n)$ have GIMs. The classification in [29] is ultimately incomplete; however, the partial results in that work suggest some plausible conjectures as to a complete answer, which is the subject of future work.

Chapter 2

A Frobenius-Schur indicator for unipotent characters

This chapter represents a revised and expanded version of the semi-expository paper [80].

2.1 Introduction

Each finite, irreducible Coxeter system (W, S) possesses a set of "unipotent characters" $\mathrm{Uch}(W)$, introduced by Lusztig in [66]. When (W, S) is crystallographic, $\mathrm{Uch}(W)$ arises from Lusztig's set of "unipotent representations" of a corresponding finite reductive group; Lusztig's result that this set depends only on (W, S) (and not on the root datum) was a primary motivation for the definition. By construction, $\mathrm{Uch}(W)$ always contains as a subset the set $\mathrm{Irr}(W)$ of complex irreducible characters of the Coxeter group W. However, we typically view the elements $\Phi \in \mathrm{Uch}(W)$ not as characters but simply as formal objects with three defining attributes:

- A polynomial $\mathrm{FakeDeg}(\Phi) \in \mathbb{N}[x]$ with nonnegative integer coefficients, called the *fake degree.*

- A nonzero polynomial $\mathrm{Deg}(\Phi) \in \mathbb{R}[x]$ with real coefficients, called the *(generic) degree.*

- A root of unity $\mathrm{Eig}(\Phi) \in \mathbb{C}^\times$, called the *Frobenius eigenvalue.*

It takes some care to adequately describe $\mathrm{Uch}(W)$ for all finite, irreducible Coxeter systems (W, S), and this description is not so well-known as that of, say, $\mathrm{Irr}(W)$. Section 2.2.2 supplies these missing details, which for the moment we can work without.

This chapter concerns an interesting way of making sense of the question: is there a well-defined notion of a Frobenius-Schur indicator for a "unipotent character" $\Phi \in \mathrm{Uch}(W)$? Recall that if G is a finite group, then the Frobenius-Schur indicator of an irreducible character $\Phi \in \mathrm{Irr}(G)$ is the number

$$\epsilon(\Phi) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \Phi \text{ is the character of a representation of } G \text{ in a real vector space,} \\ 0, & \text{if the values of } \Phi \text{ are not all real,} \\ -1, & \text{otherwise, in which case } \Phi \text{ is called quaternionic.} \end{cases} \tag{2.1.1}$$

Alternatively, $\epsilon(\Phi)$ is also the average value of $\Phi(g^2)$ over $g \in G$ (as is explained, for example, in the proof of [51, Theorem 23.14]). Since $\mathrm{Uch}(W)$ consists of formal objects and not characters, one cannot apply this definition directly, and it is not at all obvious what kind of definition could serve as an appropriate substitute.

If (W, S) is crystallographic, there is an easy way of circumventing this difficulty. In this case the elements of $\mathrm{Uch}(W)$ are in bijection with the unipotent characters of various finite reductive groups having W as Weyl group. While not clear *a priori*, it turns out that all of the actual characters corresponding to a given $\Phi \in \mathrm{Uch}(W)$ have the same indicator—an observation of Lusztig [68] which we state precisely as Proposition 2.2.3. There is thus a logical definition of ϵ on $\mathrm{Uch}(W)$ when (W, S) is crystallographic, which happens to have the following simple description: define $\epsilon(\Phi)$ to be 1 if $\mathrm{Eig}(\Phi)$ is real and 0 otherwise.

Things become more interesting in the case when (W, S) is non-crystallographic. In this situation, to describe the Frobenius-Schur indicator in a satisfactory way, we require a heuristic definition which is consistent with the crystallographic case but which makes sense for all types. A series of papers appearing in the past decade, beginning with Kottwitz [55] and Casselman [31] and proceeding through Lusztig and Vogan [72], suggests such a definition, surprisingly, in terms of the irreducible multiplicities of a certain W-representation. As we will see in a moment, this approach unexpectedly leads to an extension of the Frobenius-Schur indicator to the non-crystallographic case which forces us to assign an indicator of -1 to some elements of $\mathrm{Uch}(W)$; that is, which suggests the existence of quaternionic "unipotent characters."

The W-representation of interest is given as follows. It is interesting to note that Adin, Postnikov, and Roichman studied exactly this representation in type A_n in their paper [1].

Definition 2.1.1. Given a finite Coxeter system (W, S) with length function $\ell : W \to \mathbb{N}$, let

$$\mathrm{Invol}(W) = \mathbb{Q}\text{-span}\{a_w : w \in W \text{ such that } w^2 = 1\}$$

be a vector space with a basis indexed by the involutions in W, and define $\varrho_W : S \to \mathrm{GL}(\mathrm{Invol}(W))$ by the formula

$$\varrho_W(s)a_w = \begin{cases} -a_w, & \text{if } sw = ws \text{ and } \ell(ws) < \ell(w), \\ a_{sws}, & \text{otherwise,} \end{cases} \qquad \text{for } s \in S \text{ and } w \in W \text{ with } w^2 = 1.$$

The map ϱ_W extends to a representation of W (a nontrivial fact, whose derivation from results of Lusztig and Vogan [72, 70] will be explained in Section 2.2.1), and each conjugacy class of involutions in W spans a ϱ_W-invariant subspace in $\mathrm{Invol}(W)$. Let $\varrho_{W,\sigma}$ denote this subrepresentation on the space spanned by the conjugacy class of the involution $\sigma \in W$.

Notation. We write χ_W and $\chi_{W,\sigma}$ (when $\sigma \in W$ and $\sigma^2 = 1$) for the characters of ϱ_W and $\varrho_{W,\sigma}$.

Kottwitz [55] found a formula for the multiplicities of the irreducible constituents of ϱ_W in the case that (W, S) is a Weyl group, in terms of Lusztig's "non-abelian Fourier transform" [63]. (More precisely, Kottwitz computed the irreducible constituents of certain representations induced from the centralizers of involutions in W. The sum of these induced representations is isomorphic to ϱ_W, although this is not an obvious fact; see [43, Remark 2.2] for a detailed explanation.) Kottwitz proved this formula in the classical cases, while Casselman [31] carried out the calculations necessary to check it in the exceptional ones. In more recent work, Lusztig and Vogan re-encountered ϱ_W as a specialization of a certain Hecke algebra representation, and noted a way of restating Kottwitz's results to involve the Frobenius-Schur indicator [72, §6.4].

Our main object is to describe how this last formulation extends even to the case when (W, S) is non-crystallographic. To this end, we must briefly introduce the *Fourier transform matrix* of $\mathrm{Uch}(W)$. For any finite, irreducible Coxeter system (W, S), this is a real symmetric matrix $\mathbf{M}$, with rows and columns indexed by $\mathrm{Uch}(W)$, which possesses the following distinguishing properties (among others; see [42, Theorem 6.9] and also Section 2.5.4):

(P1) $\mathbf{M}$ transforms the vector of fake degrees of $\mathrm{Uch}(W)$ to the vector of (generic) degrees, permuted by a certain involution (see Theorem 2.5.3).

(P2) $\mathbf{M}$ is block diagonal with respect to the division of $\mathrm{Uch}(W)$ into families (see Section 2.2.4).

(P3) $\mathbf{M}$ fixes each of the vectors indexed by $\mathrm{Uch}(W)$ whose entries are the irreducible multiplicities, extended by zeros, of the left cell representations of W (see Theorem 2.6.3).

(P4) $\mathbf{M}$ and the diagonal matrix of Frobenius eigenvalues of $\mathrm{Uch}(W)$ determine a representation of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ (see Theorem 2.5.6); in particular, $\mathbf{M}^2 = 1$.

Section 2.5 provides, with accompanying references, a careful description of the matrix $\mathbf{M}$ attached to each finite, irreducible Coxeter system. While the literature often tends to present these matrices as a somewhat heuristic construction, work in preparation of Broué, Malle, and Michel shows that when W is a primitive (complex) reflection group, $\mathbf{M}$ is uniquely determined under a suitable set of natural axioms [22].

As a final preliminary, we recall that an element $\Phi \in \mathrm{Uch}(W)$ is *special* if the largest powers of x dividing the polynomials $\mathrm{FakeDeg}(\Phi) \in \mathbb{N}[x]$ and $\mathrm{Deg}(\Phi) \in \mathbb{R}[x]$ are equal. Every special $\Phi \in \mathrm{Uch}(W)$ belongs to the subset $\mathrm{Irr}(W)$. Such characters play an important role in the theory of unipotent characters for finite reductive groups and have been classified by Lusztig [63, Chapter 4].

The Fourier transform $\mathbf{M}$ acts on functions $f : \mathrm{Uch}(W) \to \mathbb{C}$ by matrix multiplication, since we may view f as a vector whose entries are indexed by $\mathrm{Uch}(W)$. The following theorem, which is our main result, extends [55, Theorem 1] and [72, §6.4] to define a function on $\mathrm{Uch}(W)$ which we might naturally view as the Frobenius-Schur indicator. The proof of this result appears at the end of Section 2.5.

Theorem 2.1.2. Suppose (W, S) is a finite, irreducible Coxeter system with associated Fourier transform matrix $\mathbf{M}$. There then exists a *unique* function $\epsilon : \mathrm{Uch}(W) \to \mathbb{R}$ such that

(1) $\epsilon(\Phi) \in \{-1, 0, 1\}$ for all $\Phi \in \mathrm{Uch}(W)$.

(2) $\epsilon(\Phi) = 0$ if and only if the Frobenius eigenvalue of $\Phi \in \mathrm{Uch}(W)$ is not real.

(3) $(\mathbf{M}\epsilon)(\Phi)$ is the multiplicity of Φ in χ_W for each special $\Phi \in \mathrm{Irr}(W) \subset \mathrm{Uch}(W)$.

For this function ϵ, it in fact holds that

$$\chi_W = \sum_{\psi \in \mathrm{Irr}(W)} (\mathbf{M}\epsilon)(\psi)\psi. \tag{2.1.2}$$

Furthermore, if (W, S) is crystallographic, then ϵ coincides with the Frobenius-Schur indicator on $\mathrm{Uch}(W)$, i.e., $\epsilon(\Phi)$ is 1 or 0 according to whether $\mathrm{Eig}(\Phi)$ is real or non-real for $\Phi \in \mathrm{Uch}(W)$.

Conditions (1) and (2) are basic properties one would desire of a prospective Frobenius-Schur indicator. In particular we can restate (2) as the requirement that $\epsilon(\Phi) = 0$ if and only if Φ is not fixed by an operator on $\mathrm{Uch}(W)$ which we reasonably view as complex conjugation (see Proposition 2.2.4). The simplicity of condition (3) and the surprising fact that it implies (2.1.2) make the function defined in the theorem an attractive extension of the Frobenius-Schur indicator to the non-crystallographic case. Admittedly, this is not evidence that ϵ is the "right" extension, only that it is a suitable and interesting one.

This Frobenius-Schur indicator gives rise to two quaternionic "unipotent characters" in type H_4. We record this phenomenon and some other properties of ϵ which become evident in the proof of Theorem 2.1.2 in the following proposition.

Proposition 2.1.3. Let $\epsilon : \mathrm{Uch}(W) \to \{-1, 0, 1\}$ be the function defined in Theorem 2.1.2.

(a) $\epsilon(\Phi) = 1$ for all $\Phi \in \mathrm{Irr}(W)$. If (W, S) is classical, then $\epsilon(\Phi) = 1$ for all $\Phi \in \mathrm{Uch}(W)$.

(b) $\epsilon(\Phi) = -1$ if and only if (W, S) is of type H_4 and Φ is either of two elements of $\mathrm{Uch}(W)$ with

$$\mathrm{Deg}(\Phi) = \tfrac{1}{60}x^6 + \text{higher powers of } x.$$

(c) $(\mathbf{M}\epsilon)(\Phi)$ is a nonnegative integer for all $\Phi \in \mathrm{Uch}(W)$. If (W, S) is classical, then $(\mathbf{M}\epsilon)(\Phi)$ is nonzero if and only if Φ is special. More generally, $(\mathbf{M}\epsilon)(\Phi)$ is nonzero only if $\Phi \in \mathrm{Irr}(W)$ or if Φ a single element of $\mathrm{Uch}(W) \setminus \mathrm{Irr}(W)$ in each of the types F_4, E_8, and H_4.

Remark. A lengthy computation shows that in type H_4, there is no symmetric matrix $\mathbf{M}$ satisfying (P1) and (P2) for which there exists a function $\epsilon : \mathrm{Uch}(W) \to \{0,1\}$ such that (2.1.2) holds. Thus, in this sense even a different choice of Fourier transform matrix in type H_4 still leads to quaternionic unipotent characters.

It is an interesting open problem to describe how much of the preceding theory extends from Coxeter systems to larger classes of groups. The preprint [43] of Geck and Malle, for example, describes analogues of many statements here for Coxeter systems with an involution preserving the set of simple reflections. Moreover, for some but not all complex reflection groups, there are analogous notions of "unipotent characters" and Fourier transforms (see [20]), for which one expects some meaningful generalization of Theorem 2.1.2 to hold.

Independent of any connection to unipotent characters, the representation ϱ_W is itself an interesting thing to study (e.g., see [1] which analyzes ϱ_W in type A_n). When W is classical of type A_n, BC_n, or D_n, it is a natural problem to describe the irreducible decomposition of ϱ_W in terms of the familiar sets of partitions α, bipartitions (α, β), and unordered bipartitions $\{\alpha, \beta\}$ indexing $\mathrm{Irr}(W)$. One can find such a description in Kottwitz's paper [55]—in particular, Kottwitz shows that the irreducible constituents of χ_W are precisely the special characters of W when W is classical (note also part (c) of Proposition 2.1.3). Many proofs in [55] are abbreviated or omitted, and we take the opportunity in this thesis to provide more detailed proofs of χ_W's decomposition in types BC_n and D_n. (Proofs of the type A_n decomposition, which we de not reproduce here, already appear in several places in the literature.) This material appears in Section 2.3.

One notable corollary of the calculations in Sections 2.3 and 2.4 is the following statement, which is interesting to compare with [101, Theorem 1]. Here, a *Gelfand model* is a representation of a finite group which is the multiplicity free sum of all of the group's irreducible representations.

Theorem 2.1.4. If (W, S) is a finite, irreducible Coxeter system then the representation ϱ_W is a Gelfand model if and only if (W, S) is of type A_n, H_3, or $I_2(m)$ with m odd.

Our final results concern a conjecture of Kottwitz connecting the decomposition of χ_W to the left cells of W. (See Section 2.6 for the definition of the left cells and left cell representations of W.)

Conjecture 2.1.5 (Kottwitz [55]). Let Γ be a left cell in W and let χ_Γ denote the character of the corresponding left cell representation. Let $\sigma \in W$ be an involution and write Σ for its conjugacy class in W. Then $\langle \chi_{W,\sigma}, \chi_\Gamma \rangle = |\Sigma \cap \Gamma|$, where $\langle \cdot, \cdot \rangle$ denotes the standard L^2-inner product on functions $W \to \mathbb{C}$.

Kottwitz [55] observed that in type A_n the conjecture is true, following from the known description of the left cells of W in terms of the RSK-correspondence. Casselman [31] meanwhile verified the conjecture in types F_4 and E_6 by a computer calculation. In Section 2.6, we show ourselves that the conjecture holds in all of the non-crystallographic cases H_3, H_4, and $I_2(m)$. Two recent preprints of Geck and Bonnafé [17, 40] establish several more cases, leaving the conjecture open only in type E_8.

We note here that a weakened version of the conjecture follows immediately from Theorem 2.1.2 and recent work of Geck [39].

Theorem 2.1.6. Let Γ be a left cell in W and let χ_Γ denote the character of the corresponding left cell representation. Then the inner product $\langle \chi_W, \chi_\Gamma \rangle$ is equal to the number of involutions in Γ.

Proof. Proposition 2.1.3 and property (P3) of the Fourier transform matrix imply that $\langle \chi_W, \chi_\Gamma \rangle = \sum_{\psi \in \mathrm{Irr}(W)} \langle \psi, \chi_\Gamma \rangle$, which is the cardinality of the set $\{\sigma \in \Gamma : \sigma^2 = 1\}$ by [39, Theorem 1.1]. $\square$

We organize this chapter as follows. In Section 2.2, we note several preliminaries concerning Coxeter systems, the representation ϱ_W, and the set $\mathrm{Uch}(W)$. Section 2.3 provides a detailed survey of Kottwitz's results [55]. In Section 2.5, we describe in detail the Fourier transform matrices attached to $\mathrm{Uch}(W)$ and derive from this information the proof of Theorem 2.1.2. Finally, in Section 2.6 we prove Kottwitz's conjecture in the non-crystallographic types H_3, H_4, and $I_2(m)$. This last section explicitly describes the left cells in these types, and includes a proof of property (P3) for the Fourier transform matrix of $\mathrm{Uch}(W)$.

2.2 Preliminaries

Throughout, we adopt the following notational conventions: $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{P}$ is the set of positive integers, and $[n]$ is the set of the first n positive integers, with $[0] = \varnothing$.

2.2.1 Representing W in $\mathrm{Invol}(W)$

Let (W, S) be a finite Coxeter system with length function $\ell : W \to \mathbb{N}$, and define the vector space $\mathrm{Invol}(W)$ as in the introduction. Here, we briefly confirm that the map ϱ_W given in Definition 2.1.1 indeed extends to a representation of W. The easiest way of deriving this from known results is to prove a slightly more general fact, which goes as follows.

For each constant $k \in \mathbb{Q}$, let $\varrho_{W,k} : S \to \mathrm{GL}(\mathrm{Invol}(W))$ be the map given by the formula

$$\varrho_{W,k}(s) a_w = \begin{cases} a_w + k a_{sw}, & \text{if } sw = ws \text{ and } s \notin \mathrm{Des}_R(w), \\ -a_w, & \text{if } sw = ws \text{ and } s \in \mathrm{Des}_R(w), \\ a_{sws}, & \text{if } sw \neq ws, \end{cases}$$

for $s \in S$ and $w \in W$ with $w^2 = 1$. Here we have written $\mathrm{Des}_R(w) = \{s \in S : \ell(ws) < \ell(w)\}$ for the right descent set of $w \in W$, which coincides with the left descent set when $w^2 = 1$. In this notation, the map ϱ_W from the introduction is precisely $\varrho_{W,0}$.

Define $\mathcal{H}_{q^2}$ as the Hecke algebra with parameter q^2 corresponding to (W, S): for us, this is the unital associative $\mathbb{Q}[q]$-algebra with basis $\{T_w : w \in W\}$ and multiplication given by the rules

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{for } w, w' \in W \text{ with } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s + 1)(T_s - q^2) = 0, & \text{for } s \in S. \end{cases}$$

In [72, 70], Lusztig and Vogan show that the multiplication defined for $s \in S$ and $w \in W$ by

$$T_s a_w = \begin{cases} q a_w + (q+1) a_{sw}, & \text{if } sw = ws \text{ and } s \notin \mathrm{Des}_R(w), \\ (q^2 - q - 1) a_w + (q^2 - q) a_{sw}, & \text{if } sw = ws \text{ and } s \in \mathrm{Des}_R(w), \\ a_{sws}, & \text{if } sw \neq ws \text{ and } s \notin \mathrm{Des}_R(w), \\ (q^2 - 1) a_w + q^2 a_{sws}, & \text{if } sw \neq ws \text{ and } s \in \mathrm{Des}_R(w). \end{cases}$$

makes the $\mathbb{Q}[q]$-module $\mathrm{Invol}(W) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$ into an $\mathcal{H}_{q^2}$-module. (It is interesting to compare this with the Hecke algebra module in [1], which in type A_n is isomorphic to the one here once we replace q in [1] with q^2.) Specializing q to 1 shows that $\varrho_{W,2}$ is a well-defined W-representation, and from this we deduce the following stronger statement.

Proposition 2.2.1. Let (W, S) be a finite Coxeter system.

(1) The map $\varrho_{W,k}$ extends to a representation of W in $\mathrm{Invol}(W)$ for any $k \in \mathbb{Q}$.

(2) The representations $\varrho_{W,k}$ and $\varrho_{W,k'}$ are isomorphic for all $k, k' \in \mathbb{Q}$.

Recall that the geometric representation V of W is a real vector space with a basis $\Pi = \{\alpha_s : s \in S\}$ indexed by S and a bilinear form $(\cdot,\cdot)$ defined by linearly extending the formula

$$(\alpha_s, \alpha_t) = -\cos\left(\tfrac{\pi}{m(s,t)}\right), \qquad \text{for } s, t \in S, \text{ where } m(s,t) \text{ denotes the order of } st \text{ in } W.$$

Elements in the generating set $S \subset W$ act on V as reflections by the formula $s(v) = v - 2(\alpha_s, v)\alpha_s$ for $s \in S$ and $v \in V$, and this action extends to W, making V into a faithful W-module. The root system of (W, S) is the set $\Phi = \{w(\alpha) : \alpha \in \Pi\}$. The basis Π serves as a set of simple roots, with respect to which we let Φ^+ denote the set of positive roots in Φ. The right descent set of $w \in W$ then has the alternate characterization $\mathrm{Des}_R(w) = \{s \in S : w(\alpha_s) \notin \Phi^+\}$.

Proof of Proposition 2.2.1. Let $n(w)$ denote the dimension of the -1-eigenspace of $w \in W$ in its geometric representation. Clearly $n(sws) = n(w)$ for all $w \in W$ and $s \in S$. Suppose $w^2 = 1$ and $sw = ws$ and $s \notin \mathrm{Des}_R(w)$. Then (with respect to the geometric representation V) all eigenvalues of w and s are equal to ± 1, and s and w are simultaneously diagonalizable. Since $\alpha_s \in V$ spans the -1-eigenspace of s, it follows that α_s must be an eigenvector of w, and that $n(sw) = n(w) \pm 1$ according to whether the corresponding eigenvalue is ± 1. As $w(\alpha_s) \in \Phi^+$ since $s \notin \mathrm{Des}_R(w)$, we must have $w(\alpha_s) = \alpha_s$, so $n(sw) = n(w) + 1$.

List the involutions $w_1, \ldots, w_N \in W$ in an order such that $i \leq j$ implies $n(w_i) \leq n(w_j)$, and write a_i for the vector $a_{w_i} \in \mathrm{Invol}(W)$. By the preceding paragraph, for each $s \in S$ the matrix of $\varrho_W^k(s)$ with respect to the basis $\{a_i : 1 \leq i \leq N\}$ has the block lower triangular form

$$\varrho_W^k(s) = \begin{pmatrix} A_0 & & & & \\ kB_1 & A_1 & & & \\ & kB_2 & A_2 & & \\ & & \ddots & \ddots & \\ & & & kB_r & A_r \end{pmatrix} \tag{2.2.1}$$

where each A_i is a signed $n_i \times n_i$ permutation matrix, with n_i denoting the number of involutions $w \in W$ with $n(w) = i$, and each B_i is an $n_i \times n_{i-1}$ matrix whose entries are either 0 or 1. In particular, the matrices A_i and B_i have no dependence on k, and it is easy to see that the braid relations $\left(\varrho_W^k(s)\varrho_W^k(t)\right)^{m(s,t)} = 1$ for $s, t \in S$ hold for all $k \in \mathbb{Q}$ if and only if they hold for a single nonzero value of k.

From [72], we know that the braid relations hold when $k = 2$, which suffices to prove (i). Part (ii) follows by similar considerations: the trace of any product of matrices of the form (2.2.1) has no dependence on k, so the character of ϱ_W^k is the same for all values of $k \in \mathbb{Q}$ and we conclude that ϱ_W^k and $\varrho_W^{k'}$ are isomorphic representations. $\square$

2.2.2 References for the construction of $\mathrm{Uch}(W)$

In this section we provide references for the construction of $\mathrm{Uch}(W)$ for each finite, irreducible Coxeter system (W, S), and also for the associated data FakeDeg, Deg, and Eig.

Even before discussing $\mathrm{Uch}(W)$, we may give the general definition of the fake degree attached to a unipotent character. The *fake degree* of an irreducible character $\Phi \in \mathrm{Irr}(W)$ is the polynomial $\mathrm{FakeDeg}(\Phi) \in \mathbb{N}[x]$ whose coefficients are the multiplicities of Φ in the graded components of the coinvariant algebra of W (see [30, §2.4 and §11.1]). The set of irreducible characters $\mathrm{Irr}(W)$ always forms a subset of $\mathrm{Uch}(W)$, and we define the fake degrees of all $\Phi \in \mathrm{Uch}(W) \setminus \mathrm{Irr}(W)$ to be zero. For $\Phi \in \mathrm{Irr}(W)$, $\mathrm{Deg}(\Phi)$ and $\mathrm{Eig}(\Phi)$ are defined by

$$\mathrm{Deg}(\Phi) = \text{the } \textit{generic degree} \text{ of } \Phi \text{ (see [30, §10.11])} \qquad \text{and} \qquad \mathrm{Eig}(\Phi) = 1,$$

while the polynomial $\mathrm{FakeDeg}(\Phi)$ has the following formula [30, Proposition 11.1.1].

Proposition 2.2.2. For $\Phi \in \mathrm{Irr}(W)$ we have

$$\mathrm{FakeDeg}(\Phi) = \prod_{i=1}^{\ell}\left(1 - x^{d_i}\right) \cdot \frac{1}{|W|}\sum_{w \in W}\frac{\Phi(w)}{\det(1 - xw)},$$

where $d_1, \ldots, d_\ell$ are the degrees of the basic polynomial invariants of W (see [30, §2.4]), and the determinant of $1 - xw$ is evaluated by identifying W with the image of its geometric representation.

Explicit expressions for the right-hand side of this formula when (W, S) is classical appear in [30, §13.8]. We list the fake degrees in type $I_2(m)$ in Section 2.2.5. In the remaining exceptional and non-crystallographic types, one can readily compute the fake degrees using the proposition and the character table of W (see also [5, 6, 14]).

Now to describe $\mathrm{Uch}(W)$ itself. When (W, S) is crystallographic, $\mathrm{Uch}(W)$ corresponds to the actual set of unipotent characters of an associated finite reductive group. Carter's book [30] contains an excellent exposition of this correspondence; Geck and Malle's paper [42, §2] also serves as a useful reference. We emphasize, however, that the construction of $\mathrm{Uch}(W)$ is due originally to Lusztig [63, 66], as are the following facts. In brief, if G is a simple algebraic group defined over a finite field with q elements, with Frobenius map $F : G \to G$ and Weyl group (W, S), and if the finite group $G^F = \{g \in G : F(g) = g\}$ is split so that F acts trivially on W, then the following hold:

- The number of unipotent characters of G^F, together with their Frobenius eigenvalues (as defined in [63, Chapter 11]), depends only on the isomorphism class of (W, S).

- The degrees of the unipotent characters of G^F are given by the values at q of a set of generic degree polynomials, which also depends only on the isomorphism class of (W, S).

A complete parametrization of $\mathrm{Uch}(W)$ in these cases, together with a list of the associated degree polynomials, appears in [30, §13.8 and §13.9], while [63, Theorem 11.2] classifies in very simple way all of the Frobenius eigenvalues (see also Observation 2.2.7). From this description derives the following proposition, which summarizes some results of Lusztig [68] and shows that there is a natural definition of the Frobenius-Schur indicator on the formal set $\mathrm{Uch}(W)$ when (W, S) is a Weyl group.

Proposition 2.2.3 (Lusztig [68]). Let G be a simple algebraic group over an algebraically closed field of positive characteristic, with a Frobenius map $F : G \to G$ for which the finite group G^F is split. Then the irreducible unipotent characters of G^F all have Frobenius-Schur indicator 1 or 0, according to whether their Frobenius eigenvalues are real or non-real.

Proof. Perhaps the simplest way to extract this result from [68] is to compare [63, Theorem 11.2] with the description of $\epsilon(\Phi)$ (from [68]) given in [72, §6.4]. Alternatively, the result follows by comparing [63, Theorem 11.2] with [68, Corollary 1.12] if G is classical, or with Table 1 and Proposition 5.6 in [37] alongside the tables in [30, §13.9] or [63] if G is exceptional. $\square$

When the Coxeter system (W, S) is non-crystallographic there is no corresponding reductive group, and the definition of $\mathrm{Uch}(W)$ is instead based on heuristic arguments involving a list of postulates considered plausible desiderata. Lusztig's paper [66] lists these postulates and constructs the set $\mathrm{Uch}(W)$ with the associated degree polynomials when (W, S) is of any of the non-crystallographic types H_3, H_4, or $I_2(m)$. The corresponding Frobenius eigenvalues are given in [66] for H_3, in [75] for H_4, and in [67] for $I_2(m)$; however, this information is not presented in the literature as clearly as in the crystallographic case. Helpfully, one can access all the relevant data in types H_3 and H_4 (as well as in the exceptional types E_6, E_7, E_8, F_4, and G_2) from the `UnipotentCharacters` command in the computer algebra system CHEVIE [41]. To deal with the dihedral case, it is expedient to review an explicit construction of $\mathrm{Uch}(W)$; we do this in Section 2.2.5.

2.2.3 Two involutions Δ and j of $\mathrm{Uch}(W)$

We now define two canonical involutions Δ and j of the set $\mathrm{Uch}(W)$ which will be of importance in Section 2.5. The first of these, Δ, arises from the observation that there is a well-defined notion of formal complex conjugation on $\mathrm{Uch}(W)$. This fact derives easily from the descriptions we cited in the previous section, and we record it as the following proposition for future reference.

Proposition 2.2.4. For each finite, irreducible Coxeter system (W, S), there exists a *unique* involutory permutation Δ of $\mathrm{Uch}(W)$ with the following properties:

(i) Δ preserves $\mathrm{Deg}(\Phi)$ and inverts $\mathrm{Eig}(\Phi)$ for each $\Phi \in \mathrm{Uch}(W)$.

(ii) Δ fixes $\Phi \in \mathrm{Uch}(W)$ if and only if $\mathrm{Eig}(\Phi) = \pm 1$.

The involution Δ is the identity permutation if and only if (W, S) is of classical type. Furthermore, in all types, Δ fixes all elements of $\mathrm{Irr}(W) \subset \mathrm{Uch}(W)$

To define our second involution j, we recall that a polynomial $f(x)$ is *palindromic* if there exists a nonnegative integer $c \in \mathbb{N}$ such that $f(x) = x^c \cdot f(x^{-1})$. In particular, the zero polynomial is palindromic. As noted by Opdam [87], the fake degree of nearly every $\Phi \in \mathrm{Uch}(W)$ is palindromic and reflecting the coefficients of $\mathrm{FakeDeg}(\Phi)$ about a certain central power of x gives rise to a well-defined and meaningful involution of $\mathrm{Uch}(W)$. (For Weyl groups W, this was first observed by Beynon and Lusztig [14].) To explain what we mean, define a constant N_Φ for each $\Phi \in \mathrm{Uch}(W)$ by the formula

$$N_\Phi = \tfrac{1}{\Phi(1)} \textstyle\sum_r \Phi(r) \text{ for } \Phi \in \mathrm{Irr}(W) \qquad \text{and} \qquad N_\Phi = 0 \text{ for } \Phi \in \mathrm{Uch}(W) \setminus \mathrm{Irr}(W),$$

where the sum on the left is over all reflections $r \in W$ (i.e., those $r \in W$ conjugate to elements of the generating set S). Also, let N denote the total number of reflections in W. The following proposition now summarizes several observations of Opdam [87, Page 448]. (In comparing this proposition to Opdam's paper, the reader should note the following misprint: the exponent of T on the right-hand side of [87, Eq. (2)] should be $N_\tau - N$ and not $N - N_\tau$.)

Proposition 2.2.5 (Opdam [87]). For each finite, irreducible Coxeter system (W, S), there exists a *unique* involutory permutation j of $\mathrm{Uch}(W)$ with the following properties:

(i) The fake degree of $j(\Phi)$ is $x^{N-N_\Phi} \cdot \mathrm{FakeDeg}(\Phi)(x^{-1})$ for each $\Phi \in \mathrm{Uch}(W)$.

(ii) j fixes $\Phi \in \mathrm{Uch}(W)$ if and only if $\mathrm{FakeDeg}(\Phi)$ is palindromic.

The involution j is the identity permutation if and only if (W, S) is *not* of type E_7, E_8, H_3, or H_4. Furthermore, in all types, j fixes all elements of $\mathrm{Uch}(W) \setminus \mathrm{Irr}(W)$.

Remark. Under the assumption that W is a Weyl group, this result first appeared as Proposition A in Beynon and Lusztig's paper [14]. There is also another interpretation of the involution j in terms of rationality properties of the corresponding characters of cyclotomic Hecke algebras; see [77, Theorem 6.5].

Even in types E_7, E_8, H_3, and H_4, the permutation j is very nearly the identity. In particular, adopting Carter's notation for the elements of $\mathrm{Irr}(W)$ (see our explanations in Sections 2.4.2 and 2.4.3), we may describe the nontrivial actions of j on $\mathrm{Uch}(W)$:

- **Type** E_7. j exchanges $\phi_{512,11}$ with $\phi_{512,12}$.
- **Type** E_8. j exchanges $\phi_{4096,11}$ with $\phi_{4096,12}$ and $\phi_{4096,26}$ with $\phi_{4096,27}$.
- **Type** H_3. j exchanges $\phi_{4,3}$ with $\phi_{4,4}$.
- **Type** H_4. j exchanges $\phi_{16,3}$ with $\phi_{16,6}$ and $\phi_{16,18}$ with $\phi_{16,21}$.

One consequence of this description is that $\Phi \in \mathrm{Irr}(W)$ is special if and only if $j(\Phi \otimes \mathrm{sgn})$ is special, a fact which Carter notes as [30, Corollary 11.3.10] when W is crystallographic. Moreover, as Beynon and Lusztig observed in [14], the irreducible characters $\Phi \in \mathrm{Irr}(W)$ with $j(\Phi) \neq \Phi$ are precisely the characters corresponding to irreducible representations of the corresponding Hecke algebra of W which are not rational. For more on this property, see also [30, Section 11.3] and [61].

2.2.4 Families in $\mathrm{Uch}(W)$

The set $\mathrm{Uch}(W)$ possesses a distinguished decomposition into disjoint subsets called *families.* When (W, S) is crystallographic, these arise as the equivalence classes of a certain relation (see [30, Section 12.3]), but in general the families are defined heuristically (see [66]). To specify the Fourier transform matrix of $\mathrm{Uch}(W)$ it suffices to attach Fourier transform matrices to each of the families, and we therefore discuss some of their significant properties here.

To begin, each family $\mathcal{F} \subset \mathrm{Uch}(W)$ contains a unique special element Φ—*special*, we recall from the introduction, means that there exist an integer $e \in \mathbb{N}$ and nonzero real numbers a, b such that

$$\mathrm{FakeDeg}(\Phi) = ax^e + \text{higher order terms} \qquad \text{and} \qquad \mathrm{Deg}(\Phi) = bx^e + \text{higher order terms}.$$

Since $\mathrm{FakeDeg}(\Phi) = 0$ if $\Phi \notin \mathrm{Irr}(W)$, each special Φ necessarily belongs to the subset of irreducible characters $\mathrm{Irr}(W) \subset \mathrm{Uch}(W)$. If Φ is the unique special element of a family $\mathcal{F}$ and e is defined as above, then x^e is also the largest power of x dividing $\mathrm{Deg}(\Psi)$ for all other $\Psi \in \mathcal{F}$. Furthermore, if $x^{e'}$ is the largest power of x dividing $\mathrm{FakeDeg}(\Psi)$ for some $\Psi \in \mathcal{F} \cap \mathrm{Irr}(W)$, then $e < e'$ unless $\Phi = \Psi$.

From this discussion we see that each family $\mathcal{F}$ in $\mathrm{Uch}(W)$ has a nonempty intersection with $\mathrm{Irr}(W)$. Thus the division of $\mathrm{Uch}(W)$ into families induces a similar partition of $\mathrm{Irr}(W)$

into families. The family decomposition of $\mathrm{Irr}(W)$, in contrast to that of $\mathrm{Uch}(W)$, has a simple definition in terms of the two-sided cell representations of W which applies in all types. Namely, a family in $\mathrm{Irr}(W)$ consists of the characters appearing as constituents of two-sided cells having the same special constituent; see [30, §12.4] for a detailed explanation.

Let Δ and j be the involutions of $\mathrm{Uch}(W)$ defined in Propositions 2.2.4 and 2.2.5 above. Every family in $\mathrm{Uch}(W)$ is preserved by both of these permutations, and we make the following definition concerning the action of j on a family.

Definition 2.2.6. An element of $\mathrm{Uch}(W)$ is *exceptional* if it is not fixed by the involution j, or equivalently if its fake degree is not palindromic. (All such characters are listed at the end of Section 2.2.3.) A family in $\mathrm{Uch}(W)$ is *exceptional* if any of its elements are exceptional.

Remark. This notion of exceptionality originates in Beynon and Lusztig's paper [14] and has since become a standard definition, appearing in various places [55, 87]. All exceptional families have size 4, and they only occur in types E_7, E_8, H_3, and H_4. In types E_7 and H_3, $\mathrm{Uch}(W)$ contains exactly one exceptional family, while in types E_8 and H_4, $\mathrm{Uch}(W)$ contains two exceptional families.

As Lusztig first observed [63], a single construction provides an extremely convenient way of parametrizing nearly every family in $\mathrm{Uch}(W)$. Given a finite group Γ, let $C_\Gamma(x) = \{g \in \Gamma : gxg^{-1} = x\}$ denote the centralizer of an element $x \in \Gamma$, and define

$$\mathscr{M}(\Gamma) \tag{2.2.2}$$

as the set of equivalence classes of pairs (x, σ) for $x \in \Gamma$ and $\sigma \in \mathrm{Irr}(C_\Gamma(x))$, with respect to the relation

$$(x, \sigma) \sim (gxg^{-1}, \sigma^g) \qquad \text{for } g \in \Gamma. \tag{2.2.3}$$

Here σ^g denotes the character of $C_\Gamma(gxg^{-1})$ with the formula $z \mapsto \sigma(g^{-1}zg)$. Apart from one family in type H_4 and one family in type $I_2(m)$, each family $\mathcal{F} \subset \mathrm{Uch}(W)$ is naturally in bijection with a set $\mathscr{M}(\Gamma)$ for some finite group Γ, given either by a product of 2-element cyclic groups or a symmetric group. In particular, each $\mathcal{F}$ has one of the following sizes:

Family size $\lvert\mathcal{F}\rvert$:	2^{2k}	8	21	39	74	k^2	k^2+k+2
Corresponding group Γ:	$(\mathbb{Z}/2\mathbb{Z})^k$	S_3	S_4	S_5			

Families of size 74, k^2, and k^2+k+2 occur only in types H_4, $I_2(2k+1)$, and $I_2(2k+2)$, respectively.

If (W, S) has one of the classical types A_n, BC_n, or D_n, then the families in $\mathrm{Uch}(W)$ each correspond to sets $\mathscr{M}\left((\mathbb{Z}/2\mathbb{Z})^k\right)$ for various integers $k \geq 0$. If (W, S) has type E_6, E_7, E_8, F_4, or G_2, then the families in $\mathrm{Uch}(W)$ each correspond to $\mathscr{M}(S_k)$ for some $k \in \{1, 2, 3, 4, 5\}$. A detailed list of the families $\mathcal{F} \subset \mathrm{Uch}(W)$ when (W, S) is crystallographic, alongside the corresponding bijections $\mathcal{F} \leftrightarrow \mathscr{M}(\Gamma)$, appears in [30, §13.8 and §13.9]. With respect to these correspondences, the following observation is a consequence of [63, Theorem 11.2]:

Observation 2.2.7. Suppose (W, S) is crystallographic and $\mathcal{F} \subset \mathrm{Uch}(W)$ is a family parametrized as in [30, §13.8 and §13.9] by the set $\mathscr{M}(\Gamma)$ for a finite group Γ. Let $\Phi \in \mathcal{F}$ be the element corresponding to $(x, \sigma) \in \mathscr{M}(\Gamma)$.

(a) If $\mathcal{F}$ is not exceptional then $\mathrm{Eig}(\Phi) = \frac{\sigma(x)}{\sigma(1)}$ and $\Delta(\Phi)$ is the unique element of $\mathcal{F}$ corresponding to the equivalence class $(x, \overline{\sigma}) \in \mathscr{M}(\Gamma)$.

(b) If $\mathcal{F}$ is exceptional then $\Gamma = S_2$ and $\mathrm{Eig}(\Phi) = 1$ if $x = 1$ and $\mathrm{Eig}(\Phi) = \sigma(x) \cdot i$ otherwise. In this case, Δ fixes the elements of $\mathcal{F}$ with Frobenius eigenvalue 1, and exchanges the two elements with Frobenius eigenvalue $\pm i$.

Furthermore, in either case Φ is special if and only if $(x, \sigma) = (1, \mathbb{1})$, where $\mathbb{1}$ denotes the principal character of Γ.

Carter's book [30] does not similarly address the non-crystallographic case; however, in these types, only a small number of families exist. A description of these families appears in Lusztig's papers [67, 66], which we may summarize as follows:

- **Type H_3.** There are 7 families: 4 of size 1 and 3 of size 4.
- **Type H_4.** There are 13 families: 6 of size 1, 6 of size 4, and 1 of size 74.
- **Type $I_2(m)$.** There are 3 families: they are described by (2.2.4).

One can find an exact parametrization of the families in types H_3 and H_4 (and also for the crystallographic exceptional types) in the computer algebra system CHEVIE [41]. We will discuss these families at greater length in Section 2.5.3.

We close this subsection by considering the related combinatorial problem of counting the elements of $\mathscr{M}(\Gamma)$. If Γ is abelian, then $\mathscr{M}(\Gamma) = \Gamma \times \mathrm{Irr}(\Gamma)$ and we have $|\mathscr{M}(\Gamma)| = |\Gamma|^2$. If Γ is a symmetric group, then we have the following less trivial result.

Proposition 2.2.8. Let $a_n = |\mathscr{M}(S_n)|$ and write $\sigma(n)$ for the sum of the positive divisors of a positive integer n. The ordinary generating function of the sequence $\{a_n\}_{n=1}^\infty$ then satisfies

$$1 + \sum_{n \geq 1} a_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{\sigma(n)}}.$$

Hence, in the language of [13], $\{a_n\}_{n=1}^\infty$ is the Euler transform of the sequence $\{\sigma(n)\}_{n=1}^\infty$.

Remark. The sequence $\{a_n\}_{n=1}^\infty = (1, 4, 8, 21, 39, 92, 170, 360, \dots)$ appears as [86, A061256].

Proof. If C_λ denotes the centralizer in S_n of a permutation with cycle type λ, then a_n is equal to sum over all partitions λ of n of the number of conjugacy classes of C_λ. It is not difficult to check that if the partition λ has m_r parts of size r, then $C_\lambda \cong \prod_{r \geq 1} G(r, m_r)$, where $G(r, n) = (\mathbb{Z}/r\mathbb{Z}) \wr S_n$ denotes the wreath product of a cyclic group of order r with S_n.

The conjugacy classes of $G(r, n)$ are indexed by partitions of n whose parts are labeled by numbers $i \in [r]$. Given a sequence $(\mu_r)_{r \geq 1}$ of such labeled partitions indexing a conjugacy class in $C_\lambda \cong \prod_{r\geq 1} G(r, m_r)$, modify the r^{th} partition μ_r by multiplying its parts by r and replacing all labels i by ordered pairs (r, i). Concatenating the parts of these modified partitions yields a map from the set of conjugacy classes of C_λ to the set $\mathcal{P}_n$ of partitions of n whose parts of size k are labeled by pairs (d, i) where d is a divisor of k and $i \in [d]$. Conversely, any $\nu \in \mathcal{P}_n$ determines a sequence of labeled partitions $(\mu_r)_{r\geq 1}$ which indexes a conjugacy class of some C_λ: namely, take μ_r to be the labeled partition given by dividing by r all parts of ν which are labeled by pairs of the form (r, i). These operations determine a bijection from $\mathcal{P}_n$ to the disjoint union of the sets of conjugacy classes of C_λ over all partitions λ of n, and thus $a_n = |\mathcal{P}_n|$.

We can view $\mathcal{P}_n$ as the set of partitions of n whose parts of size k can have $\sigma(k)$ different types. The cardinality of this set is what is counted by the coefficient of x^n in the right-hand generating function above, essentially by construction (see [13]), which completes the proof. $\square$

2.2.5 Unipotent characters in type $I_2(m)$

It seems difficult to find a single reference which provides all of the data our setup requires to construct $\mathrm{Uch}(W)$ in type $I_2(m)$. We therefore briefly include the relevant details here, with the papers [67, 66, 76] serving as our primary references.

Fix $m \geq 3$ and let (W, S) be of type $I_2(m)$. The set $\mathrm{Uch}(W)$ then consists of the objects

- $\Phi_{(0,j)}$ for integers j with $0 < j < \frac{m}{2}$,
- $\Phi_{(i,j)}$ for integers i, j with $0 < i < j < i + j < m$,

together with the additional objects

$$\begin{cases} \mathbb{1},\ \mathrm{sgn}, & \text{if } m \text{ is odd} \\ \mathbb{1},\ \mathrm{sgn},\ \Phi'_{(0,\frac{m}{2})},\ \Phi''_{(0,\frac{m}{2})}, & \text{if } m \text{ is even.} \end{cases}$$

Observe that $\mathrm{Uch}(W)$ has cardinality $k^2 + 2$ if $m = 2k + 1$ is odd or $k^2 - k + 4$ if $m = 2k$ is even.

Let $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$ be the standard primitive m^{th} root of unity. The fake degrees and degrees for $\mathrm{Uch}(W)$ are then defined by

- $\mathrm{FakeDeg}\left(\Phi_{(i,j)}\right) = \begin{cases} x^j + x^{m-j}, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases}$
- $\mathrm{FakeDeg}\left(\Phi'_{(0,\frac{m}{2})}\right) = \mathrm{FakeDeg}\left(\Phi''_{(0,\frac{m}{2})}\right) = x^{m/2}$,

and

- $\mathrm{Deg}\left(\Phi_{(i,j)}\right) = \frac{\xi^i + \xi^{m-i} - \xi^j - \xi^{m-j}}{m} \cdot x \cdot \frac{(x-1)(x+1)\prod_{k\in[m]}(x-\xi^k)}{(x-\xi^i)(x-\xi^{m-i})(x-\xi^j)(x-\xi^{m-j})}$,

- $\operatorname{Deg}\left(\Phi'_{(0,\frac{m}{2})}\right) = \operatorname{Deg}\left(\Phi''_{(0,\frac{m}{2})}\right) = \frac{2}{m} \cdot x \cdot \prod_{0<k<\frac{m}{2}} (x-\xi^k)(x-\xi^{m-k})$,

with $\operatorname{FakeDeg}(\mathbb{1}) = \operatorname{Deg}(\mathbb{1}) = 1$ and $\operatorname{FakeDeg}(\operatorname{sgn}) = \operatorname{Deg}(\operatorname{sgn}) = x^m$. Note that in the right-hand expression for $\operatorname{Deg}\left(\Phi_{(i,j)}\right)$, the denominator of the last factor always divides its numerator, and so the degree does belong to $\mathbb{R}[x]$. The Frobenius eigenvalues have the formula

$$\operatorname{Eig}(\Phi) = \begin{cases} \xi^{-ij}, & \text{if } \Phi = \Phi_{(i,j)} \text{ for some } (i,j), \\ 1, & \text{otherwise.} \end{cases}$$

We note that in this type, the permutation Δ of $\operatorname{Uch}(W)$ defined in Proposition 2.2.4 fixes all elements except those of the form $\Phi_{(i,j)}$ with $i > 0$, on which it acts by $\Delta : \Phi_{(i,j)} \mapsto \Phi_{(i,m-j)}$.

Finally, $\operatorname{Uch}(W)$ always has exactly three families, given by the sets

$$\{\mathbb{1}\}, \qquad \{\operatorname{sgn}\}, \qquad \text{and} \qquad \operatorname{Uch}(W) \setminus \{\mathbb{1}, \operatorname{sgn}\}. \tag{2.2.4}$$

The special elements of $\operatorname{Uch}(W)$ are then $\mathbb{1}$, sgn, and $\Phi_{(0,1)}$.

2.3 Decomposing χ_W for classical Coxeter systems

Kottwitz's paper [55] gives the decomposition of χ_W when W is a Weyl group of one of the classical types A_n, BC_n, and D_n. This section explains Kottwitz's results in detail, and also describes the special characters and family decomposition in $\operatorname{Irr}(W)$ in the classical cases. Most of this section is expository, our main references being Lusztig's book [63], Carter's book [30], and Kottwitz's paper [55].

2.3.1 Type A_n

Suppose (W, S) is the Coxeter system of type A_n. In this situation, we identify W with the symmetric group S_{n+1} of permutations of $[n+1]$ and take S to be the set of simple transpositions

$$S = \left\{ s_i \stackrel{\text{def}}{=} (i, i+1) \in S_{n+1} : i \in [n] \right\}.$$

We view permutations as functions and multiplication in S_{n+1} as function composition, and so evaluate the product of cycles to be $(1,2)(2,3) = (1,2,3)$, for example. The involutions of W are then the permutations of $[n+1]$ whose cycles all have length one or two, and if $w \in W$ is such an involution, we have

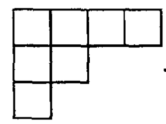
$$\operatorname{Des}_R(w) = \{s_i : i \in [n] \text{ and } w(i) > w(i+1)\}.$$

The conjugacy classes of W are the sets of permutations with the same cycle type; thus the permutations

$$\omega_m \stackrel{\text{def}}{=} (1, m+1)(2, m+2)\dots(m, 2m), \qquad \text{for } m = 1, 2, \dots, \lfloor \tfrac{n+1}{2} \rfloor \tag{2.3.1}$$

represent the distinct conjugacy classes of involutions in W.

To describe the irreducible representations of W, we adopt the convention that a *partition* of an integer n is a weakly decreasing sequence of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots)$ with $|\alpha| \overset{\text{def}}{=} \sum_i \alpha_i = n$. We write $\alpha \vdash n$ to indicate that α is a partition of n, and treat all partitions α as infinite sequences (so that α_i is defined for all $i \in \mathbb{P}$). The *Young diagram* of a partition α is the subset of $\mathbb{P}^2$ given by $\{(i,j) : 1 \leq j \leq \lambda_i\}$, which we typically represent in "English notation" as in the following example:

$\alpha = (4, 2, 1, 0, 0, \dots) \vdash 6$ has Young diagram  .

We often identify a partition α with its Young diagram; for example, we write $\alpha \subset \beta$ when the Young diagram of α is contained in that of β, which is equivalent to the condition that $\alpha_i \leq \beta_i$ for all $i \in \mathbb{P}$. We denote the *transpose* of $\alpha \vdash n$ by α'; recall that $\alpha'_i = |\{j \in \mathbb{P} : \alpha_j \leq i\}|$, so that the Young diagram of α' is the transpose of the Young diagram of α.

The isomorphism classes of irreducible representations of W are indexed by the partitions of $n+1$. In particular, for each partition $\alpha \vdash n+1$, it is possible to choose an irreducible W-representation ρ^α with the property that ρ^α restricted to $S_{\alpha_1} \times S_{\alpha_2} \times \cdots \subset S_{n+1}$ contains the trivial representation and ρ^α restricted to the $S_{\alpha'_1} \times S_{\alpha'_2} \times \cdots \subset S_{n+1}$ contains the sign representation. No two such representations are isomorphic, and any irreducible W-representation is isomorphic to ρ^α for a unique partition $\alpha \vdash n+1$. We write χ^α for the character of ρ^α, so that $\mathrm{Irr}(W) = \{\chi^\alpha : \alpha \vdash n+1\}$.

Using this notation, we may state the following theorem assembling several facts from Lusztig's book [63].

Theorem 2.3.1 (See Chapter 4, Section 4 in [63]). Suppose (W, S) is of type A_n.

(1) Every irreducible character of W is special.

(2) Each irreducible character of W belongs to its own family with one element.

Concluding this subsection, we have these two results which Kottwitz derives in [55]. Adin, Postnikov, and Roichman independently studied the representation ϱ_W in type A_n in in [1], and the combinatorial formulation of the first theorem is taken from that paper.

Theorem 2.3.2 (Corollary A.1 in [1]; see also [55]). Assume (W, S) is of type A_n.

(1) If $\sigma \in W$ is an involution with f fixed points then the character $\chi_{W,\sigma}$ is the multiplicity free sum $\sum_\alpha \chi^\alpha$ over all partitions $\alpha \vdash n+1$ whose Young diagrams have exactly f odd columns.

(2) A partition of $n+1$ has f odd columns if and only if it appears as the shape of the standard Young tableaux assigned by the RSK correspondence to an involution in S_{n+1} with f fixed points.

Since $\chi_W = \sum_m \chi_{W,\omega_m}$ where the sum is over $m = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, the first part of the preceding theorem leads immediately to this corollary.

Corollary 2.3.3 (Theorem 1.2 in [1]; see also [55]). If (W, S) is of type A_n then

$$\chi_W = \sum_{\lambda \vdash n+1} \chi^\lambda$$

is the multiplicity free sum of all irreducible characters of W.

2.3.2 Type BC_n

For algebraic groups, the two types B_n and C_n are distinct, but the Weyl groups (W, S) of the simple groups of types B_n and C_n are isomorphic, and for our purposes, all the data of interest attached to $\mathrm{Irr}(W)$ via G is the same in both types. We therefore refer to a single Coxeter system of type BC_n.

An explicit construction of the Coxeter system of type BC_n goes as follows. Let $\mathbb{F}_2 = \{0, 1\}$ denote the finite field with two elements and write $\mathbb{F}_2^n$ for additive group of n-tuples of elements of $\mathbb{F}_2$. The symmetric group S_n acts on $\mathbb{F}_2^n$ by permuting the coordinates of elements, and we denote the action of a permutation $\pi \in S_n$ on a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_2^n$ by

$$\pi(x) \overset{\text{def}}{=} \Big(x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \ldots, x_{\pi^{-1}(n)}\Big).$$

We let B_n denote the group of pairs $(x, \pi) \in \mathbb{F}_2^n \times S_n$ with multiplication given by

$$(x, \pi)(y, \sigma) = (\sigma^{-1}(x) + y, \pi\sigma), \qquad \text{for } x, y \in \mathbb{F}_2^n \text{ and } \pi, \sigma \in S_n.$$

This is usually referred to as the *hyperoctahedral group* or the wreath product $\mathbb{F}_2 \wr S_n$. We view $\mathbb{F}_2^n$ and S_n as subgroups of B_n, by identifying $x \in \mathbb{F}_2^n$ with the element $(x, 1) \in B_n$ and $\pi \in S_n$ with the element $(0, \pi) \in B_n$. It is helpful to observe that the map sending (x, π) to the $n \times n$ matrix with $(-1)^{x_i}$ in position $(\pi(i), i)$ for $i = 1, 2, \ldots, n$ and zeros in all other positions defines an isomorphism from B_n to the group of $n \times n$ signed permutation matrices.

Suppose (W, S) is the Coxeter system of type BC_n. In this situation, we identify W with the hyperoctahedral group B_n and take $S = \{s_1, \ldots, s_{n-1}, t_n\}$ where

$$s_i = (i, i+1) \in S_n \subset B_n \quad \text{for } i \in [n-1] \qquad \text{and} \qquad t_n = (0, \ldots, 0, 1) \in \mathbb{F}_2^n \subset B_n. \quad (2.3.2)$$

An element $(x, \pi) \in W$ is an involution if and only if π is an involution in S_n and $x_i = x_j$ whenever $\pi(i) = j$. If $(x, \pi) \in W$ is an involution then $t_n \in \mathrm{Des}_R(x, \pi)$ if and only if $x_n = 1$, and $s_i \in \mathrm{Des}_R(x, \pi)$ for $i \in [n-1]$ if and only if one of the following conditions holds:

- $\pi(i) > \pi(i+1)$ and $x_i = x_{i+1} = 0$;
- $\pi(i+1) > \pi(i)$ and $x_i = x_{i+1} = 1$;

- $x_i = 1$ and $x_{i+1} = 0$.

Given nonnegative integers k, ℓ, m with $2m + k + \ell = n$, let

$$\omega_{k,\ell,m} \stackrel{\text{def}}{=} \Big((\underbrace{0,\ldots,0}_{2m \text{ times}}, \underbrace{0,\ldots,0}_{k \text{ times}}, \underbrace{1,\ldots,1}_{\ell \text{ times}}),\ \omega_m \Big) \in W, \tag{2.3.3}$$

where we define $\omega_m \in S_n$ as in (2.3.1). This element is an involution, and the elements $\omega_{k,\ell,m}$ with k, ℓ, m ranging over all nonnegative integers with $2m + k + \ell = n$ represent the distinct conjugacy classes of involutions in W.

The isomorphism classes of irreducible representations of W are indexed by *bipartitions* of n, by which we mean pairs of partitions (α, β) with $|\alpha| + |\beta| = n$. We write $(\alpha, \beta) \vdash n$ to indicate that (α, β) is a bipartition of n. One typically constructs an irreducible representation $\rho^{(\alpha,\beta)}$ belonging to the isomorphism class of each bipartition $(\alpha, \beta) \vdash [n]$ in the following manner. Any representation ρ of S_n (in some vector space V) extends to representations ρ_+ and ρ_- of B_n (in the same vector space V) by the formulas

$$\rho_+(x, \pi) = \rho(\pi) \qquad \text{and} \qquad \rho_-(x, \pi) = (-1)^{x_1 + \cdots + x_n} \rho(\pi), \qquad \text{for } (x, \pi) \in B_n. \tag{2.3.4}$$

If (α, β) is a bipartition of n, then ρ^α and ρ^β are irreducible representations of symmetric groups, and the external tensor product $\rho_+^\alpha \otimes \rho_-^\beta$ defines a representation of $B_{|\alpha|} \times B_{|\beta|}$. We define $\rho^{(\alpha,\beta)}$ as the induced representation

$$\rho^{(\alpha,\beta)} \stackrel{\text{def}}{=} \operatorname{Ind}_{B_{|\alpha|} \times B_{|\beta|}}^{B_n} \left(\rho_+^\alpha \otimes \rho_-^\beta \right).$$

This representation is irreducible with degree

$$\deg \rho^{(\alpha,\beta)} = \frac{n!}{|\alpha|!|\beta|!} (\deg \rho^\alpha)(\deg \rho^\beta),$$

and any irreducible W-representation is isomorphic to $\rho^{(\alpha,\beta)}$ for a unique bipartition $(\alpha, \beta) \vdash n$. We write $\chi^{(\alpha,\beta)}$ for the character of $\rho^{(\alpha,\beta)}$, so that $\operatorname{Irr}(W) = \left\{ \chi^{(\alpha,\beta)} : (\alpha, \beta) \vdash n \right\}$.

The following constructions are taken from Lusztig's book [63]. Given a nonnegative integer m, let $\mathcal{S}_m^{BC}$ denote the set of pairs (λ, μ), where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+1})$ and $\mu = (\mu_1, \ldots, \mu_m)$ are strictly increasing nonnegative integer sequences of length $m + 1$ and m, respectively. The elements $(\lambda, \mu) \in \mathcal{S}_m^{BC}$ are often represented as two-line arrays

$$(\lambda, \mu) = \left(\begin{array}{ccccccccc} \lambda_1, & & \lambda_2, & & \cdots, & & \lambda_m, & & \lambda_{m+1} \\ & \mu_1, & & \mu_2, & & \cdots, & & \mu_m & \end{array} \right).$$

Lusztig [63] defines an equivalence relation $\sim$ on the disjoint union $\mathcal{S}^{BC} \stackrel{\text{def}}{=} \bigcup_{m \geq 0} \mathcal{S}_m^{BC}$ by setting

$$\left(\begin{array}{ccccccccc} \lambda_1, & & \lambda_2, & & \cdots, & & \lambda_m, & & \lambda_{m+1} \\ & \mu_1, & & \mu_2, & & \cdots, & & \mu_m & \end{array} \right) \sim \left(\begin{array}{ccccccccccc} 0, & & \lambda_1 + 1, & & \lambda_2 + 1, & & \cdots, & & \lambda_m + 1, & & \lambda_{m+1} + 1 \\ & 0, & & \mu_1 + 1, & & \mu_2 + 1, & & \cdots, & & \mu_m + 1 & \end{array} \right)$$

and then extending $\sim$ reflexively and transitively. Following Lusztig [63], we call the equivalence classes in $\mathcal{S}^{BC}$ under this relation *symbols.* Accompanying this notation, we have the following terminology also from Lusztig's book [63].

Definition 2.3.4. The *symbol* of a bipartition $(\alpha, \beta) \vdash n$ is the symbol of $(\lambda, \mu) \in \mathcal{S}_n^{BC}$, where

$$\begin{aligned} \lambda_i &= \alpha_{n+1-(i-1)} + (i-1), && \text{for } i \in [n+1], \\ \mu_i &= \beta_{n-(i-1)} + (i-1), && \text{for } i \in [n]. \end{aligned} \tag{2.3.5}$$

(a) We say that the bipartition (α, β) is *special* if its symbol contains for some m a representative $(\lambda, \mu) \in \mathcal{S}_m^{BC}$ such that $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for all $i \in [m]$.

(b) Two symbols Λ, Λ' in $\mathcal{S}^{BC}$ are said to *belong to the same family* if for some m there exist representatives $(\lambda, \mu) \in \mathcal{S}_m^{BC} \cap \Lambda$ and $(\gamma, \nu) \in \mathcal{S}_m^{BC} \cap \Lambda'$ such that the concatenated sequences

$$(\lambda_1, \lambda_2, \ldots, \lambda_{m+1}, \mu_1, \mu_2, \ldots, \mu_m) \qquad \text{and} \qquad (\gamma_1, \gamma_2, \ldots, \gamma_{m+1}, \nu_1, \nu_2, \ldots, \nu_m)$$

are permutations of each other. Two bipartitions of n belong to the same family if and only if their symbols belong to the same family.

(c) In any $(\lambda, \mu) \in \mathcal{S}^{BC}$, a certain number of entries of μ do not appear in λ and a certain number of entries of λ do not appear in μ. Denote the former number by $d(\lambda, \mu)$; the latter number is then necessarily equal to $d(\lambda, \mu) + 1$.

The number $d(\lambda, \mu)$ is the same for all elements in a given symbol and we therefore define

$$d(\alpha, \beta) \overset{\text{def}}{=} d(\lambda, \mu) \text{ for any } (\lambda, \mu) \in \mathcal{S}^{BC} \text{ which represents the symbol of } (\alpha, \beta).$$

Example 2.3.5. The bipartitions $(\ydiagram{2,2}, \ydiagram{3,1})$, $(\ydiagram{2}, \ydiagram{3,3})$, and $(\ydiagram{2,2,1}, \ydiagram{3})$ of 8 constitute a family as their symbols are respectively represented by

$$\begin{pmatrix} 0, & 3, & 4 \\ & 1, & 4 \end{pmatrix} \quad \begin{pmatrix} 0, & 1, & 4 \\ & 3, & 4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1, & 3, & 4 \\ & 0, & 4 \end{pmatrix}$$

The bipartition $(\ydiagram{2,2}, \ydiagram{3,1})$ is special, and we have $d(\ydiagram{2,2}, \ydiagram{3,1}) = d(\ydiagram{2}, \ydiagram{3,3}) = d(\ydiagram{2,2,1}, \ydiagram{3}) = 1$.

We may now state the type BC_n analog of Theorem 2.3.1, also due to Lusztig [63].

Theorem 2.3.6 (See Chapter 4, Section 5 in [63]). Suppose (W, S) is of type BC_n.

(1) A character $\chi^{(\alpha,\beta)}$ of W is special if and only if (α, β) is a special bipartition of n.

(2) Two irreducible characters of W belong to the same family if and only if they are indexed by bipartitions of n in the same family.

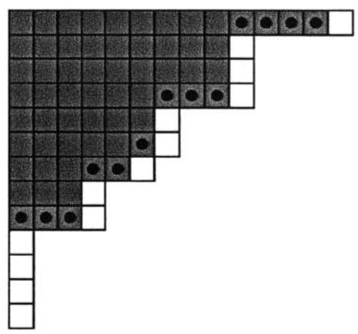
The following lemma provides another of understanding the special bipartitions (α, β) and the integer $d(\alpha, \beta)$. The first part of this result is similar to Spaltenstein's characterization of the special representations of Weyl groups of type BC_n in [88, Section 4]. Here, we write α' for the *transpose* of a partition α, which by definition is the unique partition whose Young diagram is the transpose of the Young diagram of α.

Lemma 2.3.7. Let (α, β) be a bipartition of n.

(1) The bipartition (α, β) is special if and only if $\beta_i \leq \alpha_i + 1$ and $\alpha'_i \leq \beta'_i + 1$ for all $i \in \mathbb{P}$.

(2) If (α, β) is special then $d(\alpha, \beta) = \left|\{(\alpha'_j, j) : j \in \mathbb{P}\} \cap \{(i, \beta_i) : i \in \mathbb{P}\}\right|$, which is the number of cells which appear both in the last row of a column of α and in the last column of a row of β.

Remark. The condition for a bipartition (α, β) to be special corresponds to the following picture:



If the Young diagram of α is the set of gray cells, then (α, β) is special if and only if β is formed from α by adding (a subset of) white cells and/or removing (a subset of) gray cells marked by $\bullet$'s.

Proof. If (λ, μ) and (γ, ν) belong to the same symbol in $\mathcal{S}^{BC}$ then $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for all i if and only if $\gamma_i \leq \nu_i \leq \gamma_{i+1}$ for all i. Given this, it follows that a bipartition (α, β) is special if and only if $\alpha_{i+1} \leq \beta_i \leq \alpha_i + 1$ for all $i \in \mathbb{P}$. In turn, it is straightforward to check that $\alpha_{i+1} \leq \beta_i$ for all $i \in \mathbb{P}$ if and only if $\alpha'_i \leq \beta'_i + 1$ for all $i \in \mathbb{P}$, and this gives part (1).

If (α, β) is a special bipartition and $(\lambda, \mu) \in \mathcal{S}^{BC}_n$ is defined by (2.3.5), then $d(\alpha, \beta)$ is equal to the number of $i \in [n]$ with $\lambda_i < \mu_i < \lambda_{i+1}$. This is the number of $i \in [n]$ with $\alpha_{i+1} < \beta_i < \alpha_i + 1$. In light of part (1), we deduce that $d(\alpha, \beta)$ is the number of $i \in [n]$ with $\alpha_{i+1} < \beta_i \leq \alpha_i$. Part (2) follows immediately as $(i, \beta_i) = (\alpha'_j, j)$ for some $j \in \mathbb{P}$ if and only if $\alpha_{i+1} < \beta_i \leq \alpha_i$. $\square$

Recall from (2.3.1) the definition of the permutation $\omega_m = (1, m+1)(2, m+2) \cdots (m, 2m) \in S_{2m}$. View ω_m as an involution in B_{2m} and note that this involution coincides with the element $\omega_{m,0,0}$ defined by (2.3.3). It will soon be useful to adopt the following notation: given $g = (x, \pi) \in B_n$ for some n, define $|g| = \pi \in S_n$ and $g_i = x_i$ for $i \in [n]$.

Lemma 2.3.8. If (W, S) is of type BC_{2m} then the character of the subrepresentation of ϱ_W on the subspace of $\mathrm{Invol}(W)$ spanned by the conjugacy class of $\omega_m \in W$ is $\sum_{\alpha \vdash m} \chi^{(\alpha, \alpha)}$.

Remark. This result is mentioned, for example, as [85, Eq. (6)]. We include a detailed proof for completeness.

Proof. If G is a group then we write $C_G(g) = \{x \in G : xg = gx\}$ for the centralizer of an element g. In particular, let $V_m = C_{B_{2m}}(\omega_m)$ and $H_m = V_m \cap (B_m \times B_m)$. Proofs of the following facts are left as straightforward exercises:

(a) $V_m = \Big\{(x,\pi) \in B_{2m} : \pi \in C_{S_{2m}}(\omega_m) \text{ and } x_i = x_{m+i} \text{ for all } i \in [m]\Big\}$;

(b) The centralizer $C_{S_{2m}}(\omega_m)$ is isomorphic to B_m, and so $|V_m| = m! \cdot 2^{2m}$;

(c) $H_m = \Big\{(g,h) \in B_m \times B_m : |g| = |h| \text{ and } g_i = h_i \text{ for all } i \in [m]\Big\} \cong B_m$.

Let $\lambda : V_m \to \mathbb{C}$ be the linear character defined by $\varrho_W(g)a_{\omega_m} = \lambda(g)a_{\omega_m}$ for $g \in V_m$. The character of the subrepresentation of ϱ_W described in the lemma is just the induced character $\mathrm{Ind}_{V_m}^{B_{2m}}(\lambda)$ and so has degree $\frac{|B_{2m}|}{|V_m|} = \frac{(2m)!}{m!}$. Since $\deg\left(\sum_{\alpha\vdash m}\chi^{(\alpha,\alpha)}\right) = \binom{2m}{m}\sum_{\alpha\vdash m}(\deg\chi^\alpha)^2 = \frac{(2m)!}{m!}$, to prove the lemma it suffices to show that $\chi^{(\alpha,\alpha)}$ appears in $\mathrm{Ind}_{V_m}^{B_{2m}}(\lambda)$ with nonzero multiplicity for each $\alpha \vdash m$. Let χ_+^α and χ_-^α denote the characters of the representations ρ_+^α and ρ_-^α given by (2.3.4) so that we have $\chi^{(\alpha,\alpha)} = \mathrm{Ind}_{B_m\times B_m}^{B_{2m}}(\chi_+^\alpha \otimes \chi_-^\alpha)$. By applying Frobenius reciprocity and Mackey's theorem, one obtains the inequality

$$\begin{aligned}
\left\langle \mathrm{Ind}_{V_m}^{B_{2m}}(\lambda), \chi^{(\alpha,\alpha)}\right\rangle_{B_{2m}} &= \left\langle \mathrm{Res}_{B_m\times B_m}^{B_{2m}}\left(\mathrm{Ind}_{V_m}^{B_{2m}}(\lambda)\right), \chi_+^\alpha \otimes \chi_-^\alpha\right\rangle_{B_m\times B_m} && \text{(Frobenius recipr.)}\\
&\geq \left\langle \mathrm{Ind}_{H_m}^{B_m\times B_m}\left(\mathrm{Res}_{H_m}^{V_m}(\lambda)\right), \chi_+^\alpha \otimes \chi_-^\alpha\right\rangle_{B_m\times B_m} && \text{(Mackey's theorem)}\\
&= \left\langle \mathrm{Res}_{H_m}^{V_m}(\lambda), \mathrm{Res}_{H_m}^{B_m\times B_m}\left(\chi_+^\alpha \otimes \chi_-^\alpha\right)\right\rangle_{H_m} && \text{(Frobenius recipr.)}
\end{aligned}$$

and thus we need only show that the last inner product is nonzero.

To evaluate this, recall that B_{2m} is generated by the elements $s_1, s_2, \dots, s_{2m-1}, t_{2m}$, and observe that H_m is generated by the elements $s_1', s_2', \dots, s_{m-1}', t_m'$ where $s_i' = s_i s_{i+m}$ for $i \in [m-1]$ and

$$t_m' = t_{2m}\left(s_m s_{m+1}\cdots s_{2m-1}\right)t_{2m}\left(s_{2m-1}\cdots s_{m+1}s_m\right).$$

It is easy to see that $\lambda(s_i') = 1$ for all $i \in [m-1]$ as $\omega_m \neq s_{m+i}\omega_m s_{m+i}$. With slightly greater difficulty, we compute $\lambda(t_m') = -1$ as follows. Let $\nu_0 = \omega_m$; then define $\nu_{i+1} = s_{m+i}\nu_i s_{m+i}$ for $0 \leq i \leq m-1$; then let $\mu_0 = t_{2m}\nu_m t_{2m}$; and finally define $\mu_i = s_{2m-i}\mu_{i-1}s_{2m-i}$ for $1 \leq i \leq m$. The element $t_m' \in H_m$ is a product of $2m+2$ simple generators, and if we successively conjugate ω_m by these generators (beginning with the right most factor) we obtain the sequence of involutions

$$\omega_m = \nu_0,\ \nu_1,\ \nu_2,\ \dots,\ \nu_m,\ \mu_0,\ \mu_1,\ \mu_2,\ \dots,\ \mu_m,\ t_{2m}\mu_m t_{2m} = \omega_m.$$

One checks that the only pairs of equal adjacent elements in this sequence are $\nu_{m-1} = \nu_m$ and $\mu_0 = \mu_1$, and that $s_{2m-1} \in \mathrm{Des}_R(\nu_{m-1})$ while $s_{2m-1} \notin \mathrm{Des}_R(\mu_0)$. By the definition of ϱ_W, it follows that $\lambda(t_m') = -1$ as claimed, and from this we deduce that $\lambda(g) = (-1)^{g_{m+1}+\cdots+g_{2m}}$ for $g \in H_m$. On the other hand, if $g \in H_m$ and $|g| = (\sigma,\sigma) \in S_m \times S_m \subset S_{2m}$, then

$(\chi_+^\alpha \otimes \chi_-^\alpha)(g) = (-1)^{g_{m+1}+\cdots+g_{2m}}\chi^\alpha(\sigma)^2$. In the product of these two characters the factors of -1 cancel, and we obtain

$$\langle \mathrm{Res}_{H_m}^{V_m}(\lambda), \mathrm{Res}_{H_m}^{B_m\times B_m}\left(\chi_+^\alpha \otimes \chi_-^\alpha\right)\rangle_{H_m} = \frac{1}{m!}\sum_{\sigma\in S_m}\chi^\alpha(\sigma)^2 = \langle \chi^\alpha, \chi^\alpha\rangle_{S_m} = 1,$$

which completes the proof of the lemma. □

Continue to let $V_m = C_{B_{2m}}(\omega_m)$ as in the proof of the preceding lemma. If k, ℓ, m are nonnegative integers with $2m + k + \ell = n$, then we may naturally identify $C_{B_n}(\omega_{k,\ell,m})$ with the internal direct sum

$$C_{B_n}(\omega_{k,\ell,m}) = V_m \times B_k \times B_\ell \tag{2.3.6}$$

where we view the factor B_k (respectively, B_ℓ) as the subgroup of B_n consisting of all elements (x, π) with $x_i = 0$ and $\pi(i) = i$ for

$$i \notin \{2m+1, \ldots, 2m+k\} \qquad \text{(respectively, } i \notin \{2m+k+1, \ldots, 2m+k+\ell\}\text{)}.$$

In addition, if we let $\lambda_{k,\ell,m} : C_{B_n}(\omega_{k,\ell,m}) \to \mathbb{C}$ be the linear character defined by

$$\varrho_W(g)a_{\omega_{k,\ell,m}} = \lambda_{k,\ell,m}(g)a_{\omega_{k,\ell,m}}, \qquad \text{for } g \in C_{B_n}(\omega_{k,\ell,m}),$$

then with respect to the identification (2.3.6) we have $\lambda_{k,\ell,m} = \lambda_m \otimes \mathbb{1}_{B_k} \otimes \mathrm{sgn}_{B_\ell}$, where $\lambda_m = \lambda_{0,0,m}$ is the character of V_m which we denoted λ in the preceding proof. (Here, $\otimes$ denotes the external tensor product.) The character $\chi_{W,\omega_{k,\ell,m}}$ of the subrepresentation of ϱ_W on the subspace of $\mathrm{Invol}(W)$ spanned by the conjugacy class of $\omega_{k,\ell,m}$ is $\mathrm{Ind}_{V_m\times B_k\times B_\ell}^{B_n}(\lambda_{k,\ell,m})$. Using the transitivity of induction, we deduce from the preceding lemma that this character is equal to the sum of induced characters

$$\chi_{W,\omega_{k,\ell,m}} = \sum_{\alpha\vdash m} \mathrm{Ind}_{B_{2m}\times B_k\times B_\ell}^{B_n}\left(\chi^{(\alpha,\alpha)} \otimes \mathbb{1}_{B_k} \otimes \mathrm{sgn}_{B_\ell}\right).$$

To decompose this, we appeal to the following lemma.

Lemma 2.3.9. If k, ℓ, m are nonnegative integers with $2m + k + \ell = n$, and $\gamma \vdash m$, then

$$\mathrm{Ind}_{B_{2m}\times B_k\times B_\ell}^{B_n}\left(\chi^{(\gamma,\gamma)} \otimes \mathbb{1}_{B_k} \otimes \mathrm{sgn}_{B_\ell}\right) = \sum_{(\alpha,\beta)} \chi^{(\alpha,\beta)}$$

where the sum is over all bipartitions $(\alpha, \beta) \vdash n$ such that the Young diagram of α (respectively, β) is obtained from the Young diagram of γ by adding k cells in distinct columns (respectively, ℓ cells in distinct rows).

Proof. By comparing the formulas for these characters and using the fact that characters are class functions, one finds that the left hand side is the character of

$$\mathrm{Ind}_{B_{m+k}\times B_{m+\ell}}^{B_n}\left(\left(\mathrm{Ind}_{S_m\times S_k}^{S_{m+k}}(\rho^\gamma\otimes \mathbb{1}_{S_k})\right)_+\otimes\left(\mathrm{Ind}_{S_m\times S_\ell}^{S_{m+\ell}}(\rho^\gamma\otimes \mathrm{sgn}_{S_\ell})\right)_-\right),$$

where, as above, we use the notation ρ_+, ρ_- to denote two extensions of a representation ρ of S_n to a representation of B_n. By the well-known Pieri rules for the symmetric group, this representation is isomorphic to

$$\mathrm{Ind}_{B_{m+k}\times B_{m+\ell}}^{B_n}\left(\sum_{(\alpha,\beta)}\rho_+^\alpha\otimes\rho_-^\beta\right)=\sum_{(\alpha,\beta)}\mathrm{Ind}_{B_{m+k}\times B_{m+\ell}}^{B_n}\left(\rho_+^\alpha\otimes\rho_-^\beta\right)=\sum_{(\alpha,\beta)}\rho^{(\alpha,\beta)},$$

with all sums over bipartitions $(\alpha,\beta)\vdash n$ as in the statement of the lemma. □

We may now give a detailed proof of the decomposition Kottwitz describes in [55] of the subrepresentations of ϱ_W generated by the involutions $\omega_{k,\ell,m}$. In what follows, given two partitions α,β, let $\alpha\cap\beta$ be the partition with $(\alpha\cap\beta)_i=\min\{\alpha_i,\beta_i\}$ for $i\in\mathbb{P}$.

Theorem 2.3.10 (Kottwitz [55]). Suppose (W,S) is of type BC_n. If k,ℓ,m are nonnegative integers with $2m+k+\ell=n$, then the character $\chi_{W,\omega_{k,\ell,m}}$ of the subrepresentation of ϱ_W on the subspace of $\mathrm{Invol}(W)$ spanned by the conjugacy class of $\omega_{k,\ell,m}\in W$ is

$$\chi_{W,\omega_{k,\ell,m}}=\sum_{(\alpha,\beta)}\binom{d(\alpha,\beta)}{|\alpha\cap\beta|-m}\chi^{(\alpha,\beta)}$$

where the sum is over all special bipartitions $(\alpha,\beta)\vdash n$ with $|\alpha|=m+k$ and $|\beta|=m+\ell$.

Note that we evaluate the binomial coefficient $\binom{k}{i}$ to be zero if $i<0$ or $i>k$.

Proof. If (α,β) is a bipartition of n then the multiplicity of $\chi^{(\alpha,\beta)}$ in the character under consideration is, by the preceding lemma, the number of partitions $\gamma\vdash m$ to which one can add k cells in distinct columns to obtain α and ℓ cells in distinct rows to obtain β.

Suppose $\chi^{(\alpha,\beta)}$ appears with nonzero multiplicity, so that some such $\gamma\vdash m$ exists. Then necessarily $\alpha\vdash m+k$ and $\beta\vdash n+k$, and we see that (α,β) is special by the following argument. On one hand, certainly $\gamma_i\le\beta_i\le\alpha_i+1$ for any $i\in\mathbb{P}$ since $\beta_i\in\{\gamma_i,\gamma_i+1\}$ and $\alpha_i\ge\gamma_i$. On the other hand, $\alpha_{i+1}\le\gamma_i$ since there are only $\gamma_i-\gamma_{i+1}$ distinct columns to which one can add cells in the $(i+1)^{\text{th}}$ row of γ. Hence $\alpha_{i+1}\le\beta_i\le\alpha_i+1$ for all $i\in\mathbb{P}$ so (α,β) is special by Lemma 2.3.7.

Let us now consider what forms the partition $\gamma\vdash m$ may take. Certainly $\gamma\subset\alpha\cap\beta$, and the set of cells added to γ to form α is a subset of $\{(\alpha'_j,j):j\in\mathbb{P}\}$ while the set of cells added to γ to form β is a subset of $\{(i,\beta_i):j\in\mathbb{P}\}$. Consequently, any cell in the Young diagram of $\alpha\cap\beta$ which does not belong to γ must belong to the set $D\overset{\text{def}}{=}\{(\alpha'_j,j):j\in\mathbb{P}\}\cap\{(i,\beta_i):i\in\mathbb{P}\}$, which by Lemma 2.3.7 has cardinality $d(\alpha,\beta)$ as (α,β) is special. By construction, each cell in D is a corner cell in the Young diagram of $\alpha\cap\beta$, and so removing any subset of cells $S \subset D$ from (the Young diagram of) $\alpha \cap \beta$ yields a valid partition, which we denote by ν_S. Since $|\gamma| = m$, we conclude that $\gamma = \nu_S$ for some subset $S \subset D$ with $|\alpha \cap \beta| - m$ elements. This suffices to show that the multiplicity of $\chi^{(\alpha,\beta)}$ for a special bipartition (α, β) with $\alpha \vdash m + k$ and $\beta \vdash m + \ell$ is at most $\binom{d(\alpha,\beta)}{|\alpha\cap\beta|-m}$.

To show that this multiplicity is exactly $\binom{d(\alpha,\beta)}{|\alpha\cap\beta|-m}$, it is enough to prove that (under the assumption (α, β) is special) the cells in the skew shape $\alpha \setminus \nu_D$ lie in distinct columns and the cells in the skew shape $\beta \setminus \nu_D$ lie in distinct rows. This follows easily enough, as it is not hard, using the definition of special in Lemma 2.3.7, to see that the cells in the first skew shape belong to $\{(i, \beta_i) : i \in \mathbb{P}\}$ while the cells in the second skew shape belong to $\{(\alpha'_j, j) : j \in \mathbb{P}\}$. □

Corollary 2.3.11 (Kottwitz [55]). If (W, S) is of type BC_n then

$$\chi_{(W,S)} = \sum_{(\alpha,\beta)} 2^{d(\alpha,\beta)} \chi^{(\alpha,\beta)}$$

where the sum is over all special bipartitions (α, β) of n.

Proof. If $(\alpha, \beta) \vdash n$ is a bipartition, then for each m, there is a unique pair k, ℓ with $|\alpha| = m + k$ and $|\beta| = m + \ell$. Hence, summing the subrepresentations described in the preceding result over all integers $m, k, \ell \geq 0$ with $2m + k + \ell = n$ yields the given decomposition. □

2.3.3 Type D_n

Suppose (W, S) is the Coxeter system of type D_n. In this situation, we identify W with the normal subgroup $B_n^+ \lhd B_n$ given by

$$B_n^+ \overset{\text{def}}{=} \left\{(x, \pi) \in B_n : x_1 + x_2 + \cdots + x_n = 0\right\},$$

and let $S = \{s_1, \ldots, s_{n-1}, s'_{n-1}\}$ where s_i is defined as in (2.3.2) and $s'_{n-1} = t_n s_{n-1} t_n$, with t_n as in (2.3.2). The involutions of B_n^+ are just the involutions of B_n belonging to B_n^+. Let $(x, \pi) \in W$ be such an involution. Then $s_i \in \mathrm{Des}_R(x, \pi)$ for $i \in [n-1]$ if and only if s_i belongs to the descent set of (x, π) viewed as an element of the Coxeter system of type BC_n, i.e., if and only if one of the following conditions holds:

- $\pi(i) > \pi(i + 1)$ and $x_i = x_{i+1} = 0$;
- $\pi(i + 1) > \pi(i)$ and $x_i = x_{i+1} = 1$;
- $x_i = 1$ and $x_{i+1} = 0$.

On the other hand, the element $s'_{n-1} \in \mathrm{Des}_R(x, \pi)$ if and only if one of these conditions holds:

- $\pi(n - 1) > \pi(n)$ and $x_{n-1} = 0$ and $x_n = 1$;

- $\pi(n) > \pi(n-1)$ and $x_{n-1} = 1$ and $x_n = 0$;
- $x_{n-1} = x_n = 1$.

When n is odd, the distinct conjugacy classes of involutions in W are represented by the elements $\omega_{k,\ell,m}$ with k, ℓ, m ranging over all nonnegative integers with $m = 2m + k + \ell$ and ℓ even. When n is even, the distinct conjugacy classes of involutions in W are represented by the same elements $\omega_{k,\ell,m}$ together with the one additional involution $\omega'_{n/2}$, defined by

$$\omega'_m \overset{\text{def}}{=} \Big((1,1,0,\dots,0),\ \omega_m \Big) \in B^+_{2m}, \qquad \text{for } m \geq 1. \tag{2.3.7}$$

The conjugacy classes of these involutions are the intersection of their conjugacy classes in B_n with B_n^+, with one exception: if $n = 2m$ is even then the intersection with B_n^+ of the conjugacy class of $\omega_m \in S_n \subset B_n$ splits into two classes, represented by ω_m and ω'_m.

Before even describing the irreducible representations of W, let us prove the following fact.

Lemma 2.3.12. Suppose (W, S) is the Coxeter system of type D_n and (W', S') is the Coxeter system of type BC_n, so that $W = B_n^+$ is a subgroup of $W' = B_n$. Write

$$\varrho_D \overset{\text{def}}{=} \varrho_W \qquad \text{and} \qquad \varrho_B \overset{\text{def}}{=} \varrho_{W'},$$

and suppose m, k are nonnegative integers with $2m + k = n$. The subspace V of $\mathrm{Invol}(B_n^+)$ spanned by the B_n-conjugacy class of $\omega_{k,0,m}$ is then both ρ_B- and ρ_D-invariant, and the restriction from B_n to B_n^+ of the subrepresentation of ϱ_B on V coincides with the subrepresentation of ϱ_D on V.

Remark. If we replace $\omega_{k,0,m}$ by $\omega_{k,\ell,m}$ for an integer $\ell > 0$, then the B_n^+-representations described in the lemma are still well-defined, but generally not isomorphic. In particular, while $\mathrm{Invol}(B_n^+)$ is a ϱ_B-invariant subspace of $\mathrm{Invol}(B_n)$, the subrepresentation of ϱ_B on $\mathrm{Invol}(B_n^+)$ does not restrict to a representation isomorphic to ϱ_D.

Proof. It is enough to demonstrate that if $\omega = (x, \pi) \in B_n^+$ is an involution in the B_n-conjugacy class of $\omega_{k,0,m}$, then $\varrho_B(s)a_\omega = \varrho_D(s)a_\omega$ for $s \overset{\text{def}}{=} s'_{n-1} = t_n s_{n-1} t_n \in B_n^+$. In particular, this suffices because, as noted above, s_i for $i \in [n-1]$ is a right descent of an involution $\omega \in B_n^+$ with respect to the type D_n length function if and only it is a descent with respect to the type BC_n length function.

To this end, note that we have $s\omega = \omega s$ if and only if $(n-1, n)$ is a cycle of π or if $n-1, n$ are fixed points of π such that $x_{n-1} = x_n$. In either case, $x_{n-1} = x_n \in \{0, 1\}$ and one checks that

$$\varrho_B(s)a_\omega = \varrho_D(s)a_\omega = (-1)^{x_n} a_\omega.$$

If $s\omega \neq \omega s$ then either (i) $n-1, n$ are fixed points of π but $x_{n-1} \neq x_n$, (ii) $\pi(n) \notin \{n-1, n\}$, or (iii) $\pi(n-1) \notin \{n-1, n\}$. The B_n-conjugacy class of $\omega_{k,0,m}$ has no elements satisfying (i), since if i is a fixed point of π then $x_i = 0$. Using this fact, one checks that in cases (ii) and (iii) we have $\varrho_B(s)a_\omega = \varrho_D(s)a_\omega = a_{s\omega s}$, as required. $\square$

A consequence of Clifford theory is that if α, β are partitions with $|\alpha| + |\beta| = n$, then the restricted representations

$$\mathrm{Res}^{B_n}_{B_n^+}\left(\rho^{(\alpha,\beta)}\right) \cong \mathrm{Res}^{B_n}_{B_n^+}\left(\rho^{(\beta,\alpha)}\right)$$

are isomorphic. Let $\rho^{\{\alpha,\beta\}}$ denote a representation of B_n^+ isomorphic to these restrictions; here, we view the superscript index $\{\alpha,\beta\}$ as a set of partitions, so that *a priori* we have $\rho^{\{\alpha,\beta\}} = \rho^{\{\beta,\alpha\}}$. When $\alpha = \beta$, the representation $\rho^{\{\alpha\}} = \rho^{\{\beta\}} = \rho^{\{\alpha,\beta\}}$ decomposes as a sum of two non-isomorphic irreducible representations, which we denote (in some order) by $\rho^{\{\alpha\},1}$ and $\rho^{\{\alpha\},2}$. To distinguish between these two types of representations, we give the index of $\rho^{\{\alpha,\beta\}}$ with $\alpha \neq \beta$ a special name.

Definition 2.3.13. An *unordered bipartition* of n is a set $\{\alpha,\beta\}$ consisting of two partitions $\alpha \neq \beta$ with $|\alpha| + |\beta| = n$. We write $\{\alpha,\beta\} \vdash n$ to indicate that $\{\alpha,\beta\}$ is an unordered bipartition of n.

When n is odd, the representations $\rho^{\{\alpha,\beta\}}$ with $\{\alpha,\beta\}$ ranging over all unordered bipartitions of n represent the distinct isomorphism classes of irreducible representations of B_n^+. When n is even, the representations $\rho^{\{\alpha,\beta\}}$ for unordered bipartitions $\{\alpha,\beta\} \vdash n$ together with $\rho^{\{\alpha\},1}$ and $\rho^{\{\alpha\},2}$ for partitions $\alpha \vdash n/2$ represent the distinct isomorphism classes of irreducible representations of B_n^+. We write $\chi^{\{\alpha\},1}$, $\chi^{\{\alpha\},2}$, and $\chi^{\{\alpha,\beta\}}$ for the characters of $\rho^{\{\alpha\},1}$, $\rho^{\{\alpha\},2}$, and $\rho^{\{\alpha,\beta\}}$, so that

$$\mathrm{Irr}(W) = \left\{\chi^{\{\alpha\},1}, \chi^{\{\alpha\},2} : \alpha \vdash n/2\right\} \cup \left\{\chi^{\{\alpha,\beta\}} : \{\alpha,\beta\} \vdash n\right\}$$

where the first set in the right hand union is empty is n is odd.

Observation 2.3.14. It is useful to note that with respect to this notation, we have

(a) $\mathrm{Ind}^{B_n}_{B_n^+}\left(\chi^{\{\alpha,\beta\}}\right) = \chi^{(\alpha,\beta)} + \chi^{(\beta,\alpha)}$ if $\{\alpha,\beta\}$ is an unordered bipartition of n;

(b) $\mathrm{Ind}^{B_n}_{B_n^+}\left(\chi^{\{\alpha\},1}\right) = \mathrm{Ind}^{B_n}_{B_n^+}\left(\chi^{\{\alpha\},2}\right) = \chi^{(\alpha,\alpha)}$ if n is even and $\alpha \vdash n/2$.

Proof. These facts are consequences of Frobenius reciprocity. ☐

Our labeling of the two irreducible constituents of $\mathrm{Res}^{B_{2n}}_{B_{2n}^+}\left(\rho^{(\alpha,\alpha)}\right)$ for $\alpha \vdash n$ is not canonical: it depends on an arbitrary choice for which representation to call $\rho^{\{\alpha\},1}$ and which to call $\rho^{\{\alpha\},2}$. The following result shows that there is a natural way of making these choices; this is connected to the discussion in Geck's recent work [40, Section 3.9].

Proposition 2.3.15. Suppose (W,S) is of type D_n where $n = 2m$ is even. It is possible to arrange the indices of the characters $\chi^{\{\alpha\},1}$, $\chi^{\{\alpha\},2}$ for partitions $\alpha \vdash m$ so that the characters of the subrepresentations of ϱ_W on the subspaces of $\mathrm{Invol}(W)$ spanned by the conjugacy classes of ω_m and ω'_m, respectively, are the multiplicity free sums

$$\sum_{\alpha \vdash m} \chi^{\{\alpha\},1} \qquad \text{and} \qquad \sum_{\alpha \vdash m} \chi^{\{\alpha\},2}.$$

Proof. Let V and V' be the subspaces of $\mathrm{Invol}(W)$ spanned by the conjugacy classes of ω_m and ω'_m. Recall that the centralizer of ω_m in B_n is the group V_m defined in the previous section. Since $V_m \subset B_n^+$, V_m is also the centralizer of ω_m in B_n^+, and so if $\lambda : V_m \to \mathbb{C}$ is the linear character defined by $\varrho_W(g)a_{\omega_m} = \lambda(g)a_{\omega_m}$ for $g \in V_m$, then the character of the subrepresentation of ϱ_W on V is $\mathrm{Ind}_{V_m}^{B_n^+}(\lambda)$. Since by Lemmas 2.3.8 and 2.3.12 we have $\mathrm{Ind}_{B_n^+}^{B_n}\left(\mathrm{Ind}_{V_m}^{B_n^+}(\lambda)\right) = \sum_{\alpha\vdash m}\rho^\alpha$, it follows from the observations (a) and (b) above that the character of subrepresentation of ϱ_W on V contains exactly one of $\chi^{\{\alpha\},1}$ or $\chi^{\{\alpha\},2}$ as a constituent for each $\alpha \vdash m$. This suffices to prove the proposition since the character of the subrepresentation of $\rho_{(W,S)}$ on $V \oplus V'$ is $\mathrm{Res}_{B_n^+}^{B_n}\left(\sum_{\alpha\vdash m}\chi^{(\alpha,\alpha)}\right) = \sum_{\alpha\vdash m}\left(\chi^{\{\alpha\},1}+\chi^{\{\alpha\},2}\right)$ by Lemmas 2.3.8 and 2.3.12. $\square$

The following constructions are type D versions of the "symbols" introduced in the preceding section. This material is again originally due to Lusztig [63]. Given a nonnegative integer m, let $\mathcal{S}_m^D$ denote the set of pairs (λ,μ) of strictly increasing nonnegative integer sequences of length m, represented as two-line arrays

$$\left(\begin{array}{cccc} \lambda_1, & \lambda_2, & \cdots, & \lambda_m \\ \mu_1, & \mu_2, & \cdots, & \mu_m \end{array}\right).$$

Lusztig [63] defines an equivalence relation $\sim$ on the disjoint union $\mathcal{S}^D \overset{\text{def}}{=} \bigcup_{m\geq 0}\mathcal{S}_m^D$ by setting

$$\left(\begin{array}{cccc} \lambda_1, & \lambda_2, & \cdots, & \lambda_m \\ \mu_1, & \mu_2, & \cdots, & \mu_m \end{array}\right) \sim \left(\begin{array}{cccc} \mu_1, & \mu_2, & \cdots, & \mu_m \\ \lambda_1, & \lambda_2, & \cdots, & \lambda_m \end{array}\right)$$

and

$$\left(\begin{array}{cccc} \lambda_1, & \lambda_2, & \cdots, & \lambda_m \\ \mu_1, & \mu_2, & \cdots, & \mu_m \end{array}\right) \sim \left(\begin{array}{ccccc} 0, & \lambda_1+1, & \lambda_2+1, & \cdots, & \lambda_m+1 \\ 0, & \mu_1+1, & \mu_2+1, & \cdots, & \mu_m+1 \end{array}\right)$$

and then extending $\sim$ reflexively and transitively. As with $\mathcal{S}^{BC}$, we call the equivalence classes in $\mathcal{S}^D$ under this relation *symbols*. We now recall the following definitions from [63].

Definition 2.3.16. The *symbol* of an unordered bipartition $\{\alpha,\beta\}$ of n is the symbol of $(\lambda,\mu) \in \mathcal{S}_n^D$, where

$$\begin{aligned} \lambda_i &= \alpha_{n-(i-1)} + (i-1), && \text{for } i \in [n], \\ \mu_i &= \beta_{n-(i-1)} + (i-1), && \text{for } i \in [n]. \end{aligned} \tag{2.3.8}$$

(a) The unordered bipartition $\{\alpha,\beta\}$ is *special* if its symbol contains for some m a representative $(\lambda,\mu) \in \mathcal{S}_m^D$ with $\lambda_i \leq \mu_i$ for all $i \in [m]$ and $\mu_i \leq \lambda_{i+1}$ for all $i \in [m-1]$.

(b) Two symbols Λ, Λ' in $\mathcal{S}^D$ are said to *belong to the same family* if for some m there exist $(\lambda,\mu) \in \mathcal{S}_m^D \cap \Lambda$ and $(\gamma,\nu) \in \mathcal{S}_m^D \cap \Lambda'$ such that the concatenated sequences

$$(\lambda_1,\lambda_2,\ldots,\lambda_m,\mu_1,\mu_2,\ldots,\mu_m) \qquad \text{and} \qquad (\gamma_1,\gamma_2,\ldots,\gamma_m,\nu_1,\nu_2,\ldots,\nu_m)$$

are permutations of each other. Two unordered bipartitions of n *belong to the same family* if and only if their symbols belong to the same family.

(c) In any $(\lambda, \mu) \in \mathcal{S}^D$, a certain number of entries of μ do not appear in λ and a certain number of entries of λ do not appear in μ. Denote these (equal) numbers by $e(\lambda, \mu)$.

The number $e(\lambda, \mu)$ is the same for all elements in a given symbol and we therefore define

$$e(\alpha, \beta) \stackrel{\text{def}}{=} e(\lambda, \mu) \text{ for any } (\lambda, \mu) \in \mathcal{S}^D \text{ which represents the symbol of } \{\alpha, \beta\}.$$

Note that $e(\alpha, \beta) = e(\beta, \alpha) \geq 1$ by construction since $\alpha \neq \beta$. In general, $d(\alpha, \beta) \neq e(\alpha, \beta)$, as shall soon become clear.

Example 2.3.17. The unordered bipartitions $\{\ydiagram{2,1}, \ydiagram{3,2}\}$, $\{\ydiagram{2,2}, \ydiagram{3,1}\}$, and $\{\ydiagram{1,1}, \ydiagram{3,3}\}$ of 8 constitute a family, since their symbols are respectively represented by

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

The unordered bipartition $\{\ydiagram{2,1}, \ydiagram{3,2}\}$ is special, and $e\left(\ydiagram{2,1}, \ydiagram{3,2}\right) = e\left(\ydiagram{2,2}, \ydiagram{3,1}\right) = e\left(\ydiagram{1,1}, \ydiagram{3,3}\right) = 2$.

In analogy with Theorems 2.3.1 and 2.3.6, Lusztig [63] shows that the families and special characters in type D_n are given as follows.

Theorem 2.3.18 (See Chapter 4, Section 6 in [63]). Suppose (W, S) is of type D_n.

(1) If n is even then the irreducible characters $\chi^{\{\alpha\},1}$ and $\chi^{\{\alpha\},2}$ of W are special for all $\alpha \vdash n/2$. The irreducible character $\chi^{\{\alpha,\beta\}}$ of W is special if and only if the unordered bipartition $\{\alpha, \beta\} \vdash n$ is special.

(2) If n is even then the irreducible characters $\chi^{\{\alpha\},1}$ and $\chi^{\{\alpha\},2}$ for $\alpha \vdash n/2$ all belong to their own families with one element. Irreducible characters indexed by unordered bipartitions of n belong to the same family if and only if their indices belong to the same family.

As in type BC_n, one can define the set of special unordered bipartitions $\{\alpha, \beta\}$ and the numbers $e(\alpha, \beta)$ while avoiding the notion of symbols entirely, and this ability, in the form of the following lemma will prove useful later.

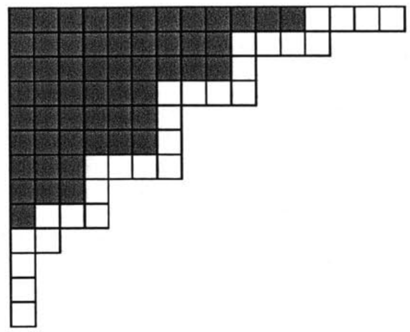
Lemma 2.3.19. Let $\{\alpha, \beta\}$ be an unordered bipartition of n.

(1) An unordered bipartition is special if and only if it is given by $\{\alpha, \beta\}$ where $\alpha \subsetneq \beta$ and the skew diagram $\beta \setminus \alpha$ contains no 2×2 squares.

(2) If $\{\alpha, \beta\}$ is special and $\alpha \subsetneq \beta$, then $e(\alpha, \beta)$ is the number of connected components in the skew diagram $\beta \setminus \alpha$.

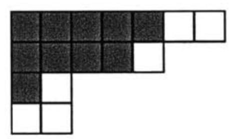
Before giving the proof we make two remarks.

Remarks. The condition for an unordered bipartition $\{\alpha, \beta\}$ to be special corresponds to the following picture:



If the Young diagram of α is the set of gray cells and $\alpha \subset \beta$, then $\{\alpha, \beta\}$ is special if and only if β is formed by adding (a subset of) white cells to α.

Concerning the second part of the lemma, we recall that the connected components of a skew diagram are the equivalences classes of cells under the relation of (left-right or up-down) adjacency. For example, if $\beta = (7, 5, 2, 2)$ and $\alpha = (5, 4, 1)$ then the skew diagram $\beta \setminus \alpha$, shown as the white cells in the picture



has $e(\alpha, \beta) = 3$ connected components.

Proof of Lemma 2.3.19. Noting (2.3.8), one finds that the definition of a an unordered bipartition $\{\alpha, \beta\}$ is equivalent to the condition that $\alpha_i \leq \beta_i$ and $\beta_{i+1} \leq \alpha_i + 1$ for all $i \in \mathbb{P}$ (or the same condition with α and β interchanged). It is straightforward to check, remembering that we assume $\alpha \neq \beta$, that this holds if and only if $\alpha \subsetneq \beta$ and $\beta \setminus \alpha$ contains no 2×2 squares.

Suppose $\{\alpha, \beta\}$ is special and $\alpha \subsetneq \beta$. Then, defining λ_i and μ_i by (2.3.8), one has $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_n \leq \mu_n$ and $e(\alpha, \beta)$ is equal to the number of $i \in [n]$ with $\lambda_i < \mu_i < \lambda_{i+1}$, where by convention $\lambda_{n+1} = \infty$. Equivalently, $e(\alpha, \beta)$ is equal to the number of $i \in \mathbb{P}$ with $\alpha_i < \beta_i \leq \alpha_{i-1}$, where we define $\alpha_0 = \infty$. It becomes clear on consulting the picture in the remarks above that the upper right most cells of the connected components of $\beta \setminus \alpha$ are precisely those of the form (i, β_i) for $i \in \mathbb{P}$ with $\alpha_i < \beta_i \leq \alpha_{i-1}$. Hence $e(\alpha, \beta)$ is the number of connected components in the skew diagram $\beta \setminus \alpha$. $\square$

Continue to suppose (W, S) is of type D_n, and let k, ℓ, m be nonnegative integers with $2m + k + \ell = n$ such that ℓ is even and k or ℓ is nonzero. Recalling (2.3.6), we note that the centralizer of $\omega_{k,\ell,m}$ in $W = B_n^+$ is the index two subgroup of $C_{B_n}(\omega_{k,\ell,m}) = V_m \times B_k \times B_\ell \subset B_n$ consisting of all elements $(x, \pi) \in C_{B_n}(\omega_{k,\ell,m})$ with $\sum_{i=1}^n x_i = \sum_{i=m+1}^n x_i = 0$. In other words, we may identify

$$C_W(\omega_{k,\ell,m}) = V_m \times U_{k,\ell}$$

where

$$U_{k,\ell} \stackrel{\text{def}}{=} \{(x,\pi) \in B_k \times B_\ell \subset B_{k+\ell} : x_1 + \cdots + x_{k+\ell} = 0\}.$$

Thus the linear character $\widetilde{\lambda} : C_W(\omega_{k,\ell,m}) \to \mathbb{C}$ defined by

$$\varrho_W(g)a_{\omega_{k,\ell,m}} = \widetilde{\lambda}(g)a_{\omega_{k,\ell,m}}, \qquad \text{for } g \in C_W(\omega_{k,\ell,m}),$$

decomposes as the external tensor product $\widetilde{\lambda} = \lambda \otimes \tau$ where $\lambda : V_m \to \mathbb{C}$ is the linear character defined in the proof of Lemma 2.3.8 and τ is a linear character $U_{k,\ell} \to \mathbb{C}$. It is a straightforward exercise from the definition of ϱ_W to check that, writing k for the partition $(k,0,0,\dots) \vdash k$ and 1^ℓ for the partition $(1,1,\dots,1,0,0,\dots) \vdash \ell$, we have

$$\mathrm{Ind}_{V_m \times U_{k,\ell}}^{V_m \times B_k \times B_\ell}(\lambda \otimes \tau) = \left(\lambda \otimes \chi^{(k,\varnothing)} \otimes \chi^{(1^\ell,\varnothing)}\right) + \left(\lambda \otimes \chi^{(\varnothing,k)} \otimes \chi^{(\varnothing,1^\ell)}\right).$$

By Lemmas 2.3.8 and 2.3.12, it follows that

$$\begin{aligned}\mathrm{Ind}_{C_W(\omega_{k,\ell,m})}^{B_{2m} \times B_k \times B_\ell}(\lambda \otimes \tau) = &\sum_{\alpha \vdash m} \left(\mathrm{Ind}_{B_{2m} \times B_k \times B_\ell}^{B_n} \left(\chi^{(\alpha,\alpha)} \otimes \chi^{(k,\varnothing)} \otimes \chi^{(1^\ell,\varnothing)}\right)\right) \\ &+ \sum_{\alpha \vdash m} \left(\mathrm{Ind}_{B_{2m} \times B_k \times B_\ell}^{B_n} \left(\chi^{(\alpha,\alpha)} \otimes \chi^{(\varnothing,k)} \otimes \chi^{(\varnothing,1^\ell)}\right)\right).\end{aligned}$$

The induced characters in these sums decompose according to the following lemma.

Lemma 2.3.20. If k, ℓ, m are nonnegative integers with $2m + k + \ell = n$ and $\alpha \vdash m$, then

$$\mathrm{Ind}_{B_{2m} \times B_k \times B_\ell}^{B_n} \left(\chi^{(\alpha,\alpha)} \otimes \chi^{(k,\varnothing)} \otimes \chi^{(1^\ell,\varnothing)}\right) = \sum_{(\beta,\gamma)} \chi^{(\beta,\alpha)}$$

and

$$\mathrm{Ind}_{B_{2m} \times B_k \times B_\ell}^{B_n} \left(\chi^{(\alpha,\alpha)} \otimes \chi^{(\varnothing,k)} \otimes \chi^{(\varnothing,1^\ell)}\right) = \sum_{(\beta,\gamma)} \chi^{(\alpha,\beta)}$$

where both sums are over all pairs of partitions (β,γ) with $|\gamma| = m + k$ and $|\beta| = m + k + \ell$ such that the Young diagram of γ is obtained from the Young diagram of α by adding k cells in distinct columns, and the Young diagram of β is obtained from the Young diagram of γ by adding ℓ cells in distinct rows.

Proof. These identities follow from the Pieri rules for the symmetric group in essentially the same way as in the proof of Lemma 2.3.9. □

From this lemma, we may now likewise give a detailed proof of the Kottwitz's decomposition of the characters $\chi_{W,\sigma}$ in type D_n.

Theorem 2.3.21 (Kottwitz [55]). Suppose (W,S) is of type D_n and k, ℓ, m are nonnegative integers with $2m + k + \ell = n$ such that ℓ is even and either k or ℓ is nonzero. Then the character $\chi_{W,\omega_{k,\ell,m}}$ of the subrepresentation of ϱ_W on the subspace of Invol(W) spanned by the conjugacy class of $\omega_{k,\ell,m} \in W$ is

$$\chi_{W,\omega_{k,\ell,m}} = \sum_{\{\alpha,\beta\}} \binom{e(\alpha,\beta)}{k - f(\alpha,\beta)} \chi^{\{\alpha,\beta\}},$$

where the sum is over all special unordered bipartitions $\{\alpha,\beta\} \vdash n$ with $|\alpha| = m$ and $|\beta| = m + k + \ell$, and where $f(\alpha,\beta)$ is the number of positive integers i with $\alpha_i \neq \beta_i$.

Proof. Let $\chi = \chi_{W,\omega_{k,\ell,m}}$. Continuing in the notation introduced after Lemma 2.3.19, we have shown that

$$\mathrm{Ind}_W^{B_n}(\chi) = \mathrm{Ind}_{C_W(\omega_{k,\ell,m})}^{B_n}(\lambda \otimes \tau) = \sum_{(\alpha,\beta,\gamma)} \left(\chi^{(\alpha,\beta)} + \chi^{(\beta,\alpha)}\right) \tag{2.3.9}$$

where the sum is over all triples of partitions (α,β,γ) with $\alpha \vdash m$ and $\gamma \vdash m+k$ and $\beta \vdash m+k+\ell$, such that γ is obtained from α by adding k cells in distinct columns, and β is obtained from γ by adding ℓ cells in distinct rows.

It is clear from this equation that for any bipartition $(\alpha,\beta) \vdash n$, the character $\chi^{(\alpha,\beta)}$ and $\chi^{(\beta,\alpha)}$ appear in $\mathrm{Ind}_W^{B_n}(\chi)$ with equal multiplicity, and that this multiplicity is zero if $\alpha = \beta$. In light of Observation 2.3.14, the theorem will follow if we can show that the multiplicity of $\chi^{(\alpha,\beta)}$ in (2.3.9) is equal to the binomial coefficient ascribed to $\chi^{\{\alpha,\beta\}}$ in the theorem statement. To this end, we first note that if $\alpha \vdash m$ and γ is obtained by first adding k cells in distinct rows to α and β is obtained by adding ℓ cells in distinct columns to γ, then clearly $\beta \vdash m+k+\ell$ and $\alpha \subset \beta$ and $\beta \setminus \alpha$ has no 2×2 squares. It therefore follows from Lemma 2.3.19 that $\{\alpha,\beta\}$ is a special unordered bipartition.

Let $\{\alpha,\beta\}$ be a special unordered bipartition with $\alpha \vdash m$ and $\beta \vdash m+k+\ell$ (so that $\alpha \subset \beta$). To complete the theorem's proof, we must show that the number $m_{\alpha,\beta}$ of partitions $\gamma \vdash m+k$ such that the cells of $\gamma \setminus \alpha$ lie in distinct columns and the cells of $\beta \setminus \gamma$ lie in distinct rows is $\binom{e(\alpha,\beta)}{k-f(\alpha,\beta)}$. If $\gamma \vdash m+k$ is such a partition, then every cell in $\gamma \setminus \alpha$ belongs to $\{(\alpha'_j+1, j) : j \in \mathbb{P}\}$ and every cell in $\beta \setminus \gamma$ belongs to $\{(i,\beta_i) : i \in \mathbb{P}\}$. Hence if ν is the partition of $m + f(\alpha,\beta)$ whose Young diagram is formed by removing from β all cells (i,β_i) which do not belong to α, then $\nu \subset \gamma$ and every cell in $\gamma \setminus \nu$ must belong to $D \overset{\text{def}}{=} \{(\alpha'_j+1,j) : j \in \mathbb{P}\} \cap \{(i,\beta_i) : i \in \mathbb{P}\}$. For any subset $S \subset D$ there exists a valid partition ν_S of $m + f(\alpha,\beta) + |S|$ whose Young diagram is given by adding the cells in S to ν. From the preceding observations, we conclude that γ must be equal to ν_S for some subset $S \subset D$ with $k - f(\alpha,\beta)$ elements.

Because the unordered bipartition $\{\alpha,\beta\}$ is special, the cells in $\nu \setminus \alpha$ lie in the set $\{(\alpha'_j+1,j) : j \in \mathbb{P}\}$. As such, it follows by construction that each partition ν_S for $S \subset D$ has the property that the cells in $\nu_S \setminus \alpha$ lie in distinct columns and the cells in $\beta \setminus \nu_S$ lie in distinct rows. Hence, the possible choices for γ are in bijection with the subsets $S \subset D$ with $k - f(\alpha,\beta)$ elements, so $m_{\alpha,\beta} = \binom{|D|}{k-f(\alpha,\beta)}$. Since $\beta \setminus \alpha$ contains no 2×2 squares, each connected component of $\beta \setminus \alpha$ contains exactly one cell in D (namely, the upper right most cell in the component). Thus $|D| = e(\alpha, \beta)$ by Lemma 2.3.19, which completes our proof. $\square$

Corollary 2.3.22 (Kottwitz [55]). If (W, S) is of type D_n then

$$\chi_{(W,S)} = \sum_{\gamma \vdash n/2} \left(\chi^{\{\gamma\},1} + \chi^{\{\gamma\},2}\right) + \sum_{\{\alpha,\beta\}} 2^{e(\alpha,\beta)-1}\chi^{\{\alpha,\beta\}}$$

where we omit the sum over $\gamma \vdash n/2$ if n is odd and where the second sum is over all special unordered bipartitions $\{\alpha, \beta\}$ of n.

Proof. If we sum the subrepresentations described in the preceding theorem over all integers $k, \ell, m \geq 0$ with $2m + k + \ell = n$ such that ℓ is even and k or ℓ is nonzero, then the resulting multiplicity of $\rho^{\{\alpha,\beta\}}$ is a sum of binomial coefficients of the form $\left(\binom{e(\alpha,\beta)}{0} + \binom{e(\alpha,\beta)}{2} + \dots\right)$ or $\left(\binom{e(\alpha,\beta)}{1} + \binom{e(\alpha,\beta)}{3} + \dots\right)$. These still give familiar powers of two, but with the exponent diminished by one. $\square$

2.4 Decomposing χ_W for exceptional Coxeter systems

Casselman computes in [31] the decomposition of χ_W when W is an exceptional Weyl group of type of type E_6, E_7, E_8, G_2, or F_4. Using the computer algebra system MAGMA, we have in turn computed the decomposition of χ_W when W is one of the remaining exceptional finite Coxeter systems of type H_3 or H_4. We take the opportunity in this thesis to present all of this data in one place–namely, as Tables A.2, A.3, A.4, A.5, A.6, A.7, A.8, A.9, A.10, and A.11 in the appendix. We have structured these tables to serve the additional purpose of providing a convenient lexicon for various notations used in the literature for the irreducible characters of the exceptional finite Coxeter groups.

2.4.1 Format of tables

Tables A.2–A.11 are structured as follows: each row corresponds to an individual character $\psi \in \mathrm{Irr}(W)$, and each collection of rows grouped together represents a family of characters $\mathcal{F} \subset \mathrm{Irr}(W)$. In each such family, the unique special character is listed first. The first column gives the multiplicity of ψ in χ_W. The second and third columns give two different names used for the character ψ. The remaining columns list data parametrizing the characters within a family (which is used in Section 2.5 to indicate how the Fourier transform acts on each ψ).

In all of our tables, the second column lists the name of ψ in the notation of Carter's book [30]. In this notation, each irreducible character of W is generally denoted $\phi_{d,e}$, were d and e are such that

$$\phi_{d,e}(1) = d \qquad \text{and} \qquad \mathrm{FakeDeg}(\phi_{d,e}) = (\text{nonzero constant}) \cdot x^e + \text{higher order terms.}$$

The two numbers d, e uniquely identify all of the irreducible characters of W in types E_6, E_7, E_8, and H_3. In types F_4 and G_2, Carter addresses a handful of ambiguities by labeling certain pairs of characters as $\phi_{d,e}$ and $\phi'_{d,e}$; we follow his example. In type H_4, the notation $\phi_{d,e}$ fails to distinguish the two irreducible characters of W of degree 30. Since we cannot refer to Carter's labeling of these characters (as he only considers the crystallographic case), we denote these characters by $\phi_{30,10,12}$ and $\phi_{30,10,14}$, where $\phi_{30,10,f}$ indicates the irreducible character with

$$\phi_{30,10,f}(1) = 30 \qquad \text{and} \qquad \mathrm{FakeDeg}(\phi_{30,10,f}) = x^{10} + x^f + \text{higher order terms.}$$

This data in the columns after the third in our tables is formatted in two different ways according to whether (W, S) has one of the crystallographic types E_6, E_7, E_8, G_2, F_4 and or one of the non-crystallographic types H_3, H_4. In Tables A.10 and A.11 addressing the non-crystallographic types, the fourth and fifth columns indicate the Fourier transform matrix attached to a given family and the index in this matrix which is assigned to a given character. A detailed explanation accompanying these assignments appears in Sections 2.5.3.

When (W, S) is crystallographic, to each family $\mathcal{F} \subset \mathrm{Irr}(W)$ there corresponds a group Γ and a set $\mathscr{M}_{\mathcal{F}} = \mathscr{M}(\Gamma)$ as defined by (2.2.2). To each $\psi \in \mathcal{F}$ there is then assigned a pair $(x, \sigma) \in \mathscr{M}_{\mathcal{F}}$, which also indexes the unipotent irreducible character Φ_ψ corresponding to ψ. (Our reference for these assignments is [30, §13.2].) In Tables A.2 to A.9, the fourth column lists the group Γ for which $\mathscr{M}_{\mathcal{F}} = \mathscr{M}(\Gamma)$, and the fifth, sixth, and seventh columns indicate the pair $(x, \sigma) \in \mathscr{M}_{\mathcal{F}}$ (as well as the centralizer $C_\Gamma(x)$ for clarity) which corresponds to ψ, as listed in [30, §13.9] and the appendix of [63].

The groups Γ occurring in the fourth column are all symmetric groups S_n with $n \leq 5$. Accordingly, the elements $x \in \Gamma$ in column five are permutations written in cycle notation. The centralizers of these elements which appear are each either

- a symmetric group S_n with $n \leq 5$;
- a cyclic group $\mathbb{Z}_n$ with $n \in \{3, 4, 5\}$;
- a direct product of the form $S_2 \times S_2$ or $S_2 \times S_3$ or $\mathbb{Z}_3 \times S_2$;
- the dihedral group of order eight Dih_8.

To indicate the characters σ of these centralizers which appear, we employ the following notation. Let $\mathbb{1}$ denote the trivial character of any group, and let sgn denote the sign character of a symmetric group. Other characters of the symmetric group are indicated by listing the Young diagram of the corresponding partition; e.g., $\yng(3,1)$ denotes the reflection representation of S_4. We will view Dih_8 as the centralizer of the permutation $(1,2)(3,4)$ in

S_4, and refer to its irreducible characters as they are labeled in the following character table:

	1	(1,2)(3,4)	(1,3)(2,4)	(1,2)	(1,4,2,3)
$\mathbb{1}$	1	1	1	1	1
ε'	1	1	1	-1	-1
ε	1	1	-1	-1	1
ε''	1	1	-1	1	-1
r	2	-2	0	0	0

Finally, we label the characters of direct products as tensor products of characters of their factors, in the obvious way. This should be ambiguous in only one case, when $x = (1,2) \in \Gamma = S_4$ and $C_\Gamma(x) = \{1,\ (1,2),\ (3,4),\ (1,2)(3,4)\} \cong \langle(1,2)\rangle \times \langle(3,4)\rangle \cong S_2 \times S_2$ and $\sigma = \mathbb{1}\otimes\mathrm{sgn}$. In this situation, $\mathbb{1}$ corresponds to the factor group $\langle(1,2)\rangle$ while sgn corresponds to the the factor group $\langle(3,4)\rangle$, so that $\sigma(1,2) = 1$ and $\sigma(3,4) = -1$.

2.4.2 Type H_3

We devote the rest of this section to describing the irreducible decomposition of the characters $\chi_{W,\sigma}$ in type H_3, H_4, and $I_2(m)$, since this will be needed in Section 2.6.

Let (W,S) be the Coxeter system of type H_3, so that W is isomorphic to the direct product of S_2 and the alternating subgroup of S_5, and $S = \{a,b,c\}$ consists of three elements, which we label according to the Dynkin diagram

$$a \overset{5}{\text{——}} b \text{——} c$$

(so that a,b,c satisfy the relations $(ab)^5 = (ac)^2 = (bc)^3 = a^2 = b^2 = c^2 = 1$). There are then four conjugacy classes of involutions in W, represented by the elements 1, a, ac, and $(abc)^5$, and W has 10 irreducible characters.

Decomposing ϱ_W and its subrepresentations is a routine calculation which we have carried out in the computer algebra system MAGMA [18]. From this computation, we obtain the following:

Proposition 2.4.1. If (W,S) is of type H_3 then the characters $\chi_{W,\sigma}$ decompose as follows:

(1) $\chi_{W,1} = \phi_{1,0}$.

(2) $\chi_{W,(abc)^5} = \phi_{1,15}$.

(3) $\chi_{W,a} = \phi_{3,1} + \phi_{3,3} + \phi_{4,3} + \phi_{5,5}$.

(4) $\chi_{W,ac} = \phi_{3,6} + \phi_{3,8} + \phi_{4,4} + \phi_{5,2}$.

Consequently, $\chi_W = \sum_{\psi\in\mathrm{Irr}(W)} \psi$ and ϱ_W is a Gelfand model.

2.4.3 Type H_4

Let (W, S) now be the Coxeter system of type H_4, so that W is a finite group of order 14400 generated by four elements $S = \{a, b, c, d\}$, which we label according to the Dynkin diagram

$$a \overset{5}{\text{——}} b \text{——} c \text{——} d\,.$$

There are five conjugacy classes of involutions in W, represented by 1, a, ac, $(abc)^5$, and $(abcd)^{15}$, and $\mathrm{Irr}(W)$ has 34 elements. Again, decomposing χ_W and $\chi_{W,\sigma}$ is a routine computation, which we have carried out in MAGMA.

Proposition 2.4.2. If (W, S) is of type H_4 then the characters $\chi_{W,\sigma}$ decompose as follows:

(1) $\chi_{W,1} = \phi_{1,0}$.

(2) $\chi_{W,(abcd)^{15}} = \phi_{1,60}$.

(3) $\chi_{W,a} = \phi_{4,1} + \phi_{4,7} + \phi_{16,3} + \phi_{36,5}$.

(4) $\chi_{W,(abc)^5} = \phi_{4,31} + \phi_{4,37} + \phi_{16,21} + \phi_{36,15}$.

(5) $$\begin{aligned}\chi_{W,ac} = {} & \phi_{9,2} + \phi_{9,6} + \phi_{9,22} + \phi_{9,26} + \phi_{16,6} + \phi_{16,18} + \phi_{25,4} + \phi_{25,16} \\ & + 2\phi_{24,6} + 2\phi_{24,12} + 2\phi_{18,10} + 2\phi_{30,10,12} + 2\phi_{30,10,14} + 2\phi_{40,8}.\end{aligned}$$

2.4.4 Type $I_2(m)$

Fix $m \geq 3$ and let (W, S) be the Coxeter system of type $I_2(m)$, so that W is the dihedral group of order $2m$, generated by the two elements $S = \{r, s\}$ subject to the relations $r^2 = s^2 = (rs)^m = 1$. The involutions in W are the elements 1 and $(rs)^j r$ for $0 \leq j \leq m-1$, along with $(rs)^{m/2}$ if m is even. We always have $\mathrm{Des}_R(1) = \varnothing$, and one checks that

$$\mathrm{Des}_R\left((rs)^{m/2}\right) = \{r, s\} \qquad \text{and} \qquad \mathrm{Des}_R\left((rs)^j r\right) = \begin{cases} \{r\}, & \text{if } 2j+1 < m \\ \{r, s\}, & \text{if } 2j+1 = m \\ \{s\}, & \text{if } 2j+1 > m. \end{cases} \tag{2.4.1}$$

Write w_0 for the longest element of W, given by $(rs)^{\frac{m-1}{2}} r$ if m is odd or $(rs)^{m/2}$ if m is even. The group W has either two or four conjugacy classes of involutions, represented by 1 and r if m is odd and by 1, w_0, r, and s if m is even.

The irreducible characters of W are given as follows. There are two linear characters when m is odd, given by $\phi_{1,0} = \mathbb{1}$ and $\phi_{1,m} = \mathrm{sgn}$, and four linear characters when m is even, given by

$$\phi_{1,0} = \mathbb{1}, \qquad \phi_{1,m} = \mathrm{sgn}, \qquad \phi'_{1,m/2} : r^j s^k \mapsto (-1)^k, \qquad \text{and} \qquad \phi''_{1,m/2} : r^j s^k \mapsto (-1)^j.$$

There are in addition $\lfloor \frac{m-1}{2} \rfloor$ distinct irreducible characters of degree two given by the functions

$$\phi_{2,k}: \begin{array}{lcl} (rs)^j r & \mapsto & 0, \\ (rs)^j & \mapsto & 2\cos(2\pi jk/m), \end{array} \qquad \text{for integers } k \text{ with } 0 < k < \frac{m}{2}.$$

These constructions exhaust all elements of $\mathrm{Irr}(W)$. We have labeled the characters of W following our convention in type H_3 and H_4: the first index of $\phi_{d,e}$ indicates the character's degree while the second index is the largest power of x dividing the character's fake degree. In the notation of Section 2.2.5, we have

$$\phi_{1,0} = \mathbb{1}, \qquad \phi_{1,m} = \mathrm{sgn}, \qquad \phi'_{1,m/2} = \Phi'_{(0,\frac{m}{2})}, \qquad \phi''_{1,m/2} = \Phi''_{(0,\frac{m}{2})}, \qquad \phi_{2,k} = \Phi_{(0,k)}.$$

Table A.12 describes the irreducible decomposition of the characters $\chi_{W,\sigma}$ and χ_W in type $I_2(m)$. The rows in this table correspond to individual irreducible characters of W, while the columns list the multiplicity of each row in the characters $\chi_{W,\sigma}$.

Three distinct patterns arise according to the residue class of m modulo 4. The proof of the given decompositions is a simple exercise using Frobenius reciprocity and the fact that $\chi_{W,\sigma}$ is induced from a linear character λ of the centralizer $C_W(\sigma)$ of σ in W. The character λ is determined by formula $\varrho_W(g)a_\sigma = \lambda(g)a_\sigma$ for $g \in C_W(\sigma)$, which may be explicitly evaluated using (2.4.1). In addition, one checks that if m is odd and σ is one of the representative involutions 1 or r then $C_W(\sigma)$ is W or $\{1,r\} \cong S_2$, and that if m is even and σ is 1, w_0, r, or s then $C_W(\sigma)$ is W, W, $\{1, w_0, r, w_0 r\} \cong S_2 \times S_2$, or $\{1, w_0, s, w_0 s\} \cong S_2 \times S_2$. Evaluating the inner product of λ with the elements of $\mathrm{Irr}(W)$ restricted to these subgroups provides the desired multiplicities of $\chi_{W,\sigma}$.

Summarizing Table A.12, we have the following proposition.

Proposition 2.4.3. Suppose (W,S) is of type $I_2(m)$ with $m \geq 3$.

(1) If m is odd then $\chi_W = \phi_{1,0} + \phi_{1,m} + \sum_{k=1}^{\frac{m-1}{2}} \phi_{2,k} = \sum_{\psi \in \mathrm{Irr}(W)} \psi$.

(2) If $m \equiv 2 \pmod 4$ then $\chi_W = \phi_{1,0} + \phi_{1,m} + \phi_{1,m/2} + \phi'_{1,m/2} + \sum_{k=1}^{\frac{m-2}{4}} 2\phi_{2,2k-1}$.

(3) If $m \equiv 0 \pmod 4$ then $\chi_W = \phi_{1,0} + \phi_{1,m} + \sum_{k=1}^{\frac{m}{4}} 2\phi_{2,2k-1}$.

2.5 Fourier transforms and proof of main theorem

Here we describe the Fourier transform matrices associated to $\mathrm{Uch}(W)$ for each finite, irreducible Coxeter system (W,S), and derive from this setup the proof of Theorem 2.1.2. In the crystallographic case, the relevant definitions are well-established and due originally to Lusztig [63]. We must take more care to describe the associated matrices for the non-crystallographic types, as the literature [19, 67, 75, 76] presenting this heuristic theory is not nearly as cohesive or extensive.

We first describe how to attach to each family in $\mathrm{Uch}(W)$ (see Section 2.2.4) a Fourier transform matrix $\mathbf{M}$. The Fourier transform matrix of $\mathrm{Uch}(W)$ is subsequently constructed as the direct sum of such matrices $\mathbf{M}$ over all families. In Section 2.5.4 we describe some notable properties of this Fourier transform and discuss in what sense these properties indicate the choices of matrices $\mathbf{M}$ to be "canonical."

2.5.1 Fourier transform matrices in crystallographic types

As described in Section 2.2.4, when (W, S) is crystallographic, every family in $\mathrm{Uch}(W)$ is parametrized by a set $\mathscr{M}(\Gamma)$ for some finite group Γ. (Recall from (2.2.3) the definition of this set.) Lusztig [63] defines the Fourier transform associated to a family indexed by $\mathscr{M}(\Gamma)$ as the matrix

$$\mathbf{M}_\Gamma \overset{\text{def}}{=} (\{m, m'\})_{m,m' \in \mathscr{M}(\Gamma)} \tag{2.5.1}$$

whose entries are the numbers

$$\{(x,\sigma),(y,\tau)\} \overset{\text{def}}{=} \frac{1}{|C_\Gamma(x)|}\frac{1}{|C_\Gamma(y)|} \sum_{\substack{g\in\Gamma \\ x\cdot gyg^{-1}=gyg^{-1}\cdot x}} \sigma\left(gyg^{-1}\right)\tau\left(g^{-1}x^{-1}g\right)$$

for $(x,\sigma),(y,\tau) \in \mathscr{M}(\Gamma)$. Carter helpfully provides an explicit description of the sets $\mathscr{M}(\Gamma)$ and the accompanying matrices $\mathbf{M}_\Gamma$ in the cases when $\Gamma = S_n$ and $n \leq 4$ (as well a partial matrix in the case $\Gamma = S_5$) [30, §13.6]; see also the overview in [30, §12.3]. We review the frequently occurring case $\Gamma = S_2$ in the following example.

Example 2.5.1. If s denotes the nontrivial element of S_2, then

$$\mathscr{M}(S_2) = \{(1,\mathbb{1}),(1,\mathrm{sgn}),(s,\mathbb{1}),(s,\mathrm{sgn})\},$$

and with respect to the order in which we just listed $\mathscr{M}(S_2)$, the corresponding matrix is

$$\mathbf{M}_{S_2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

2.5.2 Fourier transform matrices in type $I_2(m)$

The description of the Fourier transform in this case comes from Lusztig's paper [67]. Assume (W, S) is of type $I_2(m)$ for an integer $m \geq 3$ and recall the explicit construction of $\mathrm{Uch}(W)$ given in Section 2.2.5. As noted there, $\mathrm{Uch}(W)$ has only three families, two of which have size one: $\{\mathbb{1}\}$ and $\{\mathrm{sgn}\}$. The Fourier transform of both 1-element families is the 1×1 identity matrix.

Let $\mathcal{F} = \mathrm{Uch}(W)\setminus\{\mathbb{1},\mathrm{sgn}\}$ denote the remaining family, and define $X = X' \cup X''$, where

X' and X'' are the disjoint sets given by

$$X' = \left\{\text{Pairs of integers } (i,j) \text{ with } i+j<m \text{ and either } 0<i<j<m \text{ or } 0=i<j<\tfrac{m}{2}\right\},$$

$$X'' = \begin{cases} \{(0,\frac{m}{2})', (0,\frac{m}{2})''\}, & \text{if } m \text{ is even,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

As is clear from our notation in Section 2.2.5, X naturally parametrizes $\mathcal{F}$. Write $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$ for the standard m^{th} root unity. The Fourier transform of $\mathcal{F}$, as defined by Lusztig [67], is then the matrix

$$\mathbf{D}_m \stackrel{\text{def}}{=} (\{x,x'\})_{x,x'\in X}$$

whose entries are the numbers $\{x,x'\}$ given by

$$\{(i,j),(k,l)\} = \tfrac{1}{m}\left(\xi^{jk-il} + \xi^{-jk+il} - \xi^{-ik+jl} - \xi^{ik-jl}\right), \qquad \text{for } (i,j),(k,l)\in X',$$

and if m is even and $(i,j)\in X'$, by

$$\{(i,j),(0,\tfrac{m}{2})'\} = \{(i,j),(0,\tfrac{m}{2})''\} = \{(0,\tfrac{m}{2})',(i,j)\} = \{(0,\tfrac{m}{2})'',(i,j)\} = \tfrac{(-1)^i-(-1)^j}{m},$$

$$\{(0,\tfrac{m}{2})',(0,\tfrac{m}{2})'\} = \{(0,\tfrac{m}{2})'',(0,\tfrac{m}{2})''\} = \tfrac{1-(-1)^{m/2}}{2m} + \tfrac{1}{2},$$

$$\{(0,\tfrac{m}{2})',(0,\tfrac{m}{2})''\} = \{(0,\tfrac{m}{2})'',(0,\tfrac{m}{2})'\} = \tfrac{1-(-1)^{m/2}}{2m} - \tfrac{1}{2}.$$

We have labeled this matrix $\mathbf{D}_m$ as a mnemonic for "dihedral Fourier transform."

Example 2.5.2. It is helpful to review some examples of this construction.

(i) If $m=3$ then $X=\{(0,1)\}$ and $\mathbf{D}_3$ is the 1×1 identity matrix.

(ii) If $m=4$ then $X=\{(0,1),(0,2)',(0,2)'',(1,2)\}$ and with respect to an appropriate ordering of indices we have $\mathbf{D}_4 = \mathbf{M}_{S_2}$. In a similar way, one finds that $\mathbf{D}_6 = \mathbf{M}_{S_3}$. These equalities are consistent with the fact that the Coxeter systems of types $I_2(4)$ and $I_2(6)$ are isomorphic to those of types BC_2 and G_2, whose nontrivial families of unipotent characters are parametrized by $\mathscr{M}(S_2)$ and $\mathscr{M}(S_3)$.

(iii) If $m=5$ then $X=\{(0,1),(0,2),(1,2),(1,3)\}$ and with respect to the order in which we just listed X, one computes

$$\mathbf{D}_5 = \frac{1}{\sqrt{5}}\begin{pmatrix} \widetilde{\lambda} & \lambda & 1 & 1 \\ \lambda & \widetilde{\lambda} & -1 & -1 \\ 1 & -1 & \lambda & -\widetilde{\lambda} \\ 1 & -1 & -\widetilde{\lambda} & \lambda \end{pmatrix}, \qquad \text{where } \lambda = \frac{\sqrt{5}+1}{2} \text{ and } \widetilde{\lambda} = \frac{\sqrt{5}-1}{2}.$$

This is precisely the matrix listed in [19, Eq. (7.3)] and in [67, §3.10].

2.5.3 Fourier transform matrices in types H_3 and H_4

The definition of the Fourier transform in the remaining non-crystallographic cases comes from the papers [19, 75, 76]. In type H_3, $\mathrm{Uch}(W)$ has four 1-element families given by $\{\phi_{1,0}\}$, $\{\phi_{1,15}\}$, $\{\phi_{5,2}\}$, and $\{\phi_{5,5}\}$. In type H_4, $\mathrm{Uch}(W)$ has six 1-element families given by $\{\phi_{1,0}\}$, $\{\phi_{1,60}\}$, $\{\phi_{25,4}\}$, $\{\phi_{25,16}\}$, $\{\phi_{36,5}\}$, and $\{\phi_{36,15}\}$. These families are all subsets of $\mathrm{Irr}(W)$ and we have labeled their (necessarily special) elements according to our conventions in Sections 2.4.2 and 2.4.3. To all such 1-element families, the associated Fourier transform matrix is the 1×1 identity matrix.

In type H_3 (respectively, H_4), $\mathrm{Uch}(W)$ has three (respectively, six) 4-element families. One such family is exceptional in type H_3 (see Definition 2.2.6), while two are exceptional in type H_4. The following observations concerning these families are derived from the parametrizations of $\mathrm{Uch}(W)$ provided by the `UnipotentCharacters` command in CHEVIE [41]. First, in both types the exceptional families always consist of four elements

$$\Phi_{(1,1)}, \quad \Phi_{(1,\mathrm{sgn})}, \quad \Phi_{(s,1)}, \quad \Phi_{(s,\mathrm{sgn})}$$

which can be indexed by the set $\mathscr{M}(S_2)$, such that

- $\Phi_{(1,1)}, \Phi_{(1,\mathrm{sgn})} \in \mathrm{Irr}(W)$, with $\Phi_{(1,1)}$ special and $\mathrm{Deg}\left(\Phi_{(1,1)}\right) = \mathrm{Deg}\left(\Phi_{(1,\mathrm{sgn})}\right)$.

- $\Phi_{(s,1)}, \Phi_{(s,\mathrm{sgn})} \notin \mathrm{Irr}(W)$, with $\mathrm{Deg}\left(\Phi_{(s,1)}\right) = \mathrm{Deg}\left(\Phi_{(s,\mathrm{sgn})}\right)$ and $\begin{cases} \mathrm{Eig}\left(\Phi_{(s,1)}\right) = i, \\ \mathrm{Eig}\left(\Phi_{(s,\mathrm{sgn})}\right) = -i. \end{cases}$

Remark. The computer algebra system CHEVIE stores a large amount of data associated to $\mathrm{Uch}(W)$, which can be accessed by combining the commands `Display` and `UnipotentCharacters`. The parametrization by $\mathscr{M}(S_2)$ just given, however, is not included in CHEVIE, though the listed properties uniquely determine which index $(x, \sigma) \in \mathscr{M}(S_2)$ goes to which $\Phi \in \mathcal{F}$ for any exceptional 4-element family $\mathcal{F}$.

The characters $\Phi_{(1,1)}, \Phi_{(1,\mathrm{sgn})} \in \mathrm{Irr}(W)$ may be respectively either $\phi_{4,3}$, $\phi_{4,4}$ in type H_3 or $\phi_{16,3}$, $\phi_{16,6}$ or $\phi_{16,18}$, $\phi_{16,21}$ in type H_4. There is no established notation for the formal elements $\Phi_{(s,1)}$, $\Phi_{(s,\mathrm{sgn})}$ in each family, however.

The non-exceptional 4-element families in types H_3 and H_4, on the other hand, always consist of four elements

$$\Phi_{(0,1)}, \quad \Phi_{(0,2)}, \quad \Phi_{(1,2)}, \quad \Phi_{(1,3)}$$

which can be indexed by the set X in Section 2.5.2 with $m = 5$, such that if $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{5}\right)$ is a fifth root of unity, then

- $\Phi_{(0,1)}, \Phi_{(0,2)} \in \mathrm{Irr}(W)$, with $\Phi_{(0,1)}$ special and $\mathrm{Deg}\left(\Phi_{(0,1)}\right) \neq \mathrm{Deg}\left(\Phi_{(0,2)}\right)$.

- $\Phi_{(1,2)}, \Phi_{(1,3)} \notin \mathrm{Irr}(W)$, with $\mathrm{Deg}\left(\Phi_{(1,2)}\right) = \mathrm{Deg}\left(\Phi_{(1,3)}\right)$ and $\begin{cases} \mathrm{Eig}\left(\Phi_{(1,2)}\right) = \xi^3, \\ \mathrm{Eig}\left(\Phi_{(1,3)}\right) = \xi^2. \end{cases}$

This parametrization by X, though uniquely determined for each non-exceptional family, is again not actually listed in CHEVIE. The characters $\Phi_{(0,1)}, \Phi_{(0,2)} \in \mathrm{Irr}(W)$ may be either $\phi_{3,1}$, $\phi_{3,3}$ or $\phi_{3,6}$, $\phi_{3,8}$ in type H_3 or any of the pairs $\phi_{4,1}$, $\phi_{4,7}$ or $\phi_{4,31}$, $\phi_{4,37}$ or $\phi_{9,2}$, $\phi_{9,6}$ or $\phi_{9,22}$, $\phi_{9,26}$ in type H_4. There is again no established notation for the formal elements $\Phi_{(1,2)}$, $\Phi_{(1,3)}$ in each family.

The Fourier transforms of these families are now defined thus: if $\mathcal{F} \subset \mathrm{Uch}(W)$ is a 4-element family in type H_3 or H_4, then its Fourier transform matrix is

$$\mathbf{M} = \begin{cases} \mathbf{M}_{S_2}, & \text{if } \mathcal{F} \text{ is exceptional (see Example 2.5.1)}, \\ \mathbf{D}_5, & \text{if } \mathcal{F} \text{ is not exceptional (see Example 2.5.2(iii))}. \end{cases} \tag{2.5.2}$$

Note that these assignments make sense because we have indicated how each 4-element family is indexed by the same set as the corresponding matrix.

Remark. In our definition of the Fourier transform matrix for the non-exceptional 4-element families, we follow the convention of [19, §7]. The Fourier transform of the exceptional 4-element families in types H_3 and H_4 seems less well-established in the literature. The matrix assigned to these families here is chosen to be identical to the Fourier transform matrix of the other exceptional families in types E_7 and E_8. We will say more about the "correctness" of this choice in the remarks following Theorem 2.5.6.

These conventions attach a Fourier transform matrix to all but one remaining family in type H_4. In this type, $\mathrm{Uch}(W)$ has a single family $\mathcal{F}$ of size 74; the intersection of this family with $\mathrm{Irr}(W)$ has size 16 and its unique special element is the character $\phi_{24,6} \in \mathrm{Irr}(W)$. The Fourier transform of this family is constructed by Malle as the matrix S in his paper [75]. To do calculations with this matrix, one needs to be able to access it in some computer format, and the algebra package CHEVIE fortunately provides this capability. In detail, one can obtain the 74×74 Fourier transform matrix of $\mathcal{F}$ by the following sequence of CHEVIE commands in GAP:

```
W := CoxeterGroup("H",4);
Uch := UnipotentCharacters(W);
F := Uch.families[13];
M := F.fourierMat * MatPerm(F.perm,74);
```

The odd-looking multiplication by `MatPerm(F.perm,74)` in the last line has to do with the indexing conventions of the `fourierMat` field in CHEVIE. The code given here produces a matrix `M` whose rows and columns have the same indices as `Uch`, and which is identical to the one in [75].

2.5.4 Fusion data and proof of main theorem

From the preceding three subsections, we know of a Fourier transform matrix attached to each family $\mathcal{F}$ in $\mathrm{Uch}(W)$ for each finite, irreducible Coxeter system (W, S). The *Fourier transform matrix* of $\mathrm{Uch}(W)$, we reiterate, is the direct sum of these matrices over all families

$\mathcal{F}$. By construction, this Uch(W)-indexed matrix satisfies property (P2) in the introduction; i.e., it is block diagonal with respect to the decomposition of Uch(W) into families. The following theorem explains precisely how this matrix also satisfies property (P1) in the introduction. This statement should be attributed to Lusztig and Malle via a combination of results appearing in the papers [63, 67, 75, 76]; see the proof of [42, Theorem 6.9] for a detailed bibliography.

Theorem 2.5.3. Let (W, S) be a finite, irreducible Coxeter system with associated Fourier transform matrix $\mathbf{M}$, and write j for the (permutation matrix of the) involution of Uch(W) defined in Proposition 2.2.5. Then the composition $\mathbf{M} \circ j$ transforms for the vector of fake degrees of Uch(W) to the vector of (generic) degrees.

Remark. This result is mentioned in several places, but sometimes imprecisely. For example, Carter asserts, without any mention of j, that the Fourier transform matrix of Uch(W) transforms the vector of fake degrees to the vector of actual degrees whenever (W, S) is a Weyl group [30, §13.6]. This statement, at least as we interpret it, is not strictly true in types E_7 and E_8. In particular, one can check that the Fourier transform matrix $\mathbf{M}$ that Carter attaches to the three exceptional families in these types (see Section 2.2.4) does not literally transform the vector of fake degrees to the corresponding vector of (generic) degrees as listed in [30, §13.8]. But $\mathbf{M}$ does transform a nontrivial permutation of the fake degrees to the (generic) degrees.

The preceding theorem gives one reason to consider the particular Fourier matrices assigned to Uch(W) as somehow "canonical," and our next result provides another. To state this, we first must recall Lusztig's definition of a *fusion datum* [67].

Definition 2.5.4. Let X be a finite set with a distinguished element x_0, and suppose

- Δ is the matrix of an involutory permutation of X with x_0 as a fixed point;
- $\mathbf{M}$ is a real symmetric matrix indexed by X;
- $\mathbf{F}$ is a diagonal matrix indexed by X whose diagonal entries are complex roots of unity.

The tuple $(X, x_0, \Delta, \mathbf{M}, \mathbf{F})$ is a *fusion datum* if the following axioms hold:

(Commutability). $\mathbf{M} = \Delta\mathbf{M}\Delta$ and $\mathbf{F}^{-1} = \Delta\mathbf{F}\Delta$.

(Positivity). $\mathbf{M}_{x,x_0} > 0$ for all $x \in X$ and $\mathbf{F}_{x_0,x_0} = 1$.

(Modularity). $\mathbf{M}^2 = (\mathbf{F}\Delta\mathbf{M})^3 = 1$.

(Integrality). $\sum_{w\in X} \frac{\mathbf{M}_{x,w}\mathbf{M}_{y,w}\mathbf{M}_{z,w}}{\mathbf{M}_{x_0,w}} \in \mathbb{N}$ for all $x, y, z \in X$.

Remarks. A few comments are helpful in unpacking this not altogether transparent construction.

(a) In [67], Lusztig actually presents a more general definition of a fusion datum which involves two involutions $\sharp$, $\flat$ of X in place of Δ. This definition reduces to ours when $\sharp = \flat = \Delta$.

(b) The modularity axiom is so-named as it requires that the matrices Δ, $\mathbf{F}$, $\mathbf{M}$ determine a unitary representation of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

(c) The integrality axiom allows one to define an algebra structure with nonnegative integer structure coefficients on the complex vector space generated by X. More specifically, the axiom asserts that one can give this vector space the structure of a *based ring* in the sense of [65]; see the discussion in [42, §7].

Before proceeding we note the following short lemma, which identifies a common type of fusion datum based on the set $\mathscr{M}(\Gamma)$ attached to a finite group Γ.

Lemma 2.5.5. Let Γ be a finite group equal to $(\mathbb{Z}/2\mathbb{Z})^n$ or S_n for some $n \geq 0$, and suppose

- $X = \mathscr{M}(\Gamma)$ as defined by (2.2.3) and $x_0 = (1, \mathbb{1}) \in X$;
- $\Delta : X \to X$ is the involution defined by $\Delta : (x, \sigma) \mapsto (x, \overline{\sigma})$;
- $\mathbf{M} = \mathbf{M}_\Gamma$ as in (2.5.1);
- $\mathbf{F} = \mathrm{diag}(t_x)_{x\in X}$ where $t_{(x,\sigma)} = \frac{\sigma(x)}{\sigma(1)}$ for $(x, \sigma) \in \mathscr{M}(\Gamma)$.

Then $(X, x_0, \Delta, \mathbf{M}, \mathbf{F})$ is a fusion datum in the sense of Definition 2.5.4.

Proof. This is equivalent to [67, Proposition 1.6], provided we show that $(x, \overline{\sigma}) \sim (x^{-1}, \sigma)$ for each $(x, \sigma) \in \mathscr{M}(\Gamma)$ when Γ is either $(\mathbb{Z}/2\mathbb{Z})^n$ or S_n. This is immediate in the first case, so assume $\Gamma = S_n$ is a symmetric group.

Let λ be a partition of n and let $x \in S_n$ be a permutation with cycle type λ. In the proof of Proposition 2.2.8, we noted that $C_{S_n}(x)$ is isomorphic to the direct product of wreath products $\prod_{r\geq 1} G(r, m_r)$, where m_r is the number of parts of λ with size r. A less explicit version of this statement goes as follows: the cycles of x generate an abelian group $A \cong \prod_{i\geq 1} \mathbb{Z}/\lambda_i\mathbb{Z}$, and $C_{S_n}(x)$ is isomorphic to a semidirect product of the form $(S_{m_1} \times S_{m_2} \times \dots) \ltimes A$. It is easy to see that one can choose a permutation $g \in S_n$ which commutes with the left factor of this semidirect product and which has $gxg^{-1} = x^{-1}$ and in fact $g^{-1}yg = y^{-1}$ for all $y \in A$. Noting these properties, it follows from the standard construction of the irreducible characters of a semidirect product with an abelian normal subgroup (see [59, Exercise XVIII.7]) that $\sigma^g = \overline{\sigma}$ for any $\sigma \in \mathrm{Irr}(C_{S_n}(x))$. We conclude that $(x, \overline{\sigma}) = (gxg^{-1}, \overline{\sigma}^g) = (x^{-1}, \sigma)$ for all $(x, \sigma) \in \mathscr{M}(S_n)$, as required. □

With these preliminaries in tow, we may now state the following noteworthy result due to Geck, Lusztig, and Malle [42, 67, 75], showing how each family in $\mathrm{Uch}(W)$, combined with its attached Fourier transform matrix, possesses naturally the structure of a fusion datum. This observation elaborates property (P4) of the Fourier transform matrix noted in the introduction, and is essentially a special case of [42, Theorem 6.9] (although as stated, that result drops the positivity axiom of a fusion datum and uses somewhat different terminology.)

Theorem 2.5.6. Let (W, S) be a finite, irreducible Coxeter system. Suppose

- $\mathcal{F}$ is a family in $\mathrm{Uch}(W)$ and $\Phi_0 \in \mathcal{F}$ is its unique special element;
- Δ is the restriction to $\mathcal{F}$ of the involution of $\mathrm{Uch}(W)$ given in Proposition 2.2.4;
- $\mathbf{M}$ is the Fourier transform matrix of $\mathcal{F}$, as defined in the Sections 2.5.1, 2.5.2, 2.5.3;
- $\mathbf{F} = \mathrm{diag}\,(\mathrm{Eig}(\Phi))_{\Phi \in \mathcal{F}}$ is the diagonal matrix of Frobenius eigenvalues of $\Phi \in \mathcal{F}$.

Then $(\mathcal{F}, \Phi_0, \Delta, \mathbf{M}, \mathbf{F})$ is a fusion datum.

Remark. For almost all families in $\mathrm{Uch}(W)$, our assignment of Fourier transform matrix follows exactly the convention established in the computer algebra system CHEVIE [41]. For the six exceptional families in types E_7, E_8, H_3, and H_4, however, our assigned matrix differs slightly from the one stored in CHEVIE—although, these matrices are the same after a permutation of rows and/or columns. In justification of our assignments, we can say the following: if one assumes $\mathrm{Uch}(W)$ given, then for each exceptional family, there is a *unique* matrix $\mathbf{M}$ satisfying both Theorem 2.5.3 and Theorem 2.5.6. So at least in this sense our choice of $\mathbf{M}$ is canonical.

We include a brief proof of the theorem for completeness.

Proof. In the case that (W, S) is crystallographic and $\mathcal{F}$ is a non-exceptional family of $\mathrm{Uch}(W)$, the theorem follows by combining Observation 2.2.7 with Lemma 2.5.5. If $\mathcal{F}$ is one of the six exceptional families in types E_7, E_8, H_3, or H_4, then since $\mathcal{F}$ is Δ-invariant, the theorem is equivalent to the claim that $(X, x_0, \Delta, \mathbf{M}, \mathbf{F})$ is a fusion datum for $X = \{1,2,3,4\}$ and $x_0 = 1$ and

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad \mathbf{F} = \mathrm{diag}\,(1, 1, i, -i).$$

The proof of this is a straightforward computer calculation. All families not covered by these cases occur when (W, S) is non-crystallographic, and are either singletons; non-exceptional 4-element families in type H_3 or H_4; the nontrivial family in type $I_2(m)$; or the 74-element family in type H_4. In the first case the theorem holds trivially; in the second and third case, the tuple $(\mathcal{F}, \Phi_0, \Delta, \mathbf{M}, \mathbf{F})$ coincides with dihedral fusion datum given in [67, §3]; and in the last case, $(\mathcal{F}, \Phi_0, \Delta, \mathbf{M}, \mathbf{F})$ is by construction the fusion datum which Malle describes in [75]. $\square$

We conclude this section finally by proving our main theorem from the introduction.

Proof of Theorem 2.1.2. If (W, S) is classical, then the function $\epsilon : \mathrm{Uch}(W) \to \mathbb{R}$ which is identically 1 satisfies (1)-(3) in Theorem 2.1.2 and also (2.1.2) by [55, Theorem 1]. The uniqueness of this function ϵ follows by Theorem 2.5.6 as a consequence of the positivity axiom of a fusion datum, since any function $\epsilon' : \mathrm{Uch}(W) \to \mathbb{R}$ satisfying (1) and (2) must have $\epsilon'(\Phi) \leq \epsilon(\Phi)$ for all $\Phi \in \mathrm{Uch}(W)$. The final statement in the theorem holds because the Frobenius eigenvalues of $\mathrm{Uch}(W)$ all real if (W,S) is classical by Observation 2.2.7 (noting that the group Γ in Observation 2.2.7 is $(\mathbb{Z}/2\mathbb{Z})^k$ in this case).

Assume (W,S) is of type $I_2(m)$, let $\mathbf{M}$ denote the Fourier transform matrix of $\mathrm{Uch}(W)$, and let $\epsilon : \mathrm{Uch}(W) \to \mathbb{R}$ be the function with $\epsilon(\Phi) = 0$ if $\Phi = \Phi_{(i,j)}$ for some $0 < i < j < i+j < m$ such that $j \neq \frac{m}{2}$ and with $\epsilon(\Phi) = 1$ for all other $\Phi \in \mathrm{Uch}(W)$. From the discussion in Section 2.2.5, this function satisfies conditions (1) and (2) in the theorem. To show that it also satisfies (3) and (2.1.2), it suffices by Proposition 2.4.3 to check that if $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$ and $0 < i < j < i+j < m$, then when m is odd, we have

$$\mathbf{M}\epsilon\left(\Phi_{(i,j)}\right) \stackrel{\text{def}}{=} \sum_{0<k<\frac{m}{2}} \{(i,j),(0,k)\} = \sum_{0<k<\frac{m}{2}} \tfrac{1}{m}\left(\xi^{-ik}+\xi^{ik}-\xi^{jk}-\xi^{-jk}\right) = \begin{cases} 1, & \text{if } i=0, \\ 0, & \text{otherwise,} \end{cases}$$

and when m is even, we have

$$\begin{aligned} \mathbf{M}\epsilon\left(\Phi_{(i,j)}\right) &\stackrel{\text{def}}{=} \sum_{0<k<\frac{m}{2}} \Big(\{(i,j),(0,k)\} + \{(i,j),(k,\tfrac{m}{2})\}\Big) + \{(i,j),(0,\tfrac{m}{2})'\} + \{(i,j),(0,\tfrac{m}{2})''\} \\ &= \sum_{0<k<\frac{m}{2}} \Big(\tfrac{1-(-1)^j}{m}\left(\xi^{-ik}+\xi^{ik}\right) - \tfrac{1-(-1)^i}{m}\left(\xi^{jk}+\xi^{-jk}\right)\Big) + 2\cdot\tfrac{(-1)^i-(-1)^j}{m} \\ &= \begin{cases} 2, & \text{if } i=0 \text{ and } j \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and also, for $x \in \{(0,\frac{m}{2})', (0,\frac{m}{2})''\}$,

$$\begin{aligned} \mathbf{M}\epsilon\left(\Phi_x\right) &\stackrel{\text{def}}{=} \sum_{0<k<\frac{m}{2}} \Big(\{x,(0,k)\} + \{x,(k,\tfrac{m}{2})\}\Big) + \{x,(0,\tfrac{m}{2})'\} + \{x,(0,\tfrac{m}{2})''\} \\ &= \tfrac{1-(-1)^{m/2}}{2} = \begin{cases} 1, & \text{if } m \equiv 2 \pmod 4, \\ 0, & \text{if } m \equiv 0 \pmod 4. \end{cases} \end{aligned}$$

(Note that we need to check that $\mathbf{M}\epsilon(\mathbb{1}) = \mathbf{M}\epsilon(\mathrm{sgn}) = 1$ as well, but this is obvious.) Proving each of these three identities is straightforward. Thus the function ϵ satisfies conditions (1)-(3) in the theorem, and we deduce that it is the only such function by the positivity axiom of a fusion datum, exactly as in the classical case.

Suppose finally that (W,S) has one of the remaining exceptional types. If (W,S) is not of type H_4, then one can compute directly using the tables in [31] and Proposition 2.4.1 that the function $\epsilon : \mathrm{Uch}(W) \to \mathbb{R}$ which is 1 or 0 according to whether $\mathrm{Eig}(\Phi)$ is real or non-real satisfies (1)-(3) as well as (2.1.2). That ϵ is the unique function satisfying (1)-(3) then follows as in the previous cases by Theorem 2.5.6 and the positivity axiom of a fusion datum. Similarly, if (W,S) has type H_4, then one can calculate using Proposition

2.4.2 that (1)-(3) and (2.1.2) hold for the function $\epsilon : \mathrm{Uch}(W) \to \mathbb{R}$ which is 0 on all $\Phi \in \mathrm{Uch}(W)$ with $\mathrm{Eig}(\Phi) \notin \mathbb{R}$, -1 on the two elements of $\mathrm{Uch}(W)$ whose degrees have the form $\frac{1}{60}x^6$ + higher powers of x, and 1 on all other elements of $\mathrm{Uch}(W)$. Here, the uniqueness of ϵ does not follow immediately from Theorem 2.5.6 because the values of ϵ are not all positive, but it can still be easily checked. In this case, the positivity axiom of a fusion datum implies that nearly all of the functions $\epsilon' : \mathrm{Uch}(W) \to \mathbb{R}$ satisfying (1) and (2) must have $\mathbf{M}\epsilon'(\Phi) < \mathbf{M}\epsilon(\Phi)$ for one of the special characters $\Phi \in \mathrm{Irr}(W)$. The remaining list of functions on $\mathrm{Uch}(W)$ which could possibly satisfy (1)-(3) is quite small, and a short calculation confirms that the given function ϵ is indeed the only one with these properties. □

2.6 Left cells and Kottwitz's conjecture

In this final section we investigate how the preceding material connects to the left cells of a Coxeter group, and prove some partial results related to Conjecture 2.1.5 in the introduction.

Let (W, S) be a finite Coxeter system with length function $\ell : W \to \mathbb{N}$. We denote the left descent set of an element $w \in W$ by $\mathrm{Des}_L(w) \overset{\text{def}}{=} \{s \in S : \ell(sw) < \ell(w)\}$, and for each pair of elements $y, w \in W$, we write

$$P_{y,w}(x) \in \mathbb{N}[x]$$

for the associated *Kazhdan-Lusztig polynomial*, as defined in [16, Chapter 5] or [30, §12.5] or [48, §7.9], among other places. Recall that $P_{w,w}(x) = 1$ and that $P_{y,w}(x) = 0$ unless $y \leq w$, where $\leq$ denotes the Bruhat partial order on W. From [53], we have the following sequence of definitions for $y, w \in W$:

- Write $y \prec w$ if $y < w$ and $P_{y,w}(x)$ has degree $\frac{1}{2}(\ell(w) - \ell(y) - 1)$.

- Write $y \leq_L w$ if there exist elements $x_0, x_1, \dots, x_k \in W$ such that $y = x_0$ and $w = x_k$ and for each $i \in [k]$, these two conditions hold: (1) either $x_{i-1} \prec x_i$ or $x_i \prec x_{i-1}$, and (2) the descent set $\mathrm{Des}_L(x_{i-1})$ is not contained in $\mathrm{Des}_L(x_i)$.

- Write $y \sim_L w$ if $y \leq_L w$ and $w \leq_L y$.

The *left cells* of W are the equivalence classes of the relation $\sim_L$. Let $V_\Gamma = \mathbb{Q}\text{-span}\{c_w : w \in \Gamma\}$ be a vector space with a basis indexed by a left cell Γ in W, and define a map $\varrho_\Gamma : S \to \mathrm{GL}(V_\Gamma)$ by linearly extending the formula

$$\varrho_\Gamma(s)c_w = \begin{cases} -c_w, & \text{if } s \in \mathrm{Des}_L(w), \\ c_w + \displaystyle\sum_{\substack{y \in \Gamma \\ s \in \mathrm{Des}_L(y)}} \mu(y,w) c_y, & \text{if } s \notin \mathrm{Des}_L(w), \end{cases}$$

for $s \in S$ and $w \in \Gamma$, where $\mu(y,w)$ denotes the coefficient of $x^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}(x)$ (which is zero unless $y \prec w$). This extends to a representation of W, called the *left cell representation* of Γ, whose character we denote by χ_Γ.

The following properties of left cells and left cell representations are useful to recall (see [16, Chapter 6]). First, the left cell representations decompose the regular representation of W and so the sum over left cells $\sum_\Gamma \chi_\Gamma$ is equal to $\sum_{\psi\in\mathrm{Irr}(W)} \psi(1)\psi$. Second, the singleton set $\{1\}$ is always a left cell, and its character is the trivial character $\mathbb{1}$. Finally, if w_0 denotes the longest element of W and Γ is a left cell, then $w_0\Gamma$ and Γw_0 are left cells and $\chi_{w_0\Gamma} = \chi_{\Gamma w_0} = \chi_\Gamma \cdot \mathrm{sgn}$.

We now state a few less well-known facts about the characters χ_Γ.

Theorem 2.6.1. If Γ, Γ' are left cells in a finite Coxeter group W, then $\langle \chi_\Gamma, \chi_{\Gamma'}\rangle = |\Gamma\cap\Gamma'^{-1}|$.

Proof. When W is a Weyl group, this is [63, Proposition 12.15]. Alvis notes how to extend Lusztig's proof to type H_4 [4, Proposition 3.4], and his argument remains valid in types H_3 and $I_2(m)$ (noting the explicit description of the left cells for these groups given in the following sections.) Alternatively, Geck has given a general proof of this theorem; see [39, Corollary 3.9]. □

Corollary 2.6.2. If Γ is a left cell in a finite Coxeter group W, then χ_Γ is multiplicity free if and only if every $w \in \Gamma\cap\Gamma^{-1}$ is an involution.

Proof. The character χ_Γ is multiplicity free if and only if the inequality $\langle\chi_\Gamma,\chi_\Gamma\rangle \geq \sum_{\psi\in\mathrm{Irr}(W)}\langle\chi_\Gamma,\psi\rangle$ is an equality. The left side is $|\Gamma\cap\Gamma^{-1}|$ by the previous theorem, while the right side is the number of involutions in Γ by [39, Theorem 1.1]. □

The next theorem shows that the Fourier transform matrix of $\mathrm{Uch}(W)$ indeed satisfies property (P3) in the introduction.

Theorem 2.6.3. Let (W,S) be a finite, irreducible Coxeter system with Fourier transform matrix $\mathbf{M}$, as in Section 2.5. Fix a left cell Γ of W, and let v be the vector indexed by $\mathrm{Uch}(W)$ whose entries are the irreducible multiplicities of χ_Γ, extended by zeros on $\mathrm{Uch}(W)\setminus\mathrm{Irr}(W)$. Then $\mathbf{M}v = v$.

We prove this here, even though to do so we must appeal to a few results not yet given.

Proof. When W is a Weyl group, this is [63, Theorem 12.2]. In types H_3 and H_4, the theorem follows from a computer calculation using the description of the characters χ_Γ given in Sections 2.6.1 and 2.6.2. In type $I_2(m)$ it follows by a short argument using the content of Sections 2.5.2 and 2.6.3 that the theorem is equivalent to the following identities: if $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$ and $0 < i < j < i+j < m$, then when m is odd

$$\sum_{0<k<\frac{m}{2}} \tfrac{1}{m}\left(\xi^{-ik}+\xi^{ik}-\xi^{jk}-\xi^{-jk}\right) = \begin{cases} 1, & \text{if } i=0, \\ 0, & \text{otherwise,}\end{cases}$$

and when m is even

$$\begin{cases} \frac{(-1)^i-(-1)^j}{m} + \sum_{0<k<\frac{m}{2}} \frac{1}{m}\left(\xi^{-ik}+\xi^{ik}-\xi^{jk}-\xi^{-jk}\right) = \begin{cases} 1, & \text{if } i=0, \\ 0, & \text{otherwise,} \end{cases} \\ \frac{1-(-1)^{m/2}}{2m} + \sum_{0<k<\frac{m}{2}} \frac{1-(-1)^k}{m} = \frac{1}{2}. \end{cases}$$

Proving each of these is straightforward arithmetic. □

The last object of this section is to prove the following theorem. We will accomplish this in a case-by-case fashion, by examining the left cell representations in each non-crystallographic type.

Theorem 2.6.4. Conjecture 2.1.5 holds if (W, S) has type H_3, H_4, or $I_2(m)$.

From this and the results of [31, 55], it follows that Kottwitz's conjecture holds for all finite irreducible Coxeter groups except possibly those of type BC_n, D_n, E_7, and E_8. Recent work of Bonnafé and Geck [17, 40] has established the conjecture in all of these remaining cases except E_8.

2.6.1 Left cells of the Coxeter group of type H_3

Let (W, S) be the Coxeter system of type H_3. The group W decomposes into 22 distinct left cells which we may describe as follows. (This description does not seem to appear anywhere in the literature, though one can easily compute the left cells in type H_3 directly, using for example Fokko du Cloux's program `Coxeter` [34], which is what was used to derive the following statements.) Label the generators $S = \{a, b, c\}$ as in Section 2.4.2 and let $w_0 = (abc)^5$ denote the longest element of W. Following Alvis [4], we define

$$R_J = \{w \in W : \mathrm{Des}_L(w) = J\}, \quad \text{for each subset } J \subset S, \tag{2.6.1}$$

and let $X^* = \{w_0 w : w \in X\}$ for any subset $X \subset W$. We now define 12 subsets $I_i, J_i, K_i \subset W$ as follows:

$$\begin{array}{lll} I_1 = R_{\{b,c\}} \cap R_{\{a\}}aba, & J_1 = R_{\{a,c\}} \cap R_{\{c\}}cbab, & K_1 = R_{\{a\}} - (I_4 \cup J_3), \\ I_2 = I_1 a, & J_2 = J_1 b, & K_2 = R_{\{b\}} - (I_3 \cup J_1^* \cup J_2 \cup J_4), \\ I_3 = I_2 b = I_2^*, & J_3 = J_2 a, & K_3 = R_{\{c\}} - J_5. \\ I_4 = I_3 a = I_1^*, & J_4 = J_3 b, & \\ & J_5 = J_4 c, & \end{array}$$

The reader should compute that $|I_i| = 8$ and $|J_i| = 5$ and $|K_i| = 6$. In addition, let $L = \{1\}$ so that $L^* = \{w_0\}$. We now have this computational proposition (which is closely related to the calculations summarized in [61, §5]):

Proposition 2.6.5. The left cells of the Coxeter system (W, S) of type H_3 are the 22 disjoint subsets I_i, J_i, J_i^*, K_i, K_i^*, L, L^*, and the characters of the associated cell representations are respectively

$$(\phi_{4,3} + \phi_{4,4}); \quad \phi_{5,2}; \quad \phi_{5,5}; \quad (\phi_{3,1} + \phi_{3,3}); \quad (\phi_{3,6} + \phi_{3,8}); \quad \phi_{1,0}; \quad \phi_{1,15}.$$

Proof. `Coxeter` [34] outputs a description of the left cells and the associated cell representations in type H_3 in a few seconds. We have checked that this information matches what is asserted in the proposition using MAGMA [18]. □

Table A.13 lists the sizes of the intersections of the left cells in W with the group's four conjugacy classes of involutions, which we recall from Section 2.4.2 are represented by the elements 1, a, ac, and $(abc)^5$. Each row of the table corresponds to a left cell. Each of the last four columns of the table corresponds to a conjugacy class of involutions, and lists the sizes of the intersections of the conjugacy class with the left cells. Comparing Proposition 2.4.1 with Table A.13 yields a proof of Conjecture 2.1.5 in type H_3 by inspection.

2.6.2 Left cells of the Coxeter group of type H_4

The Coxeter group of type H_4 decomposes into a disjoint union of 206 left cells. Alvis classifies these in [4], and assigns each of them one of the labels A_i, B_i, B_i^*, C_i, C_i^*, D_i, D_i^*, E_i, E_i^*, F_i, F_i^*, G_1, G_1^*. We refer to [4] for the precise definition of these sets, noting the following correction:

Remark. On page 162 of the published version of Alvis's paper [4], the left cell A_{12} is defined by the equation

$$A_{12} = A_{10}d \qquad \text{(INCORRECT)}$$

which should instead be

$$A_{12} = A_{11}d. \qquad \text{(CORRECT)}$$

Apart from this quite minor (but inevitably confusing) detail, everything else in Alvis's paper seems to be completely accurate.

Table A.14 lists the sizes of the intersections of the left cells in W with the group's five conjugacy classes of involutions, which are represented as in Section 2.4.3 by the elements 1, a, ac, $(abc)^5$, and $(abcd)^{15}$. We have structured this table exactly like Table A.13, and comparing it to Proposition 2.4.2 similarly yields an immediate proof of Conjecture 2.1.5 in type H_4.

2.6.3 Left cells of the Coxeter groups of type $I_2(m)$

Let (W, S) be the Coxeter system of type $I_2(m)$ for $m \geq 3$, with $S = \{r, s\}$ and $w_0 \in W$ defined as in Section 2.4.4. The group W then decomposes into a disjoint union of four left cells, given by the singleton sets $X = \{1\}$ and $X^* = \{w_0\}$ together with

$$Y = \{s, rs, srs, \ldots, \underbrace{(\cdots srs)}_{m-1 \text{ factors}}\} = R_{\{s\}} \qquad \text{and} \qquad Y^* = \{r, sr, rsr, \ldots, \underbrace{(\cdots rsr)}_{m-1 \text{ factors}}\} = R_{\{r\}},$$

with R_J defined by (2.6.1). The following description of the left cell representations of W can be found, for example, in [38, Section 7.1].

Proposition 2.6.6 (Geck [38]). The left cells of the Coxeter system (W, S) of type $I_2(m)$ are the 4 disjoint subsets X, X^*, Y, Y^*, and the characters of the associated cell representations are respectively

$$\begin{cases} \phi_{1,0}, \quad \phi_{1,m}, \quad \sum_{0<k<\frac{m}{2}} \phi_{2,k}, \quad \sum_{0<k<\frac{m}{2}} \phi_{2,k}, & \text{if } m \text{ is odd,} \\ \phi_{1,0}, \quad \phi_{1,m}, \quad \phi'_{1,m/2} + \sum_{0<k<\frac{m}{2}} \phi_{2,k}, \quad \phi''_{1,m/2} + \sum_{0<k<\frac{m}{2}} \phi_{2,k}, & \text{if } m \text{ is even.} \end{cases}$$

The description of the left cells is also noted in [48, §7.15]. It is easy to compute the characters of the left cell representations directly, and we have included a short argument for completeness.

Proof. We have $\chi_X = \phi_{1,0} = \mathbb{1}$ and $\chi_{X^*} = \phi_{1,m} = \mathrm{sgn}$ automatically, and the remaining left cell characters are multiplicity free by Corollary 2.6.2. Since the left cell representations decompose the regular representation and since $\chi_{Y^*} = \chi_Y \cdot \mathrm{sgn}$, we must have $\chi_Y = \chi_{Y^*} = \sum_k \phi_{2,k}$ if m is odd, and if m is even, we must have either $\chi_Y = \phi'_{1,m/2} + \sum_k \phi_{2,k}$ and $\chi_{Y^*} = \phi''_{1,m/2} + \sum_k \phi_{2,k}$ or the reverse assignments. One resolves the ambiguity when m is even by computing the values of the characters at r or s. □

Table A.15 finally lists the sizes of the intersections of the four left cells in W with the group's two or four conjugacy classes of involutions (see Section 2.4.4). We have structured this table exactly like Tables A.13 and A.14, and comparing it to Table A.12 completes the proof of Theorem 2.6.4.

Chapter 3

Positivity conjectures for Kazhdan-Lusztig theory on twisted involutions

This final chapter combines and expands some of the results, background, and proofs in the preprints [81, 82].

3.1 Introduction

3.1.1 Overview

Let (W, S) be any Coxeter system, and write $\mathcal{H}_{q^2}$ for the associated *generic Hecke algebra with parameter* q^2: this is the usual Hecke algebra (namely, a certain $\mathbb{Z}[q^{\pm 1/2}]$-algebra with a basis $(T_w)_{w \in W}$ indexed by W), but with q replaced by q^2 in its defining relations; skip to Section 3.1.4 for the precise definition. Next, fix an automorphism $* : W \to W$ with order one or two which preserves the set of simple generators S. Write $\mathbf{I}_*$ for the corresponding set of elements $w \in W$ with $w^{-1} = w^*$, which one refers to as *twisted involutions*, and let $\mathcal{M}_{q^2}$ be the free $\mathbb{Z}[q^{\pm 1/2}]$-module which this set generates.

Lusztig [70] has shown that $\mathcal{M}_{q^2}$ has an $\mathcal{H}_{q^2}$-module structure which serves as a natural and interesting analogue of the regular representation of $\mathcal{H}_{q^2}$ on itself. (Section 3.1.4 contains the details of this construction.) The regular representation of $\mathcal{H}_{q^2}$ possesses a distinguished *Kazhdan-Lusztig basis* $(C_w)_{w \in W}$, whose transition matrix from the standard basis $(T_w)_{w \in W}$ defines the much-studied family of *Kazhdan-Lusztig polynomials* $(P_{y,w})_{y,w \in W} \subset \mathbb{Z}[q]$. Lusztig's work [70] indicates that one may repeat much of this theory for the module $\mathcal{M}_{q^2}$: it too has a "Kazhdan-Lusztig basis" whose transition matrix from the standard basis defines a family of "twisted Kazhdan-Lusztig polynomials" $(P^\sigma_{y,w})_{y,w \in \mathbf{I}_*} \subset \mathbb{Z}[q]$.

Many remarkable properties of the Kazhdan-Lusztig basis of $\mathcal{H}_{q^2}$ appear to have "twisted" analogs for the module $\mathcal{M}_{q^2}$. For example, one of the most famous aspects of the original Kazhdan-Lusztig polynomials $(P_{y,w})_{y,w \in W}$ is that their coefficients are always nonnegative.

(This statement, while known in many cases from the work of a number of people, has only recently been proved for all Coxeter systems by Elias and Williamson [36].) The twisted Kazhdan-Lusztig polynomials $(P^\sigma_{y,w})_{y,w\in \mathbf{I}_*}$ can have negative coefficients. However, Lusztig and Vogan [72] have shown by geometric arguments that the modified polynomials $\frac{1}{2}(P_{y,w} \pm P^\sigma_{y,w})$ for $y, w \in \mathbf{I}_*$ have nonnegative coefficients whenever W is crystallographic. In fact, for any choice of (W, S) and $*$, the polynomials $\frac{1}{2}(P_{y,w} \pm P^\sigma_{y,w})$ belong to $\mathbb{Z}[q]$, and Lusztig [70] has conjectured that their coefficients are always nonnegative.

Section 3.1.5 presents two other positivity conjectures for the "Kazhdan-Lusztig basis" of the twisted involution $\mathcal{H}_{q^2}$-module $\mathcal{M}_{q^2}$. These serves as analogues of longstanding conjectures related to the ordinary Kazhdan-Lusztig polynomials $P_{y,w}$. After stating these "twisted" conjectures, we devote the rest of this chapter to proving them in two significant special cases: for Coxeter systems (W, S) which are *universal* (i.e., such that $st \in W$ has infinite order for all distinct $s, t \in S$), and for finite Coxeter systems. In the universal case it is possible to derive explicit formulas for the polynomials $P_{y,w}$ and $P^\sigma_{y,w}$ to establish the conjectures of interest. To treat the finite case, we combine Lusztig and Vogan's results for Weyl groups [72] with computer calculations. A more detailed summary of the main results in this chapter is given in Section 3.1.6, after some necessary preliminaries.

3.1.2 Setup

Throughout we write $\mathbb{Z}$ for the integers and $\mathbb{N} = \{0, 1, 2, \dots\}$ for the nonnegative integers. We also adopt the following conventions:

- Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$.
- Let $\leq$ denote the Bruhat order on W.
- Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate v. The ring $\mathcal{A}$ will now occupy the role which $\mathbb{Z}[q^{\pm 1/2}]$ played in the previous section
- Let $q = v^2$. In the sequel, we will refer to v in place of the parameter $q^{1/2}$ used in Section 3.1.1.

Thus W is a group and $S \subset W$ is a nonempty finite set of elements of order two which generate W. Recall that the *rank* of (W, S) is given by the cardinality $|S|$, and the length $\ell(w)$ is the minimum integer k such that $w = s_1 s_2 \cdots s_k$ for some choice of generators $s_i \in S$. The Bruhat order is defined by the condition that $y \leq w$ for $y, w \in W$ if and only if whenever $w = s_1 s_2 \cdots s_{\ell(w)}$ for some $s_i \in S$, there are indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq \ell(w)$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_m}$. In particular, if $y < w$ then $\ell(y) < \ell(w)$.

For further background on Coxeter systems and the Bruhat order, see for example [16, 48, 69].

3.1.3 Kazhdan-Lusztig theory

Here we briefly recall the definition of the Kazhdan-Lusztig polynomials attached to (W,S). Let $\mathcal{H}_q$ denote the free $\mathcal{A}$-module with basis $\{t_w : w \in W\}$. This module has a unique $\mathcal{A}$-algebra structure with respect to which the multiplication rule

$$t_s t_w = \begin{cases} t_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ q t_{sw} + (q-1)t_w & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

holds for each $s \in S$ and $w \in W$. The element $t_w \in \mathcal{H}_q$ is more often denoted in the literature by the symbol T_w, but here we reserve the latter notation for the Hecke algebra $\mathcal{H}_{q^2}$, to be introduced in the next section.

We refer to the algebra $\mathcal{H}_q$ as the *Hecke algebra of* (W,S) *with parameter* q. A number of good references exist for this much-studied object; see for example [16, 48, 53, 69]. The Hecke algebra possesses a unique ring involution $\bar{\ } : \mathcal{H}_q \to \mathcal{H}_q$ with $\overline{v^n} = v^{-n}$ and $\overline{t_w} = (t_{w^{-1}})^{-1}$ for all $n \in \mathbb{Z}$ and $w \in W$, referred to as the *bar operator*, and this gives rise to the following theorem-definition from Kazhdan and Lusztig's seminal paper [53].

Theorem-Definition 3.1.1 (Kazhdan and Lusztig [53]). For each $w \in W$ there is a unique family of polynomials $(P_{y,w})_{y\in W} \subset \mathbb{Z}[q]$ with the following three properties:

(a) The element $c_w \overset{\text{def}}{=} v^{-\ell(w)} \cdot \sum_{y \in W} P_{y,w} \cdot t_y$ in $\mathcal{H}_q$ has $\overline{c_w} = c_w$.

(b) $P_{y,w} = \delta_{y,w}$ if $y \not< w$ in the Bruhat order.

(c) $P_{y,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ as a polynomial in q whenever $y < w$.

Remark. Here and elsewhere, the Kronecker delta $\delta_{y,w}$ has the usual meaning of $\delta_{y,w} = 1$ if $y = w$ and $\delta_{y,w} = 0$ otherwise.

The polynomials $(P_{y,w})_{y,w\in W}$ are the *Kazhdan-Lusztig polynomials* of the Coxeter system (W,S). Property (b) implies that the elements $(c_w)_{w\in W}$ form an $\mathcal{A}$-basis for $\mathcal{H}_q$, which one calls the *Kazhdan-Lusztig basis.* For more information on the Kazhdan-Lusztig polynomials and methods of computing them, see, for example, [48, Chapter 7] or [16, Chapter 5].

3.1.4 Twisted Kazhdan-Lusztig theory

We now present Lusztig's definition of the module $\mathcal{M}_{q^2}$ and the polynomials $P^\sigma_{y,w}$ mentioned at the start of this introduction. To begin, we let $\mathcal{H}_{q^2}$ denote the *Hecke algebra of* (W,S) *with parameter* q^2: this is the free $\mathcal{A}$-module with basis $\{T_w : w \in W\}$, equipped with the unique $\mathcal{A}$-algebra structure with respect to which the multiplication rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ q^2 T_{sw} + (q^2-1)T_w & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

holds for each $s \in S$ and $w \in W$. Like $\mathcal{H}_q$, this algebra possesses a unique ring involution $\bar{\ } : \mathcal{H}_{q^2} \to \mathcal{H}_{q^2}$ with $\overline{v^n} = v^{-n}$ and $\overline{T_w} = (T_{w^{-1}})^{-1}$ for all $n \in \mathbb{Z}$ and $w \in W$. This bar operator fixes each of the elements

$$C_w \overset{\text{def}}{=} q^{-\ell(w)} \cdot \sum_{y \in W} P_{y,w}(q^2) \cdot T_y \qquad \text{for } w \in W.$$

The elements $(C_w)_{w \in W}$ form an $\mathcal{A}$-basis of $\mathcal{H}_{q^2}$ which one refers to as the *Kazhdan-Lusztig basis*. The use of the capitalized symbols T_w, C_w is intended to distinguished elements of $\mathcal{H}_{q^2}$ from the basis elements t_w, c_w of the usual Hecke algebra $\mathcal{H}_q$.

The following Theorem-Definition of Lusztig [70] defines $\mathcal{M}_{q^2}$ explicitly as a certain module of the algebra $\mathcal{H}_{q^2}$. This statement requires a few additional ingredients:

- Fix an automorphism $w \mapsto w^*$ of W with order ≤ 2 such that $s^* \in S$ for each $s \in S$.
- Set $\mathbf{I}_* = \{w \in W : w^* = w^{-1}\}$. One calls elements of this set *twisted involutions.*
- Given $s \in S$ and $w \in \mathbf{I}_*$, let $s \ltimes w$ denote the unique element in the intersection of $\{sw, sws^*\}$ and $\mathbf{I}_* \setminus \{w\}$. Note that while $s \ltimes (s \ltimes w) = w$, the operation $\ltimes : S \times \mathbf{I}_* \to \mathbf{I}_*$ generally does not extend to a group action of W on $\mathbf{I}_*$.

We now have Lusztig's result. This statement first appeared in Lusztig and Vogan's paper [72] in the special case that W is a Weyl group or affine Weyl group and $*$ is trivial.

Theorem-Definition 3.1.2 (Lusztig and Vogan [72]; Lusztig [70]). Let $\mathcal{M}_{q^2}$ be the free $\mathcal{A}$-module with basis

$$\{a_w : w \in \mathbf{I}_*\}.$$

(a) $\mathcal{M}_{q^2}$ has a unique $\mathcal{H}_{q^2}$-module structure with respect to which the following multiplication rule holds for each $s \in S$ and $w \in \mathbf{I}_*$:

$$T_s a_w = \begin{cases} a_{s\ltimes w} + a_w & \text{if } s \ltimes w = sws^* > w \\ (q+1)a_{s\ltimes w} + qa_w & \text{if } s \ltimes w = sw > w \\ (q^2-q)a_{s\ltimes w} + (q^2-q-1)a_w & \text{if } s \ltimes w = sw < w \\ q^2 a_{s\ltimes w} + (q^2-1)a_w & \text{if } s \ltimes w = sws^* < w. \end{cases} \tag{3.1.1}$$

(b) There is a unique $\mathbb{Z}$-linear involution $\bar{\ } : \mathcal{M}_{q^2} \to \mathcal{M}_{q^2}$ such that $\overline{a_1} = a_1$ and $\overline{h \cdot m} = \overline{h} \cdot \overline{m}$ for all $h \in \mathcal{H}_{q^2}$ and $m \in \mathcal{M}_{q^2}$. This bar operator acts on the standard basis of $\mathcal{M}_{q^2}$ by the formula $\overline{a_w} = (-1)^{\ell(w)} \cdot (T_{w^{-1}})^{-1} \cdot a_{w^{-1}}$ for $w \in \mathbf{I}_*$.

The bar operator just introduced on $\mathcal{M}_{q^2}$ gives rise, in turn, to the following analogue of Theorem-Definition 3.1.1. Like the previous result, this was first shown by Lusztig and Vogan [72] in the crystallographic case (with $*$ trivial). Lusztig [70] subsequently extended the statement to all Coxeter systems.

Theorem-Definition 3.1.3 (Lusztig and Vogan [72]; Lusztig [70]). For each $w \in \mathbf{I}_*$ there is a unique family of polynomials $\left(P^\sigma_{y,w}\right)_{y\in\mathbf{I}_*} \subset \mathbb{Z}[q]$ with the following three properties:

(a) The element $A_w \overset{\text{def}}{=} v^{-\ell(w)} \cdot \sum_{y\in\mathbf{I}_*} P^\sigma_{y,w} \cdot a_y$ in $\mathcal{M}_{q^2}$ has $\overline{A_w} = A_w$.

(b) $P^\sigma_{y,w} = \delta_{y,w}$ if $y \not< w$ in the Bruhat order.

(c) $P^\sigma_{y,w}$ has degree at most $\frac{1}{2}\left(\ell(w) - \ell(y) - 1\right)$ as a polynomial in q whenever $y < w$.

Note from (b) that the elements $(A_w)_{w\in\mathbf{I}_*}$ form an $\mathcal{A}$-basis for the module $\mathcal{M}_{q^2}$. We sometimes refer to this as the "twisted Kazhdan-Lusztig basis." Likewise, we call the polynomials $P^\sigma_{y,w}$ the *twisted Kazhdan-Lusztig polynomials* of the triple $(W, S, *)$. We will discuss some general properties of these polynomials (and also address how one computes them) in Sections 3.2.2 and 3.4.3.

Before continuing to state the conjectures concerning $P^\sigma_{y,w}$ which are our main subject, let us mention a few reasons why one might care about these polynomials or the module $\mathcal{M}_{q^2}$. First, as detailed in [72], when W is a Weyl group or affine Weyl group, the module $\mathcal{M}_{q^2}$ arises from geometric considerations and in that context the polynomials $P^\sigma_{y,w}$ are expected to have importance in the theory of unitary representations of complex reductive groups.

While for more general Coxeter groups we lack such an interpretation for $\mathcal{M}_{q^2}$, there is nevertheless always a sense in which we can view the left regular representation of the Hecke algebra of a Coxeter system as a special case of (a submodule of) the module $\mathcal{M}_{q^2}$. Consequently, one can realize the ordinary Kazhdan-Lusztig polynomials of one Coxeter system as the twisted polynomials $P^\sigma_{y,w}$ corresponding to another Coxeter system with a particular choice of $*$. This is explained in Section 3.2.3.

3.1.5 Positivity conjectures

Many results in the theory of Hecke algebras depend on positivity properties of the Kazhdan-Lusztig polynomials $P_{y,w}$. In particular, we recall the following much studied conjectures:

Conjecture A. The polynomials $P_{y,w}$ have nonnegative integer coefficients.

Conjecture B. The polynomials $P_{y,w}$ are decreasing for fixed w, in the sense that the difference $P_{y,w} - P_{z,w}$ has nonnegative integer coefficients whenever $y \leq z$.

Denote the structure coefficients of $\mathcal{H}_q$ in the Kazhdan-Lusztig basis by $(h_{x,y;z})_{x,y,z\in W}$; i.e., these are the Laurent polynomials in $\mathcal{A}$ satisfying $c_x c_y = \sum_{z\in W} h_{x,y;z} c_z$ for $x, y, z \in W$.

Conjecture C. The Laurent polynomials $h_{x,y;z}$ have nonnegative coefficients.

These conjectures have been proved in the case when (W, S) is crystallographic (i.e., when W a Weyl group or affine Weyl group), finite, or universal through the work of a number of people [33, 35, 50, 54, 64, 95]. Elias and Williamson's recent proof of Soergal's conjecture

[36], finally, establishes Conjectures A and C for any Coxeter system. In this generality Conjecture B remains open.

The central topic of this work concerns "twisted" versions of the preceding conjectures. While the parallels between Theorem-Definitions 3.1.1 and 3.1.3 suggest obvious analogues of Conjectures A, B, and C in the twisted case, these statements turn out not to be the right ones. Notably, the polynomials $P^\sigma_{y,w}$ may have negative coefficients. To state the "correct" conjectures, define $P^+_{y,w}, P^-_{y,w} \in \mathbb{Q}[q]$ by

$$P^\pm_{y,w} = \tfrac{1}{2}\left(P_{y,w} \pm P^\sigma_{y,w}\right) \qquad \text{for each } y, w \in \mathbf{I}_*. \tag{3.1.2}$$

Lusztig proves that these polynomials actually have integer coefficients [70, Theorem 9.10] and conjectures the following:

Conjecture A′. The polynomials $P^+_{y,w}$ and $P^-_{y,w}$ have nonnegative integer coefficients.

This statement is a refinement of Conjecture A since $P^+_{y,w} + P^-_{y,w} = P_{y,w}$ for $y, w \in \mathbf{I}_*$. We introduce the following stronger conjecture, which is likewise a refinement of Conjecture B.

Conjecture B′. The polynomials $P^\pm_{y,w}$ are decreasing for fixed w, in the sense that the differences $P^+_{y,w} - P^+_{z,w}$ and $P^-_{y,w} - P^-_{z,w}$ have nonnegative integer coefficients whenever $y \leq z$.

Finally, to provide an analog of Conjecture C, for each $x \in W$ and $y \in \mathbf{I}_*$ define $\left(\widetilde{h}_{x,y;z}\right)_{z\in W}$ and $\left(h^\sigma_{x,y;z}\right)_{z\in\mathbf{I}_*}$ as the Laurent polynomials in $\mathcal{A}$ satisfying

$$c_x c_y c_{x^{*-1}} = \sum_{z\in W} \widetilde{h}_{x,y;z} c_z \qquad \text{and} \qquad C_x A_y = \sum_{z\in\mathbf{I}_*} h^\sigma_{x,y;z} A_z. \tag{3.1.3}$$

Note that $c_x, c_y, c_z \in \mathcal{H}_q$ while $C_x \in \mathcal{H}_{q^2}$ and $A_y \in \mathcal{M}_{q^2}$. Now, define $h^+_{x,y;z}, h^-_{x,y;z} \in \mathbb{Q}[v, v^{-1}]$ by

$$h^\pm_{x,y;z} = \tfrac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^\sigma_{x,y;z}\right), \qquad \text{for each } x \in W \text{ and } y, z \in \mathbf{I}_*. \tag{3.1.4}$$

One can show from results of Lusztig [70] that these Laurent polynomials likewise have integer coefficients (see Proposition 3.2.17 below), which leads to this conjecture.

Conjecture C′. The Laurent polynomials $h^+_{x,y;z}$ and $h^-_{x,y;z}$ have nonnegative integer coefficients.

Lusztig and Vogan's work [72] establishes Conjecture A′ when W is a Weyl group or affine Weyl group. In these cases, [72, Section 5] also mentions without proof that Conjecture C′ holds (when $*$ is trivial). Conjecture B′ is still to be open even in the crystallographic case.

3.1.6 Outline

Following Dyer [35], we say that a Coxeter system (W, S) is *universal* if the product $st \in W$ has infinite order for any distinct generators $s, t \in S$. Dyer's paper [35] derives formulas for the polynomials $P_{y,w}$ and for the decomposition of the products $c_x c_y \in \mathcal{H}_q$ in terms of the Kazhdan-Lusztig basis, thus establishing Conjectures A, B, and C in the universal case. (Dyer's results are formulated in somewhat different language than these conjectures; cf. Theorems 3.3.5 and 3.3.15 below.) Section 3.3 proceeds as something of a sequel to Dyer's work, as follows:

- In Sections 3.3.1, 3.3.2, and 3.3.3 we derive recurrence relations with coefficients in $\mathbb{N}[q]$ for the polynomials $P^\sigma_{y,z;w} \overset{\text{def}}{=} P^\sigma_{y,w} - P^\sigma_{z,w}$ and $P_{y,z;w} \overset{\text{def}}{=} P_{y,w} - P_{z,w}$ (with $y \leq z$).

- These recurrences show that in the universal case $P^\sigma_{y,w}$ and $P^\pm_{y,w}$ belong to $\mathbb{N}[q]$ and are decreasing with respect to the index $y \in \mathbf{I}_*$ and the Bruhat order; see Theorems 3.3.12 and 3.3.13 below. Thus Conjectures A′ and B′ hold for universal Coxeter systems.

- In Section 3.3.4 we describe the decomposition of the product $C_x A_y$ in terms of the distinguished basis $(A_z)_{z\in\mathbf{I}_*}$ of $\mathcal{M}_{q^2}$; see Theorem 3.3.18. This shows that $h^\sigma_{x,y,;z} \in \mathbb{N}[v, v^{-1}]$ in the universal case; see Corollary 3.3.19.

- Combining these results with Dyer's work finally affords a proof of Conjecture C′ for universal Coxeter systems; see Theorem 3.3.22.

In Section 3.4 we turn our attention to finite Coxeter systems. Most of our effort on this topic is spent in checking our conjectures for dihedral groups, where one can derive explicit formulas, and for the exceptional non-crystallographic Coxeter systems of type H_3 and H_4, for which one can resort to computer methods.

- In Section 3.4.1 we prove that our conjectures hold a finite Coxeter system if they hold for all of its irreducible factors.

- In Section 3.4.2 we work out formulas for the polynomials of interest explicitly in the case that W is a dihedral group, in order to establish Conjectures A′ and B′ (but, unfortunately, not yet C′) for rank two Coxeter systems.

- Section 3.4.3 includes pseudo-code for algorithms to compute the polynomials $(P_{y,w})_{y,w\in W}$ and $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*}$ and the structure constants $\left(\widetilde{h}_{x,y;z}\right)_{x\in W;y,z\in\mathbf{I}_*}$ and $\left(h^\sigma_{x,y;z}\right)_{x\in W;y,z\in\mathbf{I}_*}$.

- In Section 3.4.4 we discuss our implementation of the algorithms in Section 3.4.3 as extensions to `Coxeter` [34], and then summarize the outcome of our computations for Coxeter systems of rank ≤ 5.

Before carrying all this out, we provide in Section 3.2 a few preliminaries concerning the Bruhat order on $\mathbf{I}_*$, the polynomials $P_{y,w}$ and $P^\sigma_{y,w}$, and the associated bases of $\mathcal{H}_q$ and $\mathcal{M}_{q^2}$.

3.2 Preliminaries

Here, we preserve all conventions from the introduction. Thus, (W, S) is an arbitrary Coxeter system (not necessarily universal) with an S-preserving involution $* \in \mathrm{Aut}(W)$, and attached to these choices are the following structures:

- $\mathcal{H}_{q^2} = \mathcal{A}\text{-span}\{T_w : w \in W\}$ is the Hecke algebra of (W, S) with parameter q^2.
- $\mathbf{I}_* = \{w \in W : w^{-1} = w^*\}$ is the corresponding set of twisted involutions.
- $\mathcal{M}_{q^2} = \mathcal{A}\text{-span}\{a_w : w \in \mathbf{I}_*\}$ is the $\mathcal{H}_{q^2}$-module generated by $\mathbf{I}_*$.

Recall also the definitions of the special bases $(C_w)_{w\in W} \subset \mathcal{H}_{q^2}$ and $(A_w)_{w\in\mathbf{I}_*} \subset \mathcal{M}_{q^2}$, and the polynomials $(P_{y,w})_{y,w\in W}$ and $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*}$ in $\mathbb{Z}[q]$.

3.2.1 Bruhat order on twisted involutions

The set of twisted involutions $\mathbf{I}_*$ is partially ordered by the Bruhat order $\leq$ on W, and this ordering controls many important features of the basis $(A_w)_{w\in\mathbf{I}_*} \subset \mathcal{M}_{q^2}$ and the polynomials $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*}$. The subposet $(\mathbf{I}_*, \leq)$ has a more direct characterization and a number of interesting properties, which are meticulously detailed in Hultman's papers [45, 46, 47]. Hultman's work extends to arbitrary Coxeter systems many earlier observations of Richardson and Springer [89, 90, 96] concerning $\mathbf{I}_*$ when W is finite. Here we review some of this material which will be of use later on, particularly in Section 3.3.3.

Recall from Section 3.1.4 that we define

$$s \ltimes w \overset{\text{def}}{=} \begin{cases} sw & \text{if } sw = ws^* \\ sws^* & \text{if } sw \neq ws^* \end{cases} \qquad \text{for } s \in S \text{ and } w \in \mathbf{I}_*. \tag{3.2.1}$$

In [70], Lusztig uses the notation $s \bullet w$ instead of $s \ltimes w$; we prefer the symbol $\ltimes$ to emphasize that $s \in S$ acts to "twist" $w \in \mathbf{I}_*$. Although this notation does not extend to an action of W of $\mathbf{I}_*$, it does lead to the following definition, adapted from [45, 46, 47]:

Definition 3.2.1. A sequence $(s_1, s_2, \dots, s_k)$ with $s_i \in S$ is an *$\mathbf{I}_*$-expression* for a twisted involution $w \in \mathbf{I}_*$ if $w = s_1 \ltimes (s_2 \ltimes (\cdots \ltimes (s_k \ltimes 1)\cdots))$. An $\mathbf{I}_*$-expression for w is *reduced* if its length k is minimal. We consider the empty sequence () to be a reduced $\mathbf{I}_*$-expression for $w = 1$.

What we refer to as $\mathbf{I}_*$-expressions are the left-handed versions of what Hultman terms "$\underline{S}$-expressions" in [45, 46, 47]. (In consequence, all of our statements here are in fact the left-handed versions of Hultman's.) It follows by induction on $\ell(w)$ that every $w \in \mathbf{I}_*$ has a reduced $\mathbf{I}_*$-expression, and so the next statement (given as [47, Proposition 2.5]) is well-defined:

Proposition 3.2.2 (Hultman [47]). Choose a reduced $\mathbf{I}_*$-expression $(s_1, s_2, \ldots, s_k)$ for $w \in \mathbf{I}_*$ and define $w_0 = w$ and $w_i = s_i \ltimes w_{i-1}$ for $1 \leq i \leq k$. Then the number of indices $i \in \{1, 2, \ldots, k\}$ with $s_i w_i = w_i s_i^*$ depends only on w and not on the choice of $\mathbf{I}_*$-expression.

Define $\ell^* : \mathbf{I}_* \to \mathbb{N}$ by setting $\ell^*(w)$ equal to the number k in the preceding propostiion. (In particular, $\ell^*(1) = 0$ and $\ell(s) = 1$ for any $s \in S \cap \mathbf{I}_*$.) The function ℓ^* coincides with the map ϕ which Lusztig defines in [70, Proposition 4.5]. This map measures the difference in size between the (ordinary) reduced expressions and reduced $\mathbf{I}_*$-expressions for a twisted involution, in the sense of the following result, which appears as [45, Theorem 4.8].

Theorem-Definition 3.2.3 (Hultman [45]). Let $\rho : \mathbf{I}_* \to \mathbb{N}$ be the map which assigns to $w \in \mathbf{I}_*$ the common length of any of its reduced $\mathbf{I}_*$-expressions. Then the poset $(\mathbf{I}_*, \leq)$ is graded with rank function ρ, and

$$\rho = \tfrac{1}{2}(\ell + \ell^*).$$

In particular, if $w \in \mathbf{I}_*$ and $s \in S$ then $\rho(s \ltimes w) = \rho(w) - 1$ if and only if $\ell(sw) = \ell(w) - 1$.

We conclude by stating the "subword property" for the Bruhat order on $\mathbf{I}_*$, which appears for arbitrary Coxeter systems as [47, Theorem 2.8].

Theorem 3.2.4 (Hultman [47]). If $y, w \in \mathbf{I}_*$ are twisted involutions, then $y \leq w$ if and only if whenever $(s_1, s_2, \ldots, s_k)$ is a reduced $\mathbf{I}_*$-expression for w, there exist indices $1 \leq i_1 < i_2 < \cdots < i_m \leq k$ such that $(s_{i_1}, s_{i_2} \ldots, s_{i_m})$ is a reduced $\mathbf{I}_*$-expression for y.

3.2.2 Multiplication formulas and a recurrence for the twisted polynomials

While Theorem-Definition 3.1.3 establishes the existence of the distinguished basis $(A_w)_{w \in \mathbf{I}_*}$ for the $\mathcal{H}_{q^2}$-module $\mathcal{M}_{q^2}$, it gives no immediate indication of how $\mathcal{H}_{q^2}$ acts on this basis, or of how one can compute the polynomials $\left(P^\sigma_{y,w}\right)_{y,w \in \mathbf{I}_*}$. In this section we summarize the main results of Lusztig [70] addressing these problems.

The most straightforward method of computing $P^\sigma_{y,w}$ involves an intermediate family of polynomials $\left(R^\sigma_{y,w}\right)_{y,w \in \mathbf{I}_*}$.

Definition 3.2.5. For each $w \in \mathbf{I}_*$, define $\left(R^\sigma_{y,w}\right)_{y \in \mathbf{I}_*} \in \mathcal{A}$ for $y \in \mathbf{I}_*$ as the family of Laurent polynomials satisfying

$$\overline{a_w} = \sum_{y \in \mathbf{I}_*} q^{-\ell(y)} \cdot \overline{R^\sigma_{y,w}} \cdot a_y.$$

Recall here that $q = v^2$ and that if $f \in \mathcal{A}$ then $\overline{f}$ denotes the Laurent polynomial $f(v^{-1})$. Knowledge of the polynomials $R^\sigma_{y,w}$ determines all of the polynomials $P^\sigma_{y,w}$ as a consequence of the following observations. First, the defining identity

$$A_w \overset{\text{def}}{=} v^{-\ell(w)} \sum_{y \in \mathbf{I}_*} P^\sigma_{y,w} a_y = \overline{A_y}$$

implies that

$$P^\sigma_{y,w} - q^{\ell(w)-\ell(y)}\overline{P^\sigma_{y,w}} = q^{\ell(w)-\ell(y)} \sum_{\substack{z\in \mathbf{I}_* \\ y<z\leq w}} \overline{R^\sigma_{y,z}} \cdot \overline{P^\sigma_{z,w}} \tag{3.2.2}$$

for each $y, w \in \mathbf{I}_*$. Because the polynomial $P^\sigma_{y,w}$ has degree at most $\frac{\ell(w)-\ell(y)-1}{2}$ when $y < w$, we have in this case $q^{\ell(w)-\ell(y)}\overline{P^\sigma_{y,w}} \in q^{-1}\mathbb{Z}[q^{-1}]$ and so $P^\sigma_{y,w}$ is given by omitting all monomials with negative exponents from the right hand side of (3.2.2), which can be computed (knowing the R^σ-polynomials) by induction.

To compute the polynomials $R^\sigma_{y,w}$, one can appeal to the following recurrences which Lusztig derives in [70, §4]. (In Lusztig's notation in that section, one has $r_{y,w} = v^{-\ell(w)+\ell(y)}R^\sigma_{y,w}$ and $r''_{y,w} = \overline{R^\sigma_{y,w}}$.) The proof of this result in [70] is elementary, and follows by comparing both sides of (3.1.1) after applying the bar operator, using the definition of $R^\sigma_{y,w}$.

Proposition 3.2.6 (Lusztig [70]). If $y, w \in \mathrm{I}_*$, then $R^\sigma_{y,w} \in \mathbb{Z}[q]$, and moreover:

(1) $R^\sigma_{y,w} = \delta_{y,w}$ if $y \not\leq w$ in the Bruhat order.

(2) If $y < w$ and $s \in \mathrm{Des}_L(w)$ then the following holds:

(a) If $s \ltimes w = sw$ (which occurs if and only if $sw = ws^*$) then

$$(q+1)R^\sigma_{y,w} = \begin{cases} q^2 R^\sigma_{s\ltimes y, s\ltimes w} + (q^2-q-1)R^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sys^* > y \\ (q-q^2)R^\sigma_{s\ltimes y, s\ltimes w} + (q^2-1)R^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sy > y \\ (q+1)R^\sigma_{s\ltimes y, s\ltimes w} - 2qR^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sy < y \\ R^\sigma_{s\ltimes y, s\ltimes w} - qR^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sys^* < y. \end{cases}$$

(b) If $s \ltimes w = sws^*$ then

$$R^\sigma_{y,w} = \begin{cases} q^2 R^\sigma_{s\ltimes y, s\ltimes w} + (q^2-1)R^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sys^* > y \\ (q-q^2)R^\sigma_{s\ltimes y, s\ltimes w} + (q^2+q-1)R^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sy > y \\ (q+1)R^\sigma_{s\ltimes y, s\ltimes w} - qR^\sigma_{y,s\ltimes w} & \text{if } s\ltimes y = sy < y \\ R^\sigma_{s\ltimes y, s\ltimes w} & \text{if } s\ltimes y = sys^* < y. \end{cases}$$

The method of computing the polynomials $P^\sigma_{y,w}$ just outlined is not the most efficient in practice, and still does not tell us how to compute the action of $T_x \in \mathcal{H}_{q^2}$ on $A_w \in \mathcal{M}_{q^2}$. We now summarize Lusztig's results in [70] resolving such issues. To this end, we first introduce some notation to refer to the coefficients of $P^\sigma_{y,w}$ of highest possible order.

Definition 3.2.7. Given $y, w \in \mathbf{I}_*$, we let

$$\mu^\sigma(y,w) \overset{\text{def}}{=} \text{the coefficient of } q^{(\ell(w)-\ell(y)-1)/2} \text{ in } P^\sigma_{y,w}$$
$$\nu^\sigma(y,w) \overset{\text{def}}{=} \text{the coefficient of } q^{(\ell(w)-\ell(y)-2)/2} \text{ in } P^\sigma_{y,w}.$$

In turn, for each $s \in S$ define another integer $\mu^\sigma(y,w;s)$ by the following more complicated formula:

$$\mu^\sigma(y,w;s) \overset{\text{def}}{=} \nu^\sigma(y,w) + \delta_{sy,ys^*}\mu^\sigma(sy,w) - \delta_{sw,ws^*}\mu^\sigma(y,sw) - \sum_{x\in \mathbf{I}_*;\, sx<x} \mu^\sigma(y,x)\mu^\sigma(x,w).$$

As usual, the Kronecker delta here means $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

Note since $P^\sigma_{y,w}$ is a polynomial in q and not in $v = q^{\frac{1}{2}}$ that $\mu^\sigma(y,w)$ (respectively, $\nu^\sigma(y,w)$) is nonzero only if $y \leq w$ and $\ell(w) - \ell(y)$ is odd (respectively, even). The numbers $\mu^\sigma(y,w;s)$ have an analogous property, which requires a short argument to prove. Here and elsewhere, for any $w \in W$ we write

$$\mathrm{Des}_L(w) \overset{\text{def}}{=} \{s \in S : \ell(sw) < \ell(w)\} \quad \text{and} \quad \mathrm{Des}_R(w) \overset{\text{def}}{=} \{s \in S : \ell(ws) < \ell(w)\} \tag{3.2.3}$$

for the corresponding left and right descent sets.

Proposition 3.2.8. Let $y,w \in \mathbf{I}_*$ and $s \in \mathrm{Des}_L(y) \setminus \mathrm{Des}_L(w)$. Then the integer $\mu^\sigma(y,w;s)$ is nonzero only if $\ell(w) - \ell(y)$ is even and $y < s \ltimes w$.

Proof. All terms in the definition of $\mu^\sigma(y,w;s)$ are zero if $\ell(w) - \ell(y)$ is odd. Assume $y \not< s \ltimes w$. Then $y \not< w$ automatically so $\mu^\sigma(y,w;s) = \delta_{sy,ys^*}\mu^\sigma(sy,w)$. This is zero unless $sy = ys^*$, but if $sy = ys^*$ then $sy = s \ltimes y \not< w$, as $s \ltimes y < w$ would imply the contradiction $y < s \ltimes w$ by Theorem 3.2.4. (In detail, if $s \ltimes y < w$ then adding s to the beginning of any reduced $\mathbf{I}_*$-expression for $s \ltimes y$ or w forms a reduced $\mathbf{I}_*$-expression for y or $s \ltimes w$, respectively.) Thus $\mu^\sigma(s \ltimes y, w) = 0$. $\square$

Finally, define $m^\sigma(y \xrightarrow{s} w) \in \mathcal{A}$ for $y,w \in \mathbf{I}_*$ and $s \in S$ as the Laurent polynomial

$$m^\sigma(y \xrightarrow{s} w) = \begin{cases} \mu^\sigma(y,w)(v+v^{-1}) & \text{if } \ell(w)-\ell(y) \text{ is odd} \\ \mu^\sigma(y,w;s) & \text{if } \ell(w)-\ell(y) \text{ is even.} \end{cases} \tag{3.2.4}$$

Lusztig proves the following result, which explains our notation, as [70, Theorem 6.3].

Theorem 3.2.9 (Lusztig [70]). Let $w \in \mathbf{I}_*$ and $s \in S$. Then $C_s = q^{-1}(T_s + 1)$ and

$$C_s A_w = \begin{cases} (q+q^{-1})\, A_w & \text{if } s \in \mathrm{Des}_L(w) \\ (v+v^{-1})\, A_{sw} + \sum_{y\in \mathbf{I}_*;\, sy<y<sw} m^\sigma(y \xrightarrow{s} w) A_y & \text{if } s \notin \mathrm{Des}_L(w) \text{ and } sw = ws^* \\ A_{sws^*} + \sum_{y\in \mathbf{I}_*;\, sy<y<sws^*} m^\sigma(y \xrightarrow{s} w) A_y & \text{if } s \notin \mathrm{Des}_L(w) \text{ and } sw \neq ws^*. \end{cases}$$

We may equivalently rewrite this theorem as a recurrence for the polynomials $P^\sigma_{y,w}$. This provides the following "twisted" analog of one of the standard recurrences (see, e.g., [16, Theorem 5.1.7]) for the ordinary Kazhdan-Lusztig polynomials $P_{y,w}$.

Corollary 3.2.10. Let $y,w \in \mathbf{I}_*$ with $y \leq w$ and $s \in \mathrm{Des}_L(w)$.

(a) $P^\sigma_{y,w} = P^\sigma_{s\ltimes y,w}$.

(b) If $s \in \mathrm{Des}_L(y)$ and $w' = s \ltimes w$ and $c = \delta_{sw,ws^*}$ and $d = \delta_{sy,ys^*}$, then

$$(q+1)^c P^\sigma_{y,w} = (q+1)^d P^\sigma_{s\ltimes y,w'} + q(q-d)P^\sigma_{y,w'} - \sum_{\substack{z\in \mathbf{I}_*;\, sz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)+c}\cdot m^\sigma(z \xrightarrow{s} w')\cdot P^\sigma_{y,z}.$$

Proof. The corollary results from comparing coefficients of a_y on both sides of the equation in Theorem 3.2.9. Rewriting the right hand side in the standard basis $(a_w)_{w\in\mathbf{I}_*}$ is straightforward from the definitions in Section 3.1.4, while rewriting the left hand side can be done using the identities $C_s = q^{-1}(T_s+1)$ and $A_w = v^{-\ell(w)}\sum_{y\in\mathbf{I}_*} P^\sigma_{y,w}a_y$ with the multiplication rule (3.1.1). $\square$

Remark. Because of the $(q+1)^c$ factor on the left, it is not obvious from the recurrence in part (b) that $P^\sigma_{y,w}\in\mathcal{A}$ is actually a polynomial in q (but it is clear that $P^\sigma_{y,w}$ is a rational function of q). That $P^\sigma_{y,w}\in\mathbb{Z}[q]$ follows from results in [70], however, and in light of this, the recurrence shows that $(q+1)^{\ell^*(w)-\ell^*(y)}$ divides $P^\sigma_{y,w}$. Recall here the definition of ℓ^* from Proposition-Definition 3.2.2.

The recurrence in the preceding corollary leads to an algorithm for computing the polynomials $P^\sigma_{y,w}$, though there is some subtlety in formulating this. We discuss these details in Section 3.4.3.

We mention two additional properties of the polynomials $P^\sigma_{y,w}$. Both follow from Corollary 3.2.10 in a straightforward manner by induction on $\ell(w)$. Lusztig states part (b) explicitly as [70, Proposition 4.10], but this actually serves in [70] as a preliminary to the other results given here.

Corollary 3.2.11 (Lusztig [70]). Let $y,w\in\mathbf{I}_*$ with $y\leq w$.

(a) $P^\sigma_{y,w} = P^\sigma_{y^{-1},w^{-1}} = P^\sigma_{\tau(y),\tau(w)}$ for all S-preserving automorphisms $\tau\in\mathrm{Aut}(W)$ which commutes with $*$ in the sense that $\tau(x^*) = \tau(x)^*$ for all $x\in W$.

(b) $P^\sigma_{y,w}$ has constant coefficient 1.

3.2.3 Recovering untwisted Kazhdan-Lusztig theory

Here we explain how the Kazhdan-Lusztig polynomials $P_{y,w}$ can occur as instances of the twisted polynomials $P^\sigma_{y,w}$ for certain choices of (W,S) and $*$. This makes it possible to recover several important properties of the Kazhdan-Lusztig basis $(c_w)_{w\in W}\subset\mathcal{H}_q$ and the Kazhdan-Lusztig polynomials $(P_{y,w})_{w\in W}\subset\mathbb{Z}[q]$ from analogous statements for $\mathcal{M}_{q^2}$ in the previous section.

We begin with the following observation. In approaching this statement it is helpful to recall the definitions of the Laurent polynomials $P_{y,w}$, $P^\sigma_{y,w}$, $h_{x,y;z}$, $\widetilde{h}_{x,y;z}$, $h^\sigma_{x,y;z}$ from Sections 3.1.3 and 3.1.4.

Proposition 3.2.12. Suppose that (W, S) has the form

$$W = W' \times W' \qquad \text{and} \qquad S = \{(s,1) : s \in S'\} \cup \{(1,s) : s \in S'\} \tag{3.2.5}$$

for some Coxeter system (W', S'), and that $* \in \mathrm{Aut}(W)$ acts by $(x,y)^* = (y,x)$ for $y, w \in W'$. Let $\mathcal{H}'_{q^2}$ denote the Hecke algebra of (W', S') with parameter q^2.

(a) The map $w \mapsto (w, w^{-1})$ defines a poset isomorphism $(W', \leq) \xrightarrow{\sim} (\mathbf{I}_*, \leq)$.

(b) The unique $\mathcal{A}$-linear map $\mathcal{H}'_{q^2} \to \mathcal{H}_{q^2}$ with $T_w \mapsto T_{(w,1)}$ for $w \in W'$ is an injective algebra homomorphism. With respect to this embedding, the unique $\mathcal{A}$-linear map $\mathcal{H}'_{q^2} \to \mathcal{M}_{q^2}$ with $T_w \mapsto a_{(w,w^{-1})}$ for $w \in W'$ is an isomorphism of left $\mathcal{H}'_{q^2}$-modules. This map commutes with the bar operators of $\mathcal{H}'_{q^2}$ and $\mathcal{M}_{q^2}$ in the sense that $\overline{T_w} \mapsto \overline{a_{(w,w^{-1})}}$ for $w \in W'$.

(c) For all $y, w \in W'$, we have

$$P_{(y,y^{-1}),(w,w^{-1})} = (P_{y,w})^2 \qquad \text{and} \qquad P^\sigma_{(y,y^{-1}),(w,w^{-1})} = P_{y,w}(q^2).$$

(d) For all $x, y, z \in W'$, we have

$$\widetilde{h}_{(x,x^{-1}),(y,y^{-1});(z,z^{-1})} = (\widetilde{h}_{x,y;z})^2 \qquad \text{and} \qquad h^\sigma_{(x,x^{-1}),(y,y^{-1});(z,z^{-1})} = h_{x,y;z}(v^2).$$

Remark. In the four identities in parts (c) and (d), the left expressions are (Laurent) polynomials attached to the Coxeter system (W, S), while the right expressions are quantities defined in terms of the (Laurent) polynomials attached to the Coxeter system (W', S'). In particular, $h_{x,y;z}(v^2)$ denotes the Laurent polynomial obtained by replacing the parameter v with $q = v^2$ in $h_{x,y;z} \in \mathcal{A}$.

Proof. Part (a) is straightforward and has been noted previously as [46, Example 3.2] and also as [89, Example 10.1]. Since $s \ltimes w = sws^*$ for all $s \in S$ and $w \in W$, part (a) implies that the map in part (b) is an $\mathcal{A}$-linear bijection satisfying

$$T_x T_w \mapsto T_{(x,1)} a_{(w,w^{-1})} \qquad \text{for } x, w \in W'.$$

Thus our map is a left $\mathcal{H}'_{q^2}$-module isomorphism with respect to the embedding $\mathcal{H}'_{q^2} \hookrightarrow \mathcal{H}_{q^2}$. To see that this map commutes with the bar operators, note that if $w \in W'$ then $T_w = T_w \cdot T_1$ and $a_{(w,w^{-1})} = T_{(w,1)} \cdot a_1$, so

$$\overline{T_w} = T^{-1}_{w^{-1}} \cdot T_1 \mapsto T^{-1}_{(w^{-1},1)} \cdot a_1 = \overline{a_{(w,w^{-1})}}.$$

The remaining parts follow as an easy exercise from part (b) and the defining properties of $P_{y,w}$ and $P^\sigma_{y,w}$ in Theorem-Definitions 3.1.1 and 3.1.2. $\square$

As one application this observation, we may recover the standard formula for the decomposition of the product $c_s c_w$ in the Kazhdan-Lusztig basis of $\mathcal{H}_q$. Define for $y, w \in W$ the integer

$$\mu(y,w) \overset{\text{def}}{=} \text{the coefficient of } q^{(\ell(w)-\ell(y)-1)/2} \text{ in } P_{y,w}. \tag{3.2.6}$$

As with $\mu^\sigma(y,w)$, the integer $\mu(y,w)$ can only be nonzero if $y \leq w$ and $\ell(w) - \ell(y)$ is odd. If we assume the hypotheses of Proposition 3.2.12, then $sw \neq ws*$ for all $s \in S$ and $w \in \mathbf{I}_*$ and every twisted involution in $\mathbf{I}_*$ has even length. In consequence, if $s \in S'$ and $y, w \in W'$ then it follows by part (c) of the proposition that

$$m^\sigma\left((y,y^{-1}) \xrightarrow{(s,1)} (w,w^{-1})\right) = \nu^\sigma\left((y,y^{-1}),(w,w^{-1})\right) = \mu(y,w).$$

If we substitute this formula into Theorem 3.2.9 and Corollary 3.2.10 (and then replace the Coxeter system (W', S') with (W, S) and the parameter q with v) then we obtain the following "untwisted" analogue of Theorem 3.2.9:

Theorem 3.2.13 (Kazhdan and Lusztig [53]). Let $w \in W$ and $s \in S$. Then $c_s = v^{-1}(t_s + 1)$ and

$$c_s c_w = \begin{cases} (v+v^{-1})c_w & \text{if } s \in \mathrm{Des}_L(w) \\ c_{sw} + \sum_{z \in W;\, sz<z<w} \mu(z,w) c_z & \text{if } s \notin \mathrm{Des}_L(w). \end{cases}$$

Rewriting both sides of the preceding equation in the basis $\{t_w : w \in W\}$ then comparing coefficients yields the following standard recurrence for the polynomial $P_{y,w}$ which one can find also derived in [48, Chapter 7].

Corollary 3.2.14. Let $y, w \in W$ with $y \leq w$ and $s \in \mathrm{Des}_L(w)$.

(a) $P_{y,w} = P_{sy,w}$.

(b) If $s \in \mathrm{Des}_L(y)$ then

$$P_{y,w} = P_{sy,sw} + qP_{y,sw} - \sum_{\substack{z \in W;\, sz<z \\ y \leq z < w}} v^{\ell(w)-\ell(z)} \cdot \mu(z,sw) \cdot P_{y,z}.$$

From this one can derive several well-known properties of the Kazhdan-Lusztig polynomials, which we collect as the following statement paralleling Corollary 3.2.11.

Corollary 3.2.15. Let $y, w \in W$.

(a) $P_{y,w} = P_{y^{-1},w^{-1}} = P_{\tau(y),\tau(w)}$ for all S-preserving automorphisms $\tau \in \mathrm{Aut}(W)$.

(b) $P_{y,w}$ has constant coefficient 1 if $y \leq w$.

Remark. Since $\mathrm{Des}_L(w) = \mathrm{Des}_R(w^{-1})$, it follows that $P_{y,w} = P_{ys,w}$ if $s \in \mathrm{Des}_R(w)$.

3.2.4 Parity statements

Conjectures A′, B′ and C′ are statements concerning whether the Laurent polynomials

$$P^{\pm}_{y,w} = \tfrac{1}{2}\left(P_{y,w} \pm P^{\sigma}_{y,w}\right) \qquad \text{and} \qquad h^{\pm}_{x,y;z} = \tfrac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^{\sigma}_{x,y;z}\right)$$

defined by equations (3.1.2), (3.1.3), and (3.1.4) for $x \in W$ and $y,z,w \in \mathbf{I}_*$ have nonnegative integer coefficients. It is not clear *a priori* that these polynomials even have integer coefficients, and we spend this last preliminary section clarifying this property.

Here we write $f \equiv g \pmod 2$ for two Laurent polynomials $f, g \in \mathcal{A}$ if $f - g$ has only even integer coefficients; i.e., if $f - g = 2h$ for some $h \in \mathcal{A}$. Lusztig proves the following result, showing that $P^{\pm}_{y,w} \in \mathbb{Z}[q]$, as [70, Theorem 9.10].

Proposition 3.2.16 (Lusztig [70]). For all $y, w \in \mathbf{I}_*$ we have $P_{y,w} \equiv P^{\sigma}_{y,w} \pmod 2$.

The next proposition shows likewise that $h^{\pm}_{x,y;z} \in \mathbb{Z}[v, v^{-1}]$. In the case that (W,S) is a Weyl group and $*$ is trivial, this property was mentioned without proof in [72, Section 5].

Proposition 3.2.17. For all $x \in W$ and $y, z \in \mathbf{I}_*$ we have $\widetilde{h}_{x,y,z} \equiv h^{\sigma}_{x,y,z} \pmod 2$.

Proof. In what follows it is helpful to recall that we denote the bases of $\mathcal{H}_q$ using the lower case symbols t_w, c_w and the bases of $\mathcal{H}_{q^2}$ using the upper case symbols T_w, C_w. Let $w^{\dagger} = w^{*-1}$ for $w \in W$ and define $h \mapsto h^{\dagger}$ as the unique $\mathcal{A}$-algebra anti-automorphism of $\mathcal{H}_q$ such that $(t_w)^{\dagger} = t_{w^{\dagger}}$. Also write $\mathrm{proj} : \mathcal{H}_q \to \mathcal{M}_{q^2}$ for the $\mathcal{A}$-linear map with $t_w \mapsto a_w$ for $w \in \mathbf{I}_*$ and $t_w \mapsto 0$ for $w \in W \setminus \mathbf{I}_*$.

We write $m \equiv m' \pmod 2$ for $m, m' \in \mathcal{M}_{q^2}$ if $m - m' = 2m''$ for some $m'' \in \mathcal{M}_{q^2}$. With respect to this notation, Lusztig [70, 9.4(a)] proves that

$$\mathrm{proj}\left(t_x t_y (t_x)^{\dagger}\right) \equiv T_x a_y \pmod 2 \qquad \text{for all } x \in W \text{ and } y \in \mathbf{I}_*. \tag{3.2.7}$$

The current proposition derives from this fact in the following way. Let $x \in W$ and $y \in \mathbf{I}_*$ and note that $(c_w)^{\dagger} = c_{w^{\dagger}}$ by Lemma 3.2.15. The anti-automorphism $\dagger$ consequently preserves $c_x c_y c_{x^{\dagger}}$, so we must have $\widetilde{h}_{x,y,z} = \widetilde{h}_{x,y,z^{\dagger}}$ for all $z \in W$ and it follows that we can write $c_x c_y c_{x^{\dagger}} = (a + a^{\dagger}) + \sum_{z \in \mathbf{I}_*} \widetilde{h}_{x,y,z} c_z$ for an element $a \in \mathcal{H}_q$. Since $\mathrm{proj}(a + a^{\dagger}) \equiv 0 \pmod 2$ and since $\mathrm{proj}(c_z) \equiv A_z \pmod 2$ for $z \in \mathbf{I}_*$ by Proposition 3.2.16, we deduce that

$$\mathrm{proj}(c_x c_y c_{x^{\dagger}}) \equiv \sum_{z \in \mathbf{I}_*} \widetilde{h}_{x,y,z} A_z \pmod 2. \tag{3.2.8}$$

On the other hand, by definition $c_x c_y c_{x^{\dagger}} = v^{-2\ell(x) - \ell(y)} \sum_{x',x'',z \in W} P_{x',x} P_{x'',x} P_{z,y} \cdot t_{x'} t_z (t_{x''})^{\dagger}$. Since $P_{z,y} = P_{z^{\dagger},y}$ for all $z \in W$ as $y = y^{\dagger}$, the anti-automorphism $\dagger$ acts on the latter sum by exchanging the summands indexed by (x', x'', z) and $(x'', x', z^{\dagger})$. It follows by dividing the sum $\sum_{x',x'',z \in W}$ into two parts, consisting of the summands fixed and unfixed by $\dagger$, that we can write

$$c_x c_y c_{x^\dagger} = (b + b^\dagger) + v^{-2\ell(x)-\ell(y)} \sum_{x' \in W} \sum_{z \in \mathbf{I}_*} (P_{x',x})^2 P_{z,y} \cdot t_{x'} t_z (t_{x'})^\dagger.$$

for another element $b \in \mathcal{H}_q$. Since $\mathrm{proj}(b + b^\dagger) \equiv 0 \pmod 2$ and $(P_{x',x})^2 \equiv P_{x',x}(q^2) \pmod 2$ and $P_{z,y} \equiv P^\sigma_{z,y} \pmod 2$ for $y, z \in \mathbf{I}_*$, applying (3.2.7) to the preceding equation shows that

$$\mathrm{proj}\,(c_x c_y c_{x^\dagger}) \equiv C_x A_y \pmod 2 \qquad \text{for all } x \in W \text{ and } y \in \mathbf{I}_*. \tag{3.2.9}$$

The proposition now follows immediately by combining (3.2.8) and (3.2.9). □

3.3 Positivity results for universal Coxeter systems

In this section we let (W, S) be any universal Coxeter system and let $* \in \mathrm{Aut}(W)$ be any S-preserving involution of W. In this case W is the group generated by S subject only to the relations $s^2 = 1$ for $s \in S$. The elements of W consists of all words in S with distinct adjacent letters, and products of elements are given by concatenation, subject to the rule that one inductively removes all pairs of equal adjacent letters.

The involution $* \in \mathrm{Aut}(W)$ corresponds to an arbitrary choice of a permutation with order ≤ 2 of the set S. The twisted involutions $w \in \mathbf{I}_* = \{x \in W : x^{-1} = x^*\}$ each take one of two possible forms:

- If $\ell(w)$ is even then $w = x^* x^{-1}$ for some $x \in W$.
- If $\ell(w)$ is odd then $w = x^* s x^{-1}$ for some $x \in W$ and $s \in S$ with $s = s^*$.

The following observation enumerates a few other special properties of universal Coxeter systems which make them tractable test cases for general questions and conjectures. Recall here the definition of $s \ltimes w$ from (3.2.1).

Observation 3.3.1. Assume (W, S) is a universal Coxeter system.

(a) Each $w \in W$ has a unique reduced expression.

(b) Each $w \in \mathbf{I}_*$ has a unique reduced $\mathbf{I}_*$-expression.

(c) If $w \in W \setminus \{1\}$ then $|\mathrm{Des}_L(w)| = |\mathrm{Des}_R(w)| = 1$.

(d) The map $S \times \mathbf{I}_* \to \mathbf{I}_*$ given by $(s, w) \mapsto s \ltimes w$ extends to a group action of W on $\mathbf{I}_*$.

Notation. In light of part (d), it is well-defined to set $x \ltimes w \overset{\text{def}}{=} s_1 \ltimes (s_2 \ltimes (\cdots \ltimes (s_n \ltimes w) \cdots))$ where $x \in W$ and $w \in \mathbf{I}_*$ and $s_i \in S$ are such that $x = s_1 s_2 \cdots s_n$.

Before proceeding, we note as a consequence of our observation that in the special case that $*$ has no fixed points in S, one can view the ordinary Kazhdan-Lusztig theory of a universal Coxeter system as a special case of its twisted theory, in the following way.

Observation 3.3.2. Suppose (W,S) is a universal Coxeter system and $s \neq s^*$ for all $s \in S$. Then the unique $\mathcal{A}$-linear map $\mathcal{H}_{q^2} \to \mathcal{M}_{q^2}$ with $T_{w^*} \mapsto a_{w^*w^{-1}}$ for $w \in W$ defines an isomorphism of left $\mathcal{H}_{q^2}$-modules which commutes with the bar operators of $\mathcal{H}_{q^2}$ and $\mathcal{M}_{q^2}$, and consequently

$$P^\sigma_{(y^*y^{-1}),(w^*w^{-1})} = P_{y,w}(q^2) \quad \text{and} \quad h^\sigma_{x,(y^*y^{-1});(z^*z^{-1})} = h_{x,y^*;z^*}(v^2) \qquad \text{for all } w,x,y,z \in W.$$

Proof. If $s \neq s^*$ for all $s \in S$ then every $w \in \mathbf{I}_*$ has even length and the map $w^* \mapsto w^*w^{-1}$ defines a poset isomorphism $(W,\leq) \xrightarrow{\sim} (\mathbf{I}_*,\leq)$. From this, the proof of the proposition is a straightforward exercise using Theorem-Definitions 3.1.1, 3.1.2, and 3.1.3. □

3.3.1 Kazhdan-Lusztig polynomials

Dyer derived a formula for the Kazhdan-Lusztig polynomials of a universal Coxeter system [35, Theorem 3.8] which shows their coefficients to be nonnegative. We review the key parts of this result here. To begin, we note the following lemma which is a special case of [35, Lemma 3.5].

Lemma 3.3.3 (Dyer [35]). Assume (W,S) is a universal Coxeter system. Suppose $y,w \in W$ and $r,s \in S$ such that $rsw < sw < w$ and $sy > y$. Then

$$P_{y,w} = P_{y,sw} + qP_{sy,sw} - \delta \cdot qP_{y,rsw} \qquad \text{where } \delta = |\{s\} \cap \mathrm{Des}_L(rsw)|.$$

In the sequel we adopt the following notation. Given $y,z,w \in W$ with $y \leq z$, define

$$P_{y,z;w} \overset{\text{def}}{=} P_{y,w} - P_{z,w}. \tag{3.3.1}$$

We expand upon the previous lemma with the following statement.

Proposition 3.3.4. Assume (W,S) is universal. Let $y,z \in W$ with $y \leq z$ and suppose

- k is a positive integer;
- $r,s \in S$ such that $r \neq s$ and $s \notin \mathrm{Des}_L(y)$ and $s \notin \mathrm{Des}_L(z)$;
- $u \in W$ such that $\{r,s\} \cap \mathrm{Des}_L(u) = \varnothing$.

If $a,w \in W$ are defined by

$$w = \underbrace{srsrs\cdots u}_{k+1 \text{ factors}} \qquad \text{and} \qquad a = \underbrace{\cdots srsrs}_{k \text{ factors}}$$

then $P_{y,z;w} = P_{y,z;sw} + q^k P_{ay,az;aw}$.

Remark. Applying the identity $P_{y,z;w} = P_{y^{*-1},z^{*-1};w^{*-1}}$ from Lemma 3.2.15 affords a right-handed version of this proposition, which will be of use in Section 3.3.3 below.

Proof. Note that $y \leq z$ implies $ay \leq az$, since (as $sy > y$ and $sz > z$) the unique reduced expression for ay (respectively, az) is formed by concatenating $\cdots srsrs$ to the unique reduced expression for y (respectively, z). To prove the lemma, we proceed by induction on k. If $k = 1$ then the lemma reduces to Lemma 3.3.3. If $k > 1$, then since $P_{sy,sz;rsw} = P_{y,z;rsw}$ by Lemma 3.2.15, Lemma 3.3.3 asserts that $P_{y,z;w} = P_{y,z;sw} + q(P_{sy,sz;sw} - P_{sy,sz;rsw})$. By induction we may assume that $P_{sy,sz;sw} = P_{sy,sz;rsw} + q^{k-1}P_{ay,az;aw}$; substituting this identity into the preceding equation gives the desired recurrence. $\square$

From the last lemma we have an easy proof of Conjecture B (and so also of Conjecture A) for universal Coxeter systems. This result can also be deduced from [35, Theorem 3.8].

Theorem 3.3.5 (Dyer [35]). If (W, S) is a universal Coxeter system, then the polynomial $P_{y,w} - P_{z,w}$ has nonnegative integer coefficients for all $y, z, w \in W$ with $y \leq z$ in the Bruhat order. In particular, we have $P_{y,w} \in \mathbb{N}[q]$ for each $y, w \in W$.

Proof. The proof is by induction on $\ell(w)$. If $\ell(w) \leq 1$ then $P_{y,z;w} \in \{0, 1\} \subset \mathbb{N}[q]$ by Lemma 3.2.15. Assume $\ell(w) \geq 2$ so that there exists $s \in \mathrm{Des}_L(w)$. Let (y', z') be the unique pair in the set $\{(y, z), (sy, z), (y, sz), (sy, sz)\}$ which has $s \notin \mathrm{Des}_L(y')$ and $s \notin \mathrm{Des}_L(z')$. It is straightforward to check that $y' \leq z'$, and by Lemma 3.2.15 we have $P_{y,z;w} = P_{y',z';w}$. Proposition 3.3.4 applied $P_{y',z';w}$ shows that that $P_{y,z;w} \in \mathbb{N}[q]$ by induction, and it follows that $P_{y,w} \in \mathbb{N}[q]$ since $P_{w,w} = 1$. $\square$

3.3.2 Twisted Kazhdan-Lusztig polynomials

Here we initiate the proof of Conjecture B$'$ for universal Coxeter systems, to be completed in the next section. As above, (W, S) is a fixed Universal Coxeter system with a fixed S-preserving involution $*$.

Recall the definition of the Laurent polynomial $m^\sigma(y \xrightarrow{s} w) \in \mathcal{A}$ from (3.2.4).

Lemma 3.3.6. Assume (W, S) is a universal Coxeter system. If $y, w \in \mathbf{I}_*$ and $r, s \in S$ such that $y \leq w$ and $\mathrm{Des}_L(y) = \{s\} \neq \{r\} = \mathrm{Des}_L(w)$, then

$$m^\sigma(y \xrightarrow{s} w) = \begin{cases} 1 & \text{if } y = rwr^* \text{ or if } (y, w) = (s, r) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First note that since W is a universal Coxeter group and $y \notin \{1, r\}$ we have $r \ltimes y = ryr^*$ and $\ell(r \ltimes y) = \ell(y) + 2$. In addition, from Corollary 3.2.10 we have $P^\sigma_{y,w} = P^\sigma_{r \ltimes y,w}$.

We claim that $\mu^\sigma(y, w) = 0$. To prove this, note that if $r \ltimes y = w$ then $\ell(w) - \ell(y)$ is even, and if $r \ltimes y \not\leq w$ then $y \not< w$, so in either case $\mu^\sigma(y, w) = 0$. On other other hand, if $r \ltimes y < w$ then the degree of $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w}$ as a polynomial in q is at most $\frac{\ell(w) - \ell(r \ltimes y) - 1}{2}$ which is strictly less than $\frac{\ell(w) - \ell(y) - 1}{2}$, so again $\mu^\sigma(y, w) = 0$.

It thus suffices to show that $\mu^\sigma(y, w; s) = 1$ if $y = rwr^*$ or if $(y, w) = (s, r)$ and $\mu^\sigma(y, w; s) = 0$ otherwise. To this end, observe that if we apply our first claim to the definition of $\mu^\sigma(y,w;s)$, and also note that $sw \neq ws^*$ since $w \notin \{1,s\}$, we obtain

$$\mu^\sigma(y,w;s) = \nu^\sigma(y,w) + \delta_{sy,ys^*}\mu^\sigma(sy,w).$$

If $y = rwr^*$ then $P^\sigma_{y,w} = P^\sigma_{w,w} = 1$ so $\nu^\sigma(y,w) = 1$. Alternatively, if $y < w$ and $y \neq rwr^*$ then it follows as above that $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w}$ has degree strictly less than $\frac{\ell(w)-\ell(y)-2}{2}$ so $\nu^\sigma(y,w) = 0$. Hence

$$\nu^\sigma(y,w) = \begin{cases} 1 & \text{if } y = rwr^* \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.2}$$

In turn, we have $sy = ys^*$ if and only if $y = s$ (note that $y \neq 1$ by hypothesis), in which case $\mu^\sigma(sy,w) = \mu^\sigma(1,w)$. If $w = r$ then $\mu^\sigma(1,w) = 1$ and if $w \neq r$ then either $w = rr^*$ (in which case $\ell(w) - \ell(1)$ is even) or $r \ltimes 1 \neq w$ (in which case $P^\sigma_{1,w} = P^\sigma_{r\ltimes 1,w}$ has degree strictly less than $\frac{\ell(w)-\ell(1)-1}{2}$) so $\mu^\sigma(1,w) = 0$. Thus

$$\delta_{sy,ys^*}\nu^\sigma(sy,w) = \begin{cases} 1 & \text{if } (y,w) = (s,r) \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.3}$$

Combining (3.3.2) and (3.3.3) gives the desired formula for $\mu^\sigma(y,w;s)$. □

We now have the following analogue of Lemma 3.3.3.

Lemma 3.3.7. Assume (W,S) is a universal Coxeter system. Suppose $y,w \in \mathbf{I}_*$ and $r,s \in S$ such that $rs \ltimes w < s \ltimes w < w$ and $sy > y$. Then

$$P^\sigma_{y,w} = P^\sigma_{y,s\ltimes w} + q^2 P^\sigma_{s\ltimes y,s\ltimes w} - \delta \cdot q^2 P^\sigma_{s\ltimes y,rs\ltimes w^*} + \delta' \cdot q(P^\sigma_{1,s\ltimes w} - P^\sigma_{s,s\ltimes w}).$$

where we define

$$\delta = \begin{cases} 1 & \text{if } s \in \mathrm{Des}_L(rs \ltimes w) \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta' = \begin{cases} 1 & \text{if } y = 1 \text{ and } s = s^* \text{ and } w \neq srs \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Everything follows by combining Lemma 3.3.7 with Corollary 3.2.10. It is straightforward to check that the lemma holds if $y \not< w$, so assume $y < w$. Let $\delta'' = \delta_{sy,ys^*}$ and note by hypothesis that $sw \neq ws^*$. By Corollary 3.2.10 we therefore have

$$P^\sigma_{y,w} = P^\sigma_{y,s\ltimes w} + q^2 P^\sigma_{s\ltimes y,s\ltimes w} + \delta'' q(P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w}) - \sum_{\substack{z\in\mathbf{I}_*;\, sz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)} m^\sigma(z \xrightarrow{s} s\ltimes w) P^\sigma_{y,z}. \tag{3.3.4}$$

From the preceding lemma we know that $m^\sigma(z \xrightarrow{s} s\ltimes w) = 1$ if $z = rs\ltimes w$ or $(z, s\ltimes w) = (s,r)$, and $m^\sigma(z \xrightarrow{s} s \ltimes w) = 0$ otherwise. The sum in (3.3.4) includes a summand indexed by $z = rs \ltimes w$ if and only if $\delta = 1$. On the other hand, if $s \ltimes w = r$ then the sum includes a summand indexed by $z = s$ if and only if $s = s^*$. Since $P^\sigma_{y,s} = 1$ if $y \in \{1,s\}$ and $P^\sigma_{y,s} = 0$

otherwise, we conclude that

$$P^\sigma_{y,w} = P^\sigma_{y,s\ltimes w} + q^2 P^\sigma_{s\ltimes y,s\ltimes w} + \delta'' \cdot q(P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w}) - \delta \cdot q^2 P^\sigma_{y,rs\ltimes w} - \delta''' \cdot q$$

where we define

$$\delta''' = \begin{cases} 1 & \text{if } y = 1 \text{ and } s = s^* \text{ and } w = srs \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\delta = 1$ then $P^\sigma_{y,rs\ltimes w} = P^\sigma_{s\ltimes y,rs\ltimes w}$ by Corollary 3.2.10. Thus to finish our proof, it is enough to check that

$$\delta''(P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w}) - \delta''' = \delta'(P^\sigma_{1,s\ltimes w} - P^\sigma_{s,s\ltimes w})$$

This is clear if $y = 1$ and $s = s^*$ and $w \neq srs$ since then $\delta' = \delta'' = 1$ and $\delta''' = 0$. On the other hand, if $y = 1$ and $s = s^*$ but $w = srs$ then $\delta' = 0$ and $\delta'' = \delta''' = 1$ and $P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w} = P^\sigma_{1,r} = 1$, which again gives equality. Finally, if $y \neq 1$ or $s \neq s^*$ then $\delta' = \delta'' = \delta''' = 0$ and our equation again holds. □

3.3.3 Four technical propositions

To prove Conjectures A′ and B′ for the universal Coxeter system (W, S) we require an analog of Proposition 3.3.4. The requisite statement splits into four somewhat more technical propositions, which we prove here. The Coxeter system (W, S) is always assumed to be universal in this section (and we stop stating this condition in our results).

Mirroring the notation $P_{y,z;w}$ from (3.3.1), given $y, z, w \in \mathbf{I}_*$ with $y \leq z$, we define

$$P^\sigma_{y,z;w} \overset{\text{def}}{=} P^\sigma_{y,w} - P^\sigma_{z,w}. \tag{3.3.5}$$

Also, given elements $w_1, w_2, \dots, w_k \in W$ we write $\langle w_1, w_2, \dots, w_k \rangle$ for the subgroup they generate. Finally, recall from Theorem-Definition 3.2.3 that we denote the rank function on $(\mathbf{I}_*, \leq)$ by

$$\rho : \mathbf{I}_* \to \mathbb{N},$$

so that $\rho(w)$ is the length of any reduced $\mathbf{I}_*$-expression for $w \in \mathbf{I}_*$.

At least one half of the following result is well-known, being equivalent to the fact that the Kazhdan-Lusztig polynomials of dihedral Coxeter systems are all constant.

Proposition 3.3.8. Let $y, z, w \in \mathbf{I}_*$ with $y \leq z$. If $r, s \in S$ such that $w \in \langle r, s \rangle$, then

$$P^\sigma_{y,z;w} = P_{y,z;w} = \begin{cases} 1 & \text{if } y \leq w \text{ and } z \not\leq w \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If suffices to show that $P^\sigma_{y,w} = P_{y,w} = 1$ if $y \leq w$; however, this follows by a straightforward argument using induction on the length of w and Lemmas 3.3.3 and 3.3.7. In particular, the base cases for our induction are given by Corollary 3.2.10 and Lemma 3.2.15, which show that $P^\sigma_{y,w} = 1$ if $y \leq w$ and $\rho(w) \leq 1$, and that $P_{y,w} = 1$ if $y \leq w$ and $\ell(w) \leq 1$. $\square$

For the duration of this section we adopt the following specific setup: fix $y, z \in \mathbf{I}_*$ with $y \leq z$ and assume $w \in \mathbf{I}_*$ has the form

$$w = \underbrace{srsrs\cdots}_{k+1 \text{ factors}} \ltimes u \tag{3.3.6}$$

where

- k is a positive integer;
- $r, s \in S$ such that $r \neq s$ and $s \notin \mathrm{Des}_L(y)$ and $s \notin \mathrm{Des}_L(z)$;
- $u \in \mathbf{I}_*$ such that $\{r, s\} \cap \mathrm{Des}_L(u) = \varnothing$.

In addition, define $a \in \langle r, s\rangle \subset W$ as the element

$$a = \underbrace{\cdots srsrs}_{k \text{ factors}} \tag{3.3.7}$$

and let $y', z', w' \in \mathbf{I}_*$ denote the twisted involutions

$$y' = a \ltimes y \qquad \text{and} \qquad z' = a \ltimes z \qquad \text{and} \qquad w' = a \ltimes w. \tag{3.3.8}$$

Observe that $\rho(y') = \rho(a) + \rho(y)$ and $\rho(z') = \rho(a) + \rho(z)$ and $\rho(w') = \rho(u) + 1$, and that clearly $y' \leq z'$ in the Bruhat order. In addition, w' is given by either $s \ltimes u$ or $r \ltimes u$, depending on the parity of k. We now have our second proposition.

Proposition 3.3.9. Suppose $w \notin \langle r, s\rangle$ and either $y \neq 1$ or $s \neq s^*$. Then

(a) $P^\sigma_{y,z;w} = P^\sigma_{y,z;sws^*} + q^{2k} P^\sigma_{y',z';w'}$.

(b) $P_{y,z;w} = P_{y,z;sws^*} + q^{2k} P_{y',z';w'} + 2q^k P_{ay,az;aws^*}$.

Remark. The best way of making sense of this and the next two propositions is through pictures. The recurrences in each proposition are conveniently illustrated as trees whose nodes are labeled by the polynomials $P^\sigma_{y,z;w}$ or $P_{y,z;w}$ and whose edges are labeled by powers of q; see Figures B-1, B-2, B-3, B-4, B-5, and B-6. In these diagrams, the branches at each level indicate one application of Lemma 3.3.7 or Lemma 3.3.3; these lemmas add two or three children to a given node while possibly also canceling a node two levels down the tree. This cancelation accounts for the chains of k single-child nodes, which appear as dashed lines.

Proof. First consider Figure B-1. The proof of part (a) is very similar to that of Proposition 3.3.4, but using Lemma 3.3.3 in place of Lemma 3.3.7. The argument is entirely analogous because, under our current hypotheses, whenever we apply Lemma 3.3.7 the second indicator δ' defined in that result is zero.

Now consider Figure B-2. To prove part (b), we first apply the right-handed version of Proposition 3.3.4 to $P_{y,z;w}$ and then apply the left-handed version of Proposition 3.3.4 to the result. In detail, the first application gives

$$P_{y,z;w} = P_{y,z;ws^*} + q^k P_{ya^{*-1},za^{*-1};wa^{*-1}}$$

while the second gives $P_{y,z;ws^*} = P_{y,z;sws^*} + q^k P_{ay,az;aws^*}$ and

$$P_{ya^{*-1},za^{*-1};wa^{*-1}} = P_{ya^{*-1},za^{*-1};swa^{*-1}} + q^k P_{y',z';w'}.$$

Since $P_{ay,az;aws^*} = P_{ya^{*-1},za^{*-1};swa^{*-1}}$ by Lemma 3.2.15 combining the preceding equations gives the desired recurrence. □

We proceed immediately to our next proposition.

Proposition 3.3.10. Suppose $w \notin \langle r, s\rangle$ and $y = 1 \neq z$ and $s = s^*$ and $r = r^*$. Then there are elements $u_0, u_1, \ldots, u_k \in \mathbf{I}_*$ and $z_1, z_2, \ldots, z_k \in W$ with $u_i \leq u_{i+1}$ and $u_i \leq z_i$ such that

(a) $P^\sigma_{y,z;w} = P^\sigma_{y,z;sws} + q^{2k} P^\sigma_{y',z';w'} + \sum_{0\leq i<k} q^{i+k} P^\sigma_{u_i,u_{i+1};w'}$.

(b) $P_{y,z;w} = P_{y,z;sws} + q^{2k} P_{y',z';w'} + \sum_{0\leq i<k} q^{i+k} \left(P_{u_i,u_{i+1};w'} + 2P_{u_{i+1},z_{i+1};w'}\right)$.

Proof. The twisted involutions $u_0, u_1, \ldots, u_k \in \mathbf{I}_*$ are defined as follows:

- If $k - i$ is even then let $u_i = (\cdots srsrs) \ltimes 1$, where $(\cdots srsrs)$ has i factors.
- If $k - i$ is odd then let $u_i = (\cdots rsrsr) \ltimes 1$, where $(\cdots rsrsr)$ has i factors.

Consider Figure B-3. To prove part (a), we note that Lemma 3.3.7 implies

$$P^\sigma_{y,z;w} = P^\sigma_{1,z;s\ltimes w} + q^2(P^\sigma_{s,s\ltimes z;s\ltimes w} - \delta \cdot P^\sigma_{s,s\ltimes z;rs\ltimes w}) + qP^\sigma_{1,s;s\ltimes w}$$

where $\delta = 0$ if $k = 1$ and $\delta = 1$ otherwise. If $\delta = 1$ then Proposition 3.3.9 gives $P^\sigma_{s,s\ltimes z;s\ltimes w} = P^\sigma_{s,s\ltimes z;rs\ltimes w} + q^{2(k-1)} P^\sigma_{y',z';w'}$;, by substituting this into the previous equation we get in either case

$$P^\sigma_{y,z;w} = P^\sigma_{1,z;s\ltimes w} + q^{2k} P^\sigma_{y',z';w'} + qP^\sigma_{1,s;s\ltimes w}. \tag{3.3.9}$$

If $k = 1$ then this equation coincides with the recurrence in part (a), and if $k > 1$ then by induction (with the parameters (k, r, s, y, z, w) replaced by $(k-1, s, r, 1, s, s \ltimes w)$) we may assume that

$$P^\sigma_{1,s;s\ltimes w} = P^\sigma_{1,s;rs\ltimes w} + q^{2(k-1)} P^\sigma_{u_{k-1},u_k;w'} + \sum_{0\leq i<k-1} q^{i+k-1} P^\sigma_{u_i,u_{i+1};w'}.$$

Since here $P^\sigma_{1,s;rs\ltimes w} = P^\sigma_{1,1;rs\ltimes w} = 0$ as $s \in \mathrm{Des}_L(rs \ltimes w)$, substituting the previous equation into (3.3.9) establishes part (a) for all k.

Before proving part (b) we must define the elements $z_i \in W$. For this, we first define an intermediate sequence $\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{k+1} \in W$ in the following way. Set $\widetilde{z}_{k+1} = az$ where a is given by (3.3.7), and for $i \leq k$ define $\widetilde{z}_i$ inductively by these cases:

- If $k-i$ is even then let $\widetilde{z}_i$ be the element with smaller length in the set $\{\widetilde{z}_{i+1}, \widetilde{z}_{i+1}r^*\}$.
- If $k-i$ is odd then let $\widetilde{z}_i$ be the element with smaller length in the set $\{\widetilde{z}_{i+1}, \widetilde{z}_{i+1}s^*\}$.

Note by construction that $\widetilde{z}_i r^* > \widetilde{z}_i$ if $k-i$ is even and $\widetilde{z}_i s^* > \widetilde{z}_i$ if $k-i$ is odd. Finally, define $z_1, z_2, \dots, z_k \in W$ as follows:

- If $k-i$ is even then let $z_i = \widetilde{z}_i(rsrsr\cdots)^*$ where $(rsrsr\cdots)$ has $i-1$ factors.
- If $k-i$ is odd then let $z_i = \widetilde{z}_i(srsrs\cdots)^*$ where $(srsrs\cdots)$ has $i-1$ factors.

Note that by construction $\ell(z_i) = \ell(\widetilde{z}_i) + i - 1$. Note also that since we assume $s = s^*$ and $r = r^*$, the $*$'s in the preceding bullet points are superfluous; however, these will be significant in the proof of the next proposition when we refer to the definition of u_{k-1}, u_k, and z_k.

Consider Figure B-4. To prove part (b), we note from Proposition 3.3.4 that

$$P_{y,z;w} = P_{1,z;sw} + q^k P_{a,az;aw} = P_{1,z;ws} + q^k P_{as,az;aw}.$$

Here the second equality follows from properties in Lemma 3.2.15 (in particular, the fact that $P_{y,w} = P_{ys,w}$ if $ws < w$). One checks similarly that applying (the left- and right-handed versions of) Proposition 3.3.4 to the terms on the right gives

$$P_{y,z;w} = P_{1,z;sws} + q^k P_{a,\widetilde{z}_k;aws} + q^k P_{asr,\widetilde{z}_k;aws} + q^{2k} P_{y',z';w'}. \tag{3.3.10}$$

From here, it is a straightforward exercise to check the identities

$$P_{a,\widetilde{z}_k;aws} = \sum_{i=0}^{k-1} q^i P_{u_{i+1},z_{i+1};w'} \qquad \text{and} \qquad P_{asr,\widetilde{z}_k;aws} = \sum_{i=0}^{k-1} q^i P_{u_i,z_{i+1};w'}$$

which on substitution afford the desired recurrence (since $P_{u_{i-1},z_i;w'} + P_{u_i,z_i;w'} = P_{u_{i-1},u_i;w'} + 2P_{u_i,z_i;w'}$). In particular, one obtains these identities by applying the right-handed version Proposition 3.3.4 to the left hand sides, and then applying the proposition again to the term in the result with coefficient one, repeating this process until the third index of every polynomial is w'. $\square$

For this section's final proposition, it is convenient to let $y'', z'', w'' \in W$ denote the elements

$$y'' = a \qquad \text{and} \qquad z'' = \begin{cases} azr^* & \text{if } r^* \in \mathrm{Des}_R(z) \\ az & \text{otherwise} \end{cases} \qquad \text{and} \qquad w'' = awsr^*. \tag{3.3.11}$$

We remark that in the notation of the proof of the previous proposition, the element $z'' = \widetilde{z}_k$. Thus we also have $z_k = z''(rsrsr\cdots)^*$ where $(rsrsr\cdots)$ has $k-1$ factors.

Proposition 3.3.11. Suppose $y = 1 \neq z$ and $s = s^*$ and $r \neq r^*$ (so that automatically $w \notin \langle r, s \rangle$). Then, with $u_{k-1}, u_k \in \mathrm{I}_*$ and $z_k \in W$ defined as in the proof of Proposition 3.3.10, we have

(a) $P^\sigma_{y,z;w} = P^\sigma_{y,z;sws} + q^{2k} P^\sigma_{y',z';w'} + q^{2k-1} P^\sigma_{u_{k-1},u_k;w'}$.

(b) $P_{y,z;w} = P_{y,z;sws} + q^{2k} P_{y',z';w'} + q^{2k-1}\left(P_{u_{k-1},u_k;w'} + 2P_{u_k,z_k;w'}\right) + \begin{cases} 2q^k P_{y'',z'';w''} & \text{if } k > 1 \\ 0 & \text{if } k = 1. \end{cases}$

Proof. Consider Figure B-5. To prove part (a), we first note that the argument used to show (3.3.9) in the previous proposition remains valid here and gives

$$P^\sigma_{y,z;w} = P^\sigma_{1,z;s\ltimes w} + q^{2k} P^\sigma_{y',z';w'} + q P^\sigma_{1,s;s\ltimes w}.$$

If $k = 1$ then (using the definitions in the proof of Proposition 3.3.10) we have $u_0 = 1$ and $u_1 = s$ and so this equation coincides with the desired recurrence. If $k > 1$, then since $r \neq r^*$, we can apply Proposition 3.3.8 with the parameters (k, r, s, y, z, w) replaced by $(k-1, s, r, 1, s, s \ltimes w)$ to obtain

$$P^\sigma_{1,s;s\ltimes w} = P^\sigma_{1,s;rs\ltimes w} + q^{2(k-1)} P^\sigma_{u_{k-1},u_k;w'}.$$

Here $u_{k-1} = (\cdots rsrsr) \ltimes 1$ where $(\cdots rsrsr)$ has $k-1$ factors and $u_k = (\cdots srsrs) \ltimes 1$ where $(\cdots srsrs)$ has k factors. Substituting this identity into our formula for $P^\sigma_{y,z,w}$ then establishes part (a) for all k.

To prove part (b), consider Figure B-6 and observe that it follows by successive applications of Propositions 3.3.4, exactly as in the proof of Proposition 3.3.10, that

$$P_{y,z;w} = P_{1,z;sws} + q^k P_{a,z'';aws} + q^k P_{as,z'';aws} + q^{2k} P_{y',z';w'}.$$

Note that the third term on the right $q^k P_{as,z'';aws}$ differs from the analogous equation (3.3.10) above; this is because now we have $asr^* \not< as$ since $r \neq r^*$.

Now, if $k = 1$ then $u_{k-1} = u_0 = as = 1$ and $u_k = u_1 = a = s$ and $z_k = z_1 = z''$ and $w' = aws$, so the preceding formula for $P_{y,z;w}$ coincides with the desired recurrence as $P_{u_{k-1},z_k;w'} + P_{u_k,z_k;w'} = P_{u_{k-1},u_k;w'} + 2P_{u_k,z_k;w'}$. Alternatively, if $k > 1$ then the right-handed version of Proposition 3.3.4 with the parameters (k, r, s, y, z, w) replaced by $(k-1, s, r, a, z'', aws)$ or $(k-1, s, r, as, z'', aws)$ gives

$$P_{a,z'';aws} = P_{y'',z'';w''} + q^{k-1} P_{u_k,z_k;w''} \qquad \text{and} \qquad P_{as,z'';aws} = P_{as,z'';w''} + q^{k-1} P_{u_{k-1},z_k;w''}.$$

Since $w''s < w''$ as $k > 1$, we have $P_{as,z'';w''} = P_{a,z'';w''} = P_{y'',z'';w''}$, and so substituting these two identities into our previous equation gives the desired recurrence in all cases. $\square$

Our first application of these results is the following theorem, which shows that the perhaps most natural analogues of Conjectures A and B for twisted involutions (which are false in general) do hold in the universal case.

Theorem 3.3.12. If (W, S) is a universal Coxeter system and $* \in \mathrm{Aut}(W)$ is any S-preserving involution, then the difference $P^\sigma_{y,w} - P^\sigma_{z,w}$ has nonnegative integer coefficients for all $y, z, w \in \mathbf{I}_*$ with $y \leq z$ in the Bruhat order. In particular, $P^\sigma_{y,w} \in \mathbb{N}[q]$ for each $y, w \in \mathbf{I}_*$.

Proof. The proof is by induction on $\rho(w)$, and is similar to that of Theorem 3.3.5. Fix $y, z, w \in \mathbf{I}_*$ with $y < z$. If $\rho(w) \leq 1$ then the theorem follows from Proposition 3.3.8. Suppose $\rho(w) \geq 2$, and that $s \in \mathrm{Des}_L(w)$. By Corollary 3.2.10 we may assume that $s \notin \mathrm{Des}_L(y)$ and $s \notin \mathrm{Des}_L(z)$, in which case one checks that the triple (y, z, w) satisfies the hypotheses of one of Propositions 3.3.8, 3.3.9, 3.3.10, or 3.3.11. These propositions then imply $P^\sigma_{y,z;w} \in \mathbb{N}[q]$ by induction. $\square$

Next, as the main result of this section we prove that Conjectures A′ and B′ hold for universal Coxeter systems.

Theorem 3.3.13. If (W, S) is a universal Coxeter system and $* \in \mathrm{Aut}(W)$ is any S-preserving involution, then the polynomials $P^+_{y,w} - P^+_{z,w}$ and $P^-_{y,w} - P^-_{z,w}$ have nonnegative integer coefficients for all $y, z, w \in \mathbf{I}_*$ with $y \leq w$ in the Bruhat order. In particular, $P^+_{y,w} \in \mathbb{N}[q]$ and $P^-_{y,w} \in \mathbb{N}[q]$ for each $y, w \in \mathbf{I}_*$.

Proof. Recall that the coefficients of $P_{y,z;w} \pm P^\sigma_{y,z;w}$ are all even by Proposition 3.2.16. Since $P_{y,z;w}$ and $P^\sigma_{y,z;w}$ both have positive coefficients by Theorems 3.3.5 and 3.3.12, it suffices just to show that $P_{y,z;w} - P^\sigma_{y,z;w} \in \mathbb{N}[q]$ for $y, z, w \in \mathbf{I}_*$ with $y < z$. One can prove this fact by induction on $\rho(w)$ using the same argument as in the proof of Theorem 3.3.12. The same inductive argument works because the differences between parts (a) and (b) in each of our propositions in this section involves only polynomials $P_{y,z;w} \in \mathbb{N}[q]$ and differences $P_{y,z;w} - P^\sigma_{y,z;w}$. $\square$

3.3.4 Structure constants

In the rest of this paper, we redirect our focus to Conjecture C′. Continue to assume (W, S) is a universal Coxeter system. This section describes an inductive method of computing the Laurent polynomials $(h_{x,y;z})_{x,y,z \in W}$ and $\left(h^\sigma_{x,y;z}\right)_{x \in W,\, y,z \in \mathbf{I}_*}$, which we recall from (3.1.3) are the structure constants in $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ satisfying

$$c_x c_y = \sum_{z \in W} h_{x,y;z} c_z \in \mathcal{H}_q \qquad \text{and} \qquad C_x A_y = \sum_{z \in \mathbf{I}_*} h^\sigma_{x,y;z} A_z \in \mathcal{M}_{q^2}.$$

We begin by recollecting some relevant results of Dyer [35] concerning $h_{x,y;z}$ in the universal case. The following appears as [35, Definition 3.11].

Definition 3.3.14. Assume (W, S) is a universal Coxeter system. Let $w \in W$ and $n = \ell(w)$, and suppose $s_i \in S$ such that $w = s_1 s_2 \cdots s_n$. For each integer $j \in \mathbb{Z}$, define $c(w, j) \in \mathcal{H}_q$ recursively according to the following cases:

(a) If $2 \leq j \leq n-1$ (so that $n \geq 3$) and $s_{j-1} = s_{j+1}$, then set

$$c(w,j) = c_{w'} + c(w', j-1), \qquad \text{where } w' = s_1 \cdots \widehat{s}_j \widehat{s}_{j+1} \cdots s_n.$$

Here, we write $\widehat{s}_j$ to indicate that the factor s_j is omitted.

(b) Otherwise set $c(w,j) = 0$.

The following result of Dyer [35, Theorem 3.12] gives the decomposition of the product $c_x c_y$ in terms of the Kazhdan-Lusztig basis of $\mathcal{H}_q$, and shows that the Laurent polynomials $(h_{x,y;z})_{x,y,z\in W}$ have nonnegative coefficients, and are in fact polynomials in $v + v^{-1}$ with nonnegative integer coefficients. (This latter property fails for other Coxeter systems.)

Theorem 3.3.15 (Dyer [35]). Assume (W,S) is universal. Let $x, y \in W$ and $n = \ell(x)$. Then

$$c_x c_y = \begin{cases} (v+v^{-1})\,(c_{xsy} + c(xsy,n)) & \text{if } \mathrm{Des}_R(x) = \mathrm{Des}_L(y) = \{s\} \neq \varnothing \\ c_{xy} + c(xy,n) + c(xy,n+1) & \text{otherwise.} \end{cases}$$

Remark. The preceding theorem differs from the corresponding statement in [35] as a result of our notational conventions. In [35, Theorem 3.12], Dyer writes "C_w" to denote the element of $\mathcal{H}_q$ which in our notation is written

$$\sum_{y\in W} (-v)^{\ell(w)-\ell(y)} \cdot P_{y,w}(q^{-1}) \cdot v^{-\ell(y)} \cdot t_y.$$

This element is just $(-1)^{\ell(w)} \cdot \iota(c_w)$, where ι is the $\mathcal{A}$-algebra automorphism of $\mathcal{H}_q$ with $t_w \mapsto (-q)^{\ell(w)} \cdot t_{w^{-1}}^{-1}$ for $w \in W$. (When checking this, it helps to recall $\overline{c_w} = c_w$.) This observation transforms Dyer's results into what is stated here.

Moving on to the analogous decomposition of $C_x A_y$, we have this lemma. Recall from Theorem 3.2.9 that if $s \in S$ then $C_s = q^{-1}(T_s + 1) \in \mathcal{H}_{q^2}$.

Lemma 3.3.16. Assume (W,S) is a universal Coxeter system. Suppose $s \in S$ and $w \in \mathbf{I}_*$.

(a) If $s \in \mathrm{Des}_L(w)$ then $C_s A_w = (q + q^{-1})\, A_w$.

(b) If $s \notin \mathrm{Des}_L(w)$ then

$$C_s A_w = \begin{cases} A_{sws^*} + A_{rwr^*} & \text{if } \mathrm{Des}_L(w) = \{r\} \text{ and } \mathrm{Des}_L(rwr^*) = \{s\} \\ A_{sws^*} + A_s & \text{if } w \in S \text{ and } s = s^* \\ (v+v^{-1}) A_s & \text{if } w = 1 \text{ and } s = s^* \\ A_{sws^*} & \text{otherwise.} \end{cases}$$

Proof. Part (a) is immediate from Theorem 3.2.9. If $w = 1$ then $m^\sigma(y \xrightarrow{s} 1) =$ for all $y \in \mathbf{I}_*$ with $sy < y$ so by Theorem 3.2.9 we have $C_s A_1 = (v+v^{-1})^c A_{s\ltimes 1}$ where $c = \delta_{s,s^*}$. This proves part (b) when $w = 1$.

For the remaining cases, assume $w \neq 1$ and $\mathrm{Des}_L(w) = \{r\} \neq \{s\}$. Combining Theorem 3.2.9 and Lemma 3.3.6 gives $C_s A_w = A_{s \ltimes 1} + \sum_{y \in X} A_y$, where $X \subset \mathbf{I}_*$ is the subset which contains s if $s = s^*$ and $w = r \in S$, and which contains rwr^* if $rwr^* \in \mathbf{I}_*$ and $\mathrm{Des}_L(rwr^*) = \{s\}$. Since rwr^* always belongs to $\mathbf{I}_*$ and since $\mathrm{Des}_L(rwr^*) = \{s\}$ implies $w \notin S$, the set X contains at most one element and our formula $C_s A_w = A_{s \ltimes 1} + \sum_{y \in X} A_y$ reduces to the cases in the lemma. $\square$

We now make this definition, after Definition 3.3.14.

Definition 3.3.17. Assume (W, S) is a universal Coxeter system. Let $w \in \mathbf{I}_*$ and $n = \rho(w)$, and suppose $s_i \in S$ such that $w = s_1 \ltimes s_2 \ltimes \cdots \ltimes s_n \ltimes 1$. For each integer $j \in \mathbb{Z}$, define $A(w, j) \in \mathcal{M}_{q^2}$ recursively according to the following cases:

(a) If $2 \leq j \leq n-1$ (so that $n \geq 3$) and $s_{j-1} = s_{j+1}$, then set

$$A(w, j) = A_{w'} + A(w', j-1), \qquad \text{where } w' = s_1 \ltimes \cdots \ltimes \widehat{s}_j \ltimes \widehat{s}_{j+1} \ltimes \cdots \ltimes s_n.$$

Here, we again write $\widehat{s}_j$ to indicate that the factor s_j is omitted.

(b) If $j = n$ and $n \geq 2$ and $\{s_{n-1}, s_n\} \subset \mathbf{I}_*$, then set

$$A(w, j) = A_{w'} + A(w', n-1), \qquad \text{where } w' = s_1 \ltimes \cdots \ltimes s_{n-1}.$$

(c) Otherwise set $A(w, j) = 0$.

Using this notation, the following analog of Theorem 3.3.15 now decomposes the product $C_x A_y$ in terms of the distinguished basis $(A_z)_{z \in \mathbf{I}_*}$ of $\mathcal{M}_{q^2}$. This result shows that the Laurent polynomials $\left(h^\sigma_{x,y;z}\right)_{x \in W, y, z \in \mathrm{I}}$ have nonnegative coefficients, but in contrast to our previous situation, $h^\sigma_{x,y;z}$ does not typically have nonnegative coefficients when written as a polynomial in $v + v^{-1}$.

Theorem 3.3.18. Assume (W, S) is universal. If $x \in W$ and $y \in \mathbf{I}_*$ and $n = \ell(x)$, then

$$C_x A_y = \begin{cases} (v + v^{-1})\,(A_{x \ltimes 1} + A(x \ltimes 1, n)) & \text{if } x \neq 1 \text{ and } y = 1 \text{ and } \mathrm{Des}_R(x) \subset \mathbf{I}_* \\ (q + q^{-1})\,(A_{xs \ltimes y} + A(xs \ltimes y, n)) & \text{if } \mathrm{Des}_R(x) = \mathrm{Des}_L(y) = \{s\} \neq \varnothing \\ A_{x \ltimes y} + A(x \ltimes y, n) + A(x \ltimes y, n+1) & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to that of [35, Theorem 3.12], and proceeds by induction on n. If $n \in \{0, 1\}$ then the theorem reduces to Lemma 3.3.16 (checking this fact is a healthy exercise which we leave to the reader), so we may assume $\ell(x) \geq 2$ and that

$$x = x'rs \qquad \text{for some } x' \in W \text{ and } r, s \in S \text{ with } \ell(x') = \ell(x) - 2.$$

It follows from Theorem 3.3.15 (noting that the $\mathbb{Z}$-linear map $\mathcal{H}_q \to \mathcal{H}_{q^2}$ with $v^n \mapsto q^n$ and $t_w \mapsto T_w$ is a ring embedding with $c_w \mapsto C_w$) that

$$C_x = \begin{cases} C_{xs}C_s - C_{x'} & \text{if } \mathrm{Des}_R(x') = \{s\} \\ C_{xs}C_s & \text{otherwise.} \end{cases} \tag{3.3.12}$$

It suffices to consider the following five cases, exactly one of which must occur:

(i) Suppose $y = 1$. Then $A(x \ltimes y, n+1) = 0$ and so we wish to show that $C_x A_1 = (v+v^{-1})^c \cdot (A_{v\ltimes 1} + A(v \ltimes 1, n))$ where $c = |\{s\} \cap \mathbf{I}_*|$.

(ii) Suppose $s \in \mathrm{Des}_L(y)$. We then wish to show that $C_x A_y = (q+q^{-1})\,(A_{xs\ltimes y} + A(xs \ltimes y, n))$.

(iii) Suppose $y \in S$ and $s \notin \mathrm{Des}_L(y)$ and $s = s^*$. Then $A(x \ltimes y, n+1) = A_{x\ltimes 1} + A(x \ltimes 1, n)$ and so we wish to show $C_x A_y = A_{x\ltimes y} + A(x \ltimes y, n) + A_{x\ltimes 1} + A(x \ltimes 1, n)$.

(iv) Suppose $\rho(y) = 1$ and $s \notin \mathrm{Des}_L(y)$ but either $y \notin S$ or $s \neq s^*$. Then $A(x \ltimes y, n+1) = 0$ and so we wish to show $C_x A_y = A_{x\ltimes y} + A(x \ltimes y, n)$.

(v) Suppose $\rho(y) \geq 2$ and $s \notin \mathrm{Des}_L(y)$. We then want $C_x A_y = A_{x\ltimes y} + A(x \ltimes y, n) + A(x \ltimes y, n+1)$.

The proof of each case is similar, and involves substituting (3.3.12) for C_x and then applying Lemma 3.3.16 and induction. Case (v) is the most complicated, but its proof is nearly the same as that of [35, Lemma 6.2]. We demonstrate (i) as an example and leave the rest to the reader.

For case (i), suppose $y = 1$ and let $c = |\{s\} \cap \mathbf{I}_*|$; recall that $\mathrm{Des}_R(x) = \{s\}$ by assumption. If $\mathrm{Des}_R(x') \neq \{s\}$ then $C_x = C_{x'r}C_s$ by (3.3.12) and $A(x \ltimes 1, n-1) = 0$, in which case by Lemma 3.3.16 and then induction we get

$$\begin{aligned} C_x A_1 &= C_{x'r} C_s A_1 \\ &= (v+v^{-1})^c \cdot C_{x'r} A_{s\ltimes 1} \\ &= (v+v^{-1})^c \cdot (A_{x\ltimes 1} + \underbrace{A(x \ltimes 1, n-1)}_{=0} + A(x \ltimes 1, n)), \end{aligned}$$

which is what we want to show. Alternatively, if $\mathrm{Des}_R(x') = \{s\}$ then $C_x = C_{x'r}C_s - C_{x'}$ by (3.3.12) and $A(x \ltimes 1, n-1) = A_{x'\ltimes 1} + A(x' \ltimes 1, n-2)$, so by induction $C_{x'}A_1 = (v+v^{-1})^c \cdot A(x \ltimes 1, n-1)$. In this case by Lemma 3.3.16 and then induction we have

$$\begin{aligned} C_x A_1 &= (C_{x'r}C_s - C_{x'})A_1 \\ &= (v+v^{-1})^c \cdot C_{x'r} A_{s\ltimes 1} - C_{x'}A_1 \\ &= (v+v^{-1})^c \cdot (A_{x\ltimes 1} + A(x \ltimes 1, n)) + \underbrace{(v+v^{-1})^c \cdot A(x \ltimes 1, n-1) - C_{x'}A_1}_{=0} \end{aligned}$$

which is again the desired formula. $\square$

Wrapping up, we have this corollary immediately from Theorems 3.3.15 and 3.3.18.

Corollary 3.3.19. If (W, S) is a universal Coxeter system then each of the families

$$(h_{x,y;z})_{x,y,z\in W} \qquad \text{and} \qquad (\widetilde{h}_{x,y;z})_{x,y,z\in W} \qquad \text{and} \qquad (h^{\sigma}_{x,y;z})_{x\in W,\, y,z\in \mathbf{I}_*}$$

consists of Laurent polynomials in $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with nonnegative coefficients.

3.3.5 Proof of the positivity conjecture for universal structure constants

As previously, (W, S) is a universal Coxeter system with a fixed S-preserving involution $* \in \mathrm{Aut}(W)$. We devote this final section to proving Conjecture C′ for universal Coxeter systems–i.e., that the Laurent polynomials $h^{\pm}_{x,y;z} = \frac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^{\sigma}_{x,y;z}\right)$ defined in Section 3.1.5 always have nonnegative coefficients.

To begin, it is useful to recall the following notation from the proof of Proposition 3.2.17. Given $w \in W$, let $w^\dagger = w^{*-1}$ and more generally let $h \mapsto h^\dagger$ denote the $\mathcal{A}$-linear map $\mathcal{H}_q \to \mathcal{H}_q$ with $(t_w)^\dagger = t_{w^\dagger}$ for $w \in W$. Observe that $\dagger$ is an anti-automorphism (of $\mathcal{A}$-algebras) and that $(c_w)^\dagger = c_{w^\dagger}$ for all $w \in W$ by Lemma 3.2.15. We now state two technical lemmas associated with Definitions 3.3.14 and 3.3.17.

Lemma 3.3.20. Assume (W, S) is a universal Coxeter system. Suppose $u, t \in W$ such that $\ell(u \ltimes t) = 2\ell(u) + \ell(t)$ and $\ell(t) \in \{0, 1\}$ and $t = t^*$. Fix an integer $n \leq \ell(u)$. Then there exists a unique integer $k \geq 0$ and a unique sequence of elements

$$u = u_0 > u_1 > \cdots > u_k$$

in W (descending with respect to the Bruhat order), such that $c(ut, n) = \sum_{i=1}^{k} c_{u_i t}$. This sequence has the following additional properties:

(a) For each $0 \leq i \leq k$ we have $\ell(u_i \ltimes t) = 2\ell(u_i) + \ell(t)$.

(b) $c(utw, n) = \sum_{i=1}^{k} c_{u_i tw} + c(u_k tw, n-k)$ for any $w \in W$ with $\ell(utw) = \ell(u) + \ell(t) + \ell(w)$.

(c) $A(u \ltimes t, n) = \sum_{i=1}^{k} A_{u_i \ltimes t} + \delta \cdot A(u_k \ltimes t, n-k)$ where $\delta = \begin{cases} 1 & \text{if } n - k = \ell(u_k) + 1 \\ 0 & \text{otherwise.} \end{cases}$

Remark. Note that we may have $k = 0$ in this lemma; this indicates that $c(ut, n) = 0$. In this case the sums $\sum_{k=1}^{n}$ are considered to be zero, and we automatically have $\delta = 0$ in part (c) since $n < \ell(u_0) + 1$ by hypothesis.

Proof. We sketch the proof of this lemma, as everything derives from the definitions in a straightforward way by induction on $\ell(u)$. The existence of the sequence of elements $u = u_0 > u_1 > \cdots > u_k$ follows from Definition 3.3.14 by inspection, as does property (a). Property (b) holds because the first $k + 1$ terms in the expansion of $c(utw, n)$, which one gets by applying Definition 3.3.14 successively, depend only on the first $n+k$ factors in the unique reduced expression for utw. Part (c) follows from the fact that the same sequence of elements in S gives both the unique reduced expression for ut and the unique reduced $\mathbf{I}_*$-expression for $u \ltimes t$. Noting this and comparing Definitions 3.3.14 and 3.3.17 (while remembering $n \leq \ell(u)$), we deduce that $A(u \ltimes t, n) = \sum_{i=1}^{k} A_{u_i \ltimes t} + A(u_k \ltimes t, n-k)$, and that $A(u_k \ltimes t, n-k)$ is zero unless $n-k = \rho(u_k \ltimes t)$. The latter condition is equivalent to having both $\ell(t) = 1$ and $n-k = \ell(u_k)+1$; however, if $\ell(t) = 0$ while $n-k = \ell(u_k)+1$ then $A(u_k \ltimes t, n-k)$ is zero by definition. $\square$

In what follows, we let $\Phi : \mathcal{M}_{q^2} \to \mathcal{H}_q$ denote the $\mathcal{A}$-linear map with $A_w \mapsto c_w$ for $w \in \mathbf{I}_*$.

Lemma 3.3.21. Assume (W,S) is a universal Coxeter system. Suppose $x \in W$ and $s \in S \cap \mathbf{I}_*$ such that $s \notin \mathrm{Des}_R(x)$. If $n = \ell(x)$, then

$$c(x \ltimes s, n+1) = \Phi\left(A(x \ltimes s, n+1)\right).$$

Proof. If $x = 1$ or if $\mathrm{Des}_L(x) \not\subset \mathbf{I}_*$ then the lemma holds since $c(x \ltimes s, n+1)$ and $A(x \ltimes s, n+1)$ are both zero. Assume $x \neq 1$ so that $x = x'r$ for some $y \in W$ and $r \in S \cap \mathbf{I}_*$ with $\ell(x') = \ell(x)-1$. Then $c(x \ltimes s, n+1) = c_{x' \ltimes r} + c(x' \ltimes r, n)$ and $A(x' \ltimes s, n+1) = A_{x' \ltimes r} + A(x' \ltimes r, n)$, so the lemma follows by induction on n. $\square$

We may now state our final result, which establishes Conjecture C′ in the universal case.

Theorem 3.3.22. If (W,S) is a universal Coxeter system and $* \in \mathrm{Aut}(W)$ is any S-preserving involution, then the Laurent polynomials $h^{\pm}_{x,y,z}$ defined by (3.1.4) have nonnegative integer coefficients for all $x \in W$ and $y, z \in \mathbf{I}_*$.

Proof. Let $\mathcal{H}_q^+ = \mathbb{N}[v, v^{-1}]\text{-span}\{c_w : w \in W\}$ denote set of elements in $\mathcal{H}_q$ whose coefficients with respect to the Kazhdan-Lusztig basis $(c_w)_{w \in W}$ have nonnegative coefficients. Note that $\mathcal{H}_q^+$ is preserved by $\dagger$ since $(c_w)^\dagger = c_{w^\dagger}$.

Let $x \in W$ and $y \in \mathbf{I}_*$. By Theorems 3.3.15 and 3.3.18 we know that $c_x c_y c_{x^\dagger} \in \mathcal{H}_q^+$ and $\Phi(C_x A_y) \in \mathcal{H}_q^+$, and if we write $c_x c_y c_{x^\dagger} \pm \Phi(C_x A_y) = \sum_{z \in W} p_z^{\pm} c_z$ for some polynomials $p_z^{\pm} \in \mathbb{Z}[v, v^{-1}]$, then by definition $h^{\pm}_{x,y;z} = \frac{1}{2} p_z^{\pm}$ for each $z \in \mathbf{I}_*$. It is thus immediate that every $h^{+}_{x,y;z}$ has nonnegative coefficients, and to prove the theorem it is enough to show that

$$c_x c_y c_{x^{*-1}} - \Phi(C_x A_y) \in \mathcal{H}_q^+. \tag{3.3.13}$$

To this end, let $n = \ell(x)$. If $n = 0$ then (3.3.13) automatically holds since the left hand side is zero, so we may assume $n \geq 1$. There are three cases, which we consider in turn:

(a) Suppose $y = 1$. If $s \neq s^*$ then by Theorems 3.3.15 and 3.3.18 we have

$$c_x c_y c_{x^\dagger} = c_x c_{x^\dagger} = c_{x \ltimes 1} + c(x \ltimes 1, n) + c(x \ltimes 1, n+1)$$

while $C_x A_y = A_{x \ltimes 1}$. Certainly $c(x \ltimes 1, n) + c(x \ltimes 1, n+1) \in \mathcal{H}_q^+$ so (3.3.13) holds.

On the other hand if $s = s^*$ then we have

$$c_x c_y c_{x^\dagger} = c_x c_{x^\dagger} = (v + v^{-1}) \cdot (c_{x \ltimes 1} + c(x \ltimes 1, n))$$

while $C_x A_y = (v + v^{-1}) \cdot (A_{x\ltimes 1} + A(x \ltimes 1, n)$. Since $s = s^*$ we have $\Phi\left(A(x \ltimes 1, n)\right) = c(x \ltimes 1, n)$ by Lemma 3.3.21, so (3.3.13) again holds.

(b) Suppose $y \neq 1$ and $\mathrm{Des}_L(y) \neq \mathrm{Des}_R(x)$. If $y \in S$ then $c(xy, n+1) = 0$ and so we have

$$\begin{aligned} c_x c_y c_{x^\dagger} &= \left(c_{xy} + c(xy, n)\right) c_{x^\dagger} \\ &= c_{x\ltimes y} + c(xy, n) c_{x^\dagger} + c(x \ltimes y, n+1) + \left(\text{an element of } \mathcal{H}^+\right) \end{aligned}$$

while if $\ell(y) \geq 2$ then

$$\begin{aligned} c_x c_y c_{x^\dagger} &= \left(c_{xy} + c(xy, n) + c(xy, n+1)\right) c_{x^\dagger} \\ &= c_{x\ltimes y} + c(xy, n) c_{x^\dagger} + c(xy, n+1) c_{x^\dagger} + \left(\text{an element of } \mathcal{H}_q^+\right), \end{aligned}$$

since in either case $x \ltimes y = xyx^\dagger$. Consulting Theorem 3.3.18, we conclude that (3.3.13) will hold if we can prove the following claims:

(b1) If $\ell(y) \geq 1$ then we have $c(xy, n) c_{x^\dagger} - \Phi\left(A(x \ltimes y, n)\right) \in \mathcal{H}_q^+$.

(b2) If $\ell(y) \geq 2$ then we have $c(xy, n+1) c_{x^\dagger} - \Phi\left(A(x \ltimes y, n+1)\right) \in \mathcal{H}_q^+$.

To prove (b1), write $y = ztz^\dagger$ where $z, t \in W$ such that $\ell(t) \leq 1$ and $t = t^*$ and $\ell(ztz^\dagger) = 2\ell(z) + \ell(t)$. Now let $u = xz$ and let $u = u_0 > u_1 > \cdots > u_k$ be the corresponding sequence of elements in W described in Lemma 3.3.20. Using part (b) of Lemma 3.3.20 and the fact that $\dagger$ is an anti-automorphism, we then have

$$\begin{aligned} c(xy, n) c_{x^\dagger} = c(utz^\dagger, n) c_{x^\dagger} &= \left(\sum_{i=1}^{k} c_{u_i t z^\dagger} + c(u_k t z^\dagger, n-k) \right) c_{x^\dagger} \\ &= \left(c_x \sum_{i=1}^{k} c_{zt(u_i)^\dagger} \right)^\dagger + c(u_k t z^\dagger, n-k) c_{x^\dagger} \\ &= \left(\sum_{i=1}^{k} c(ut(u_i)^\dagger, n) \right)^\dagger + \left(\text{an element of } \mathcal{H}_q^+\right). \end{aligned}$$

Here, the last equality follows by applying Theorem 3.3.15 to the terms in the sum on the second line. Since each $c(ut(u_i)^\dagger, n) = \sum_{j=1}^k c_{u_j t(u_i)^\dagger} + \left(\text{an element of } \mathcal{H}_q^+\right)$ by parts (a) and (b) of Lemma 3.3.20, after collecting terms in $\mathcal{H}_q^+$ we get

$$c(xy, n) c_{x^\dagger} = \sum_{i=1}^{k} c_{u_i \ltimes t} + c(u_k \ltimes t, n-k) + \left(\text{an element of } \mathcal{H}_q^+\right).$$

By part (c) of Lemma 3.3.20, however, we have $A(x \ltimes y, n) = \sum_{i=1}^{k} A_{u_i \ltimes t} + \delta \cdot A(u_k \ltimes t, n - k)$, where $\delta \in \{0, 1\}$ is zero unless $n - k = \ell(x_k) + 1$. If $\delta = 1$ then $A(u_k \ltimes t, n-k) = 0$ unless $t \in S \cap \mathbf{I}_*$, and so (b1) follows by Lemma 3.3.21.

One proves (b2) by replacing n with $n+1$ in the preceding argument. Our applications of Lemma 3.3.20 remain valid after this substitution because we assume $\ell(y) \geq 2$, which implies $\ell(u) \geq 1$ and in turn $n + 1 \leq \ell(u)$.

(c) Suppose $y \neq 1$ and $\mathrm{Des}_R(x) = \mathrm{Des}_L(y) = \{s\}$ for some $s \in S$. Again use Theorems 3.3.15 and 3.3.18 to expand the products $(c_x c_y) c_{x^\dagger}$ and $C_x A_y$. On comparing the resulting terms (while again noting Lemma 3.3.21) one finds that to prove (3.3.13) it is enough to show

$$(v + v^{-1}) \cdot c(xsy, n) c_{x^\dagger} = (q + q^{-1}) \cdot \Phi\left(A(xs \ltimes y, n)\right) + \left(\text{an element of } \mathcal{H}_q^+\right). \quad (3.3.14)$$

If $y = s \in \mathbf{I}_*$ then by Theorem 3.3.15 we have $c(xsy, n) = c_{xys} + \left(\text{an element of } \mathcal{H}_q^+\right)$ and in turn $c(xsy, n) c_{x^\dagger} = (v + v^{-1}) \cdot c(xs \ltimes y, n) + \left(\text{an element of } \mathcal{H}_q^+\right)$. Thus (3.3.14) holds in this case by Lemma 3.3.21.

If $\ell(y) \geq 2$ then the proof of (3.3.14) is similar to the arguments in part (b). A sketch goes as follows. First write $y = ztz^\dagger$ where $z, t \in W$ such that $\ell(t) \leq 1 \leq \ell(z)$ and $t^* = t$ and $\ell(ztz^\dagger) = 2\ell(z) + \ell(t)$. Let $u = xsz$ and let $u = u_0 > u_1 > \cdots > u_k$ be the corresponding sequence of elements in W. By now rewriting $c(xsy, n)$ in terms of the elements u_i and expanding various products using the properties in Lemma 3.3.20, one obtains

$$c(xsy, n) c_{y^\dagger} = (v + v^{-1}) \left(\sum_{i=1}^{k} c_{u_i \ltimes t} + c(u_k \ltimes t, n - k) \right) + \left(\text{an element of } \mathcal{H}_q^+\right).$$

Comparing this to the formula for $A(xs \ltimes y, n)$ in part (c) of Lemma 3.3.20 then shows that (3.3.14) holds, as a consequence of Lemma 3.3.21.

$\square$

3.4 Positivity results for finite Coxeter systems

In this section we consider Coxeter systems (W, S) which are finite. Here we prove Conjectures A′, B′, and C′ for Coxeters of rank ≤ 5, either by elementary methods (in the dihedral case) or by computer calculations (in ranks three, four, and five).

3.4.1 Reduction to the irreducible case

For the moment let (W, S) be any Coxeter system. Recall that if there exists a subset $S' \subset S$ such that every $s' \in S'$ commutes with every $s'' \in S'' \overset{\text{def}}{=} S \setminus S'$, then there is an isomorphism

$W \cong W' \times W''$ where W' and W'' are the subgroups respectively generated by S' and S''. If the only such sets $S' \subset S$ are S are $\varnothing$, then the Coxeter system (W, S) is *irreducible.* If S' is nonempty and the Coxeter system (W', S') is irreducible then we say that it is an *irreducible factor* of (W, S).

Lemma 3.4.1. Let (W, S) be a Coxeter system with an S-preserving involution $* \in \mathrm{Aut}(W)$. Let $S' \subset S$ be a subset preserved by $*$, set $S'' = S \setminus S'$, and suppose every $s' \in S'$ commutes with every $s'' \in S''$. Write

$$W' = \langle S' \rangle \qquad \text{and} \qquad W'' = \langle S'' \rangle$$

for the subgroups generated by S' and S'', and let $\mathbf{I}'_* = W' \cap \mathbf{I}_*$ and $\mathbf{I}''_* = W'' \cap \mathbf{I}_*$.

(a) For each $w \in W$ there are unique elements in W' and W'', which we denote w' and w'' respectively, such that $w = w'w''$.

(b) Moreover, if $w \in W$ then $w \in \mathbf{I}_*$ if and only if $w' \in \mathbf{I}'_*$ and $w'' \in \mathbf{I}''_*$.

(c) If $y, w \in W$ then $y \leq w$ if and only if $y' \leq w'$ and $y'' \leq w''$.

(d) $P_{y,w} = P_{y',w'}P_{y'',w''}$ and $h_{x,y;z} = h_{x',y';z'}h_{x'',y'';z''}$ for all $w, x, y, z \in W$.

(e) $P^\sigma_{y,w} = P^\sigma_{y',w'}P^\sigma_{y'',w''}$ and $h^\sigma_{x,y;z} = h^\sigma_{x',y';z'}h^\sigma_{x'',y'';z''}$ for all $x \in W$ and $w, y, z \in \mathbf{I}_*$.

Remark. In part (d), we may identify $P_{y',w'}$ and $P_{y'',w''}$ with Kazhdan-Lusztig polynomials of the Coxeter systems (W', S') and (W'', S''). In part (e), likewise, we may identify $P^\sigma_{y',w'}$ and $P^\sigma_{y'',w''}$ with polynomials attached to $(W', S', *)$ and $(W'', S'', *)$. Similar identificiation apply to the structure constants.

Proof. Parts (a) and (c) follows from basic group theory and properties of the Bruhat order (see [16, Exercise 2.3]), and part (b) follows from (a) since $w = w'w'' = w''w'$. Since in the Hecke algebras $\mathcal{H}_q$ and $\mathcal{H}_{q^2}$ we have $t_w = t_{w'}t_{w''}$ and $T_w = T_{w'}T_{w''}$ for all $w \in W$, parts (d) and (e) follow from the uniqueness specified in Theorem-Definitions 3.1.1and 3.1.2. $\square$

The following proposition shows that to establish Conjectures A′, B′, C′ for all Coxeter systems, it suffices to verify Conjectures A, B, C, A′, B′, C′ for all triples $(W, S, *)$ with (W, S) an irreducible Coxeter system. Table A.16 displays an irredundant list of such triples with (W, S) finite.

Proposition 3.4.2. Suppose Conjectures A and A′ (respectively, B and B′, or C and C′) hold with respect to any choice of involution for every irreducible factor of a Coxeter system (W, S) with finite rank. Then Conjectures A and A′ (respectively, B and B′, or C and C′) hold for (W, S) with respect to any choice of involution.

Proof. We prove this result by induction on the rank of (W, S), which is assumed to be finite. Let $S' \subset S$ and $S'' = S \setminus S'$ be subsets satisfying the hypotheses in Lemma 3.4.1. Then, writing $W' = \langle S' \rangle$ and $W'' = \langle S'' \rangle$, we have in the notation of Lemma 3.4.1 that

$$P^+_{y,w} = P^+_{y',w'} P^+_{y'',w''} + P^-_{y',w'} P^-_{y',w'} \quad \text{and} \quad P^-_{y,w} = P^+_{y',w'} P^-_{y'',w''} + P^-_{y',w'} P^+_{y',w'}$$

for $y, w \in \mathbf{I}_*$, and that

$$P^\pm_{y,w} - P^\pm_{z,w} = \tfrac{1}{2}(P^\pm_{y',w'} + P^\pm_{z',w'})(P^\pm_{y'',w''} - P^\pm_{z'',w''}) + \tfrac{1}{2}(P^\pm_{y',w'} - P^\pm_{z',w'})(P^\pm_{y'',w''} + P^\pm_{z'',w''})$$

for $y, z, w \in \mathbf{I}_*$ with $y \leq z$, and that

$$h^+_{x,y;z} = h^+_{x',y';z'} h^+_{x'',y'';z''} + h^-_{x',y';z'} h^-_{x'',y'';z''} \quad \text{and} \quad h^-_{x,y;z} = h^+_{x',y';z'} h^-_{x'',y'';z''} + h^-_{x',y';z'} h^+_{x'',y'';z''}$$

for $x \in W$ and $w, y, z \in \mathbf{I}_*$. Suppose Conjecture A′ (respectively B′, C′) holds for all irreducible factors of (W, S). If S' and $S"'$ are both proper subsets of S, then we may assume by induction that Conjecture A′ (respectively B′, C′) holds for $(W', S', *)$ and $(W'', S'', *)$, in which case the same conjecture holds for $(W, S, *)$ as a consequence of the first (respectively second, third) identities above.

Suppose on the other hand that S' and S'' cannot be chosen to both be proper subsets of S. Then either (W, S) is irreducible, or there are disjoint subsets $S', S'' \subset S$ with $S = S' \cup S''$ such that $\{s^* : s \in S'\} = S''$ and such that the Coxeter systems (W', S') and (W'', S'') are both irreducible, where $W' = \langle S' \rangle$ and $W'' = \langle S'' \rangle$. In the first case the proposition holds by hypothesis. In this second case, $W' \cong W''$ and we may identify the triple $(W, S, *)$ with a Coxeter system with involution of the form in Proposition 3.2.12. From parts (c) and (d) of that proposition it follows that Conjectures A′, B′. and C′ are respectively equivalent to:

(a) $P_{y,w}(q)^2 \pm P_{y,w}(q^2) \in \mathbb{N}[q]$ for all $y, w \in W'$.

(b) $(P_{y,w}(q)^2 - P_{z,w}(q)^2) \pm (P_{y,w}(q^2) - P_{z,w}(q^2)) \in \mathbb{N}[q]$ for all $y, z, w \in W'$ with $y \leq z$.

(c) $h_{x,y;z}(v)^2 \pm h_{x,y;z}(v^2) \in \mathbb{N}[v, v^{-1}]$ for all $x, y, z \in W'$.

The proposition holds in this case because elementary properties of polynomials show that statements (a), (b), and (c) are implied respectively by Conjectures A, B, and C. For example, (b) follows from Conjecture B because if $f, g \in \mathbb{N}[x]$ such that $f - g \in \mathbb{N}[x]$ then

$$(f^2 - g^2) \pm (f(x^2) - g(x^2)) = \underbrace{(f - g)^2 \pm (f(x^2) - g(x^2))}_{\in \mathbb{N}[x]} + \underbrace{2g(f - g)}_{\in \mathbb{N}[x]} \in \mathbb{N}[x].$$

□

Since Conjectures A, B, and C hold for all finite Coxeter systems (see [33]), we have this corollary.

Corollary 3.4.3. Suppose Conjecture A′ (respectively, B′, C′) holds with respect to any choice of involution for every irreducible factor of a *finite* Coxeter system (W, S). Then Conjecture A′ (respectively, B′, C′) holds for (W, S) with respect to any choice of involution.

3.4.2 Positivity results for dihedral groups

Suppose (W, S) is of type $I_2(m)$ where $m \in \{3, 4, 5, \dots\}$ is finite. Throughout this section we let $S = \{s, t\}$; then W is the group of order $2m$, generated by S subject only to the relations $s^2 = t^2 = (st)^m = 1$. Define for $i = 1, 2, \dots$ the elements

$$[s, i) = \underbrace{ststs\cdots}_{i \text{ factors}} \qquad \text{and} \qquad [t, i) = \underbrace{tstst\cdots}_{i \text{ factors}}$$

and also

$$(i, s] = \underbrace{\cdots ststs}_{i \text{ factors}} \qquad \text{and} \qquad (i, t] = \underbrace{\cdots tstst}_{i \text{ factors}}.$$

The distinct elements of W are then precisely

$$1, \qquad [s,1), \dots, [s, m-1), \qquad [t,1), \dots, [t, m-1), \qquad \text{and} \qquad [s,m) = [t,m)$$

or alternatively

$$1, \qquad (1,s], \dots, (m-1, s], \qquad (1,t], \dots, (m-1, t], \qquad \text{and} \qquad (m,s] = (m,t].$$

The elements just listed are all reduced expressions, and the Bruhat order on W has the simple description that $y < w$ if and only if $\ell(y) < \ell(w)$. Using these facts and Corollary 3.2.15, the following well-known proposition is a straightforward exercise by induction.

Proposition 3.4.4. Suppose (W, S) is of dihedral type $I_2(m)$, with $m \in \{3, 4, \dots\}$ finite. Then $P_{y,w} = 1$ for all $y, w \in W$ with $y \leq w$.

Proof. The proposition certainly holds if $w = 1$ since $P_{w,w} = 1$ for all $w \in W$. Fix $w \neq 1$ and $y \leq w$, and suppose $P_{y',w'} = 1$ whenever we have $y' \leq w' < w$.

Choose $s \in \mathrm{Des}_L(w)$; since $P_{y,w} = P_{sy,w}$ and $sy \leq w$ if $sy > y$, we may assume $s \in \mathrm{Des}_L(y)$ as well. In this case $sy \leq sw$, so by the second part of Corollary 3.2.14 we must show that

$$qP_{y,sw} = \sum_{\substack{z \in W; sz < z \\ y \leq z < w}} v^{\ell(w)-\ell(z)} \cdot \mu(z, sw).$$

Observe that by hypothesis $\mu(z, sw)$ is 1 if $\ell(z) = \ell(w) - 2$ and is 0 otherwise. Thus it suffices to show that $y \leq sw$ if and only if there is a unique $z \in W$ such that $y \leq z$ and $sz < z$ and $\ell(z) = \ell(w) - 2$. To this end, note that if $y \leq sw$ then since $sy < y$ we must have $sy < y < sw < w$, in which case $\ell(w) \geq 3$ so the desired z is given uniquely by tsw. On the other hand, $y \not\leq sw$ only occurs if $\ell(y) = \ell(w) - 1 = \ell(sw)$ in which case clearly no such z exists, as $\ell(z)$ is both equal to $\ell(w) - 2$ and bounded below by $\ell(y)$. $\square$

There exist exactly two S-preserving involution $*$ of W: either $*$ is the identity automorphism or $*$ is the automorphism interchanging s and t. In the trivial case, $\mathbf{I}_*$ consists of the identity, the longest element, and all elements of W of odd length, namely

$$1, \qquad [s,1),\ [s,3),\ [s,5),\ \dots \qquad [t,1),\ [t,3),\ [t,5),\ \dots \qquad \text{and} \qquad [s,m) = [t,m).$$

In the nontrivial case $\mathbf{I}_*$ consists of the longest element and all elements of even length, i.e.,

$$1, \qquad [s,2),\ [s,4),\ [s,6),\ \dots \qquad [t,2),\ [t,4),\ [t,6),\ \dots \qquad \text{and} \qquad [s,m) = [t,m).$$

In the following three lemmas we assume (W,S) is of dihedral type $I_2(m)$, with $m \in \{3,4,\dots\}$ finite, and that $* \in \mathrm{Aut}(W)$ is either S-preserving involution. We also write $w_0 = [s,m) = [t,m)$ for the longest element in W.

Lemma 3.4.5. Every $w \in W$ has a unique reduced expression except w_0, which has exactly two reduced expressions given by $stst\cdots$ and $tstst\cdots$ (each with m factors).

Proof. Certainly w_0 has at least two reduced expressions and every other element has at least one. As there are only $2m+1$ distinct (possibly empty) expressions of length at most m involving s and t without equal adjacent letters, and since $|W| = 2m$, we may replace "at least" in the previous sentence by "exactly." $\square$

Lemma 3.4.6. Suppose $r \in S$ and $w \in \mathbf{I}_*$ such that $rw = wr^*$.

(a) If m is odd and $*$ is trivial then $w \in \{1, r\}$.

(b) If m is odd and $*$ is nontrivial then $w \in \{w_0, rw_0\}$.

(c) If m is even and $*$ is trivial and $w \in \{w_0, rw_0\}$.

(d) The case that m is even and $*$ is nontrivial cannot occur.

Proof. Since $rw = wr^*$ if and only if $rw' = w'r^*$ where $w' = r \ltimes w$, we may assume $rw > w$. If $\ell(w) = 0$ then $sw = ws^*$ if and only if $s = s^*$. If $0 < \ell(w) < m-1$ then it follows from the previous lemma that $rw \neq wr^*$. It remains only to consider the case when $\ell(w) = m-1$ (since when $\ell(w) = m$ it cannot hold that $rw > w$). In this situation $rw = wr^*$ if and only if $w_0 = rw = r(rw)r^* = rw_0r^*$. One checks that this holds precisely when $m = \ell(w_0)$ is odd and $*$ is nontrivial or m is even and $*$ is trivial. $\square$

Lemma 3.4.7. Suppose $y, w \in \mathbf{I}_*$ then $\ell(w) - \ell(y) = 1$.

(a) If m is odd and $*$ is trivial then $y = 1$ and $w \in S$.

(b) If m is odd and $*$ is nontrivial then $y \in \{sw_0, tw_0\}$ and $w = w_0$.

(c) If m is even and $*$ is trivial then $y \in \{sw_0, tw_0\}$ and $w = w_0$, or $y = 1$ and $w \in S$.

(d) The case that m is even and $*$ is nontrivial cannot occur.

Proof. The claims here following by inspecting the lists of elements in $\mathbf{I}_*$ given before Lemma 3.4.5, noting that the elements $[s,i)$ and $[t,i)$ have length i when $i \leq m$. $\square$

We now have the twisted analog of Proposition 3.4.4, showing that $P_{y,w} = P^\sigma_{y,w} \in \{0,1\}$ for all $y, w \in \mathbf{I}_*$ when (W,S) is a finite dihedral Coxeter system. Despite the simplicity of this statement, we know of no simpler proof than the following rather complicated inductive argument using Corollary 3.2.10.

Theorem 3.4.8. Suppose (W,S) is of dihedral type $I_2(m)$, with $m \in \{3,4,\dots\}$ finite. Let $* \in \mathrm{Aut}(W)$ be either S-preserving involution. Then $P^\sigma_{y,w} = 1$ for all $y, w \in \mathbf{I}_*$ with $y \leq w$.

Proof. Let $y, w \in \mathbf{I}_*$ with such that $y \leq w$. If $w = 1$ then $y \leq w$ implies $y = w$ so $P^\sigma_{y,w} = 1$ as desired. If $\ell(w) \in \{1,2\}$, then $w = r \ltimes 1$ for some $r \in S$, in which case $y \leq w$ if and only if $y \in \{1, w\}$, whence $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w} = 1$ by the first part of Corollary 3.2.10.

For the remainder of this proof we assume that $\ell(w) \geq 3$. We may assume that $y < w$ since $P^\sigma_{w,w} = 1$, and may take as an inductive hypothesis that $P^\sigma_{y',w'} = 1$ when $w' < w$ or when $w = w'$ and $y' > y$. Let $r \in \mathrm{Des}_L(w)$ and set $w' = r \ltimes w$. If $r \notin \mathrm{Des}_L(y)$ then $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w} = 1$ by our hypothesis, so assume $r \in \mathrm{Des}_L(y)$. This means in particular that $y \neq 1$, and that $r \ltimes y \leq w'$.

Suppose $y \not\leq w'$. Then $\ell(y) = \ell(w')$, so the only element $z \in \mathbf{I}_*$ with $y \leq z < w$ is $z = y$, and the second part of Corollary 3.2.10 becomes

$$(q+1)^c P^\sigma_{y,w} = (q+1)^d - v^{\ell(w)-\ell(y)+c} \cdot m^\sigma(y \xrightarrow{r} w')$$

where $c = \delta_{rw,wr^*}$ and $d = \delta_{ry,yr^*}$. To express $m^\sigma(y \xrightarrow{r} w')$ more simply, we note that since $\ell(y) = \ell(w')$, we have

$$\nu^\sigma(y,w') = \mu^\sigma(y,x)\mu^\sigma(x,w') = 0 \qquad \text{for all } x \in \mathbf{I}_*,$$

and also

$$\delta_{ry,yr^*}\mu^\sigma(ry,w') = \delta_{ry,yr^*} \qquad \text{and} \qquad \delta_{rw',w'r^*}\mu^\sigma(y,rw') = \delta_{rw,wr^*}\mu^\sigma(y,w).$$

Thus, by the definition (3.2.4), our previous equation becomes

$$(q+1)^c P^\sigma_{y,w} = (q+1)^d - q(d - c \cdot \mu^\sigma(y,w)).$$

If $c = 0$ then this reduces to the formula $P^\sigma_{y,w} = (q+1)^d - dq$ which is equal to 1 for all $d \in \{0,1\}$. If $c = 1$ then $\ell(y) = \ell(w') = \ell(w) - 1$ so $\mu^\sigma(y,w)$ is the constant coefficient of $P^\sigma_{y,w}$ and therefore equal to 1. In this case we must have $d = 0$ since (using Lemma 3.4.6) the only element $x \in \mathbf{I}_*$ with $rx = xr^*$ and $\ell(x) = \ell(w) - 1$ is w' which by assumption is distinct from y. Thus if $c = 1$ then $d = 0$ and our equation becomes $(q+1)P^\sigma_{y,w} = q + 1$ so $P^\sigma_{y,w} = 1$ again as desired.

From now on we assume $y \leq w' \leq w$. Since $r \in \mathrm{Des}_L(y) \setminus \mathrm{Des}_L(w')$, we must actually have $y < w'$. Further, since $y \neq 1$ and $w' \neq w_0$, it follows from Lemma 3.4.7 that $\ell(w') - \ell(y) \geq 2$.

Continuing, by the second part of Corollary 3.2.10 and our inductive hypothesis, we have

$$(q+1)^c P^\sigma_{y,w} = q^2 + 1 - \sum_{\substack{z\in \mathbf{I}_*;\, rz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)+c} m^\sigma(z \xrightarrow{r} w')$$

where $c = \delta_{rw,wr^*}$. (There are no d's here because $(q+1)^d + q(q-d) = q^2+1$ for all $d \in \{0,1\}$.) We wish to replace the right hand side of this equation with a more elementary expression. To this end, suppose $z \in \mathbf{I}_*$ such that $rz < z$ and $y \leq z < w$. We make the following observations:

(a) $\mu^\sigma(z,w') = 0$. This follows because, by hypothesis, $\mu^\sigma(z,w')$ is 1 if $\ell(w') - \ell(z) = 1$ and is zero otherwise. We cannot have $\ell(w') - \ell(z) = 1$ by Lemma 3.4.7 since $z \neq 1$ and $w' \neq w_0$.

(b) By definition and inductive hypothesis, $\nu^\sigma(z,w') = \begin{cases} 1 & \text{if } \ell(w') - \ell(z) = 2 \\ 0 & \text{otherwise.} \end{cases}$

(c) $\delta_{rz,zr^*}\mu^\sigma(rz,w') = 0$. This follows as $\mu^\sigma(rz,w') = 0$ unless $\ell(w') - \ell(rz) = 1$, which by Lemma 3.4.7 occurs only if $rz = 1$ and $w' \in S$ (since $w' \neq w_0$). By assumption, however, we have $\ell(w') \geq \ell(y) + 2 \geq 3$.

(d) $\delta_{rw',w'r^*}\mu^\sigma(z,rw') = c \cdot \mu^\sigma(z,w)$ by definition.

(e) $\mu^\sigma(z,x)\mu^\sigma(x,w') = 0$ for all $x \in \mathbf{I}_*$ with $r \in \mathrm{Des}_L(x)$. This follows as the product can only be nonzero if $z < x < w'$, in which case by hypothesis the product is 1 if and only if $\ell(x) = \ell(z) + 1 = \ell(w') - 1$ and is zero otherwise. If $\ell(x) = \ell(z) + 1$, however, then $x \neq 1$, so $\ell(x) \neq \ell(w') - 1$ as $w' \neq w_0$, by Lemma 3.4.7.

In consequence of (a), we deduce that $m^\sigma(z \xrightarrow{r} w') = 0$ if $\ell(w') - \ell(z)$ is odd, and in consequence of (b)-(e), we deduce that if $\ell(w') - \ell(z)$ is even then

$$m^\sigma(z \xrightarrow{r} w') = \nu^\sigma(z,w') - c \cdot \mu^\sigma(z,w).$$

Thus, noting that $\ell(w) + c = \ell(w') + 2$, we have

$$(q+1)^c P^\sigma_{y,w} = q^2+1 - \left(\sum_z v^{\ell(w')-\ell(z)+2} \cdot \nu^\sigma(z,w')\right) + \left(\sum_z v^{\ell(w')-\ell(z)+2} \cdot c \cdot \mu^\sigma(z,w)\right) \quad (3.4.1)$$

where both sums are over $z \in \mathbf{I}_*$ with $rz < z$ and $y \leq z < w$ and $\ell(w') - \ell(z)$ even. Recall that $\ell(w') - \ell(y) \geq 2$ and that $\ell(y) \geq 1$. From this and the description of the elements of $\mathbf{I}_*$ given before Lemma 3.4.5, we note two additional observations:

- There exists exactly one element $z \in \mathbf{I}_*$ with $y \leq z < w$ and $rz < z$ and $\ell(w') - \ell(z)$ even and $\nu^\sigma(z,w') \neq 0$. This is the element $z = r' \ltimes w'$ where $r' \in \mathrm{Des}_L(w') \subset S$ is the generator distinct from $r \in S$, for which $\ell(w') - \ell(z) = 2$ and $\nu^\sigma(z,w') = 1$ by claim (b) above. It follows that the first parenthesized sum in (3.4.1) is equal to q^2.

- If $c = 1$ then by Lemma 3.4.6 we must have $w = w_0$, since $\ell(w) \geq 3$ and $r \in \mathrm{Des}_L(w)$. In this case there exists exactly one element $z \in \mathbf{I}_*$ with $y < z < w$ (note that we exclude that case $y = z$) and $rz < z$ and $\ell(w') - \ell(z)$ even and $\mu^\sigma(z, w) \neq 0$. Namely, this element z is given by the unique twisted involution of length $m - 1$ distinct from $w' = rw$. This element has $\ell(w') - \ell(z) = 0$ and $\mu^\sigma(z, w) = 1$, by inductive hypothesis. It follows that the second parenthesized sum in (3.4.1) is equal to

$$c \cdot q + c \cdot v^{\ell(w)-\ell(y)+1} \cdot \mu^\sigma(y, w).$$

  The second term here corresponds to the summand indexed by $z = y$. Such a summand occurs if and only if $\ell(w') - \ell(y)$ is even, but our expression accounts for this circumstance because if $\ell(w') - \ell(y)$ is odd and $c \neq 0$ then nevertheless $\mu^\sigma(y, w) = 0$, as $\ell(w) - \ell(y)$ would then not be odd.

Substituting these facts into (3.4.1) gives

$$(q + 1)^c P^\sigma_{y,w} = 1 + c \cdot q + c \cdot v^{\ell(w)-\ell(y)+1} \cdot \mu^\sigma(y, w). \tag{3.4.2}$$

If $c = 0$ then it follows immediately that $P^\sigma_{y,w} = 1$. Suppose $c = 1$. If $\ell(w) - \ell(y)$ is even then $\mu^\sigma(y, w) = 0$ so the preceding equation becomes $(q + 1)P^\sigma_{y,w} = q + 1$ and we get likewise $P^\sigma_{y,w} = 1$. Assume therefore that $\ell(w) - \ell(y)$ is odd. Define

$$\mu_n = \mu^\sigma(y, w) \qquad \text{and} \qquad n = \tfrac{\ell(w)-\ell(y)-1}{2}$$

so that by definition $P^\sigma_{y,w} = \mu_n q^n + \mu_{n-1} q^{n-1} + \cdots + \mu_0$ for some integers $\mu_0, \ldots, \mu_{n-1}$. In this notation, our equation (3.4.2) becomes

$$(q + 1)(\mu_n q^n + \mu_{n-1} q^{n-1} + \cdots + \mu_0) = 1 + q + q^{n+1} \mu_n.$$

As the left hand side is equal to $\mu_n q^{n+1} + \sum_{i=1}^n (\mu_i + \mu_{i-1}) q^n + \mu_0$, equating coefficients of q^i gives $\mu_0 = 1$ and $\mu_0 + \mu_1 = 1$ and $\mu_i + \mu_{i-1} = 0$ for $i = 2, 3, \ldots, n$. The only solution to this system of equations is to set $\mu_0 = 1$ and $\mu_1 = \mu_2 = \cdots = \mu_n = 0$; hence even in this final case we get $P^\sigma_{y,w} = 1$ as desired. □

Summarizing the effect of Proposition 3.4.4 and Theorem 3.4.8 on our conjectures, we have this corollary.

Corollary 3.4.9. Conjectures A′ and B′ hold for all Coxeter systems of rank two.

Proof. The results in this section, together with Proposition 3.3.8, show that Conjectures A′ and B′ hold, for any choice of $*$, whenever (W, S) is of type $I_2(m)$ with $m \in \{3, 4, \ldots\} \cup \{\infty\}$. The only remaining Coxeter system of rank two if that of type $I_2(2) = A_1 \times A_1$, and all of our conjectures hold for this Coxeter system, by inspection and also as a consequence of Proposition 3.4.2. □

We would like to be able to comment on Conjecture C′ for Coxeter systems of rank two, but at the time of writing we have not yet worked out sufficiently tractable formulas for the Laurent polynomials $\left(\widetilde{h}_{x,y;z}\right)_{x\in W;\, y,z\in \mathbf{I}_*}$ and $\left(h^\sigma_{x,y;z}\right)_{x\in W;\, y,z\in \mathbf{I}_*}$ in the finite dihedral case to do so. This is the subject of future work. We mention that Du Cloux [33] has derived formulas for the Kahzdan-Lusztig structure constants $(h_{x,y;z})_{x,y,z\in W}$ for dihedral Coxeter systems, showing at least that Conjecture C holds for all Coxeter systems of rank two.

3.4.3 Algorithms for the finite case

We now describe how to translate the formulas in Sections 3.2.2 and 3.2.3 into algorithms for computing the various polynomials $P_{y,w}$, $P^\sigma_{y,w}$, $h_{x,y;z}$, $\widetilde{h}_{x,y;z}$, $h^\sigma_{x,y;z}$ of interest. Of course then the polynomials with which Conjectures A′, B′, and C′ are concerned are given by

$$P^{\pm}_{y,z} = \tfrac{1}{2}\left(P_{y,w} \pm P^\sigma_{y,w}\right) \qquad \text{and} \qquad h^{\pm}_{x,y;z} = \tfrac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^\sigma_{x,y;z}\right)$$

for $x \in W$ and $w, y, z \in \mathbf{I}_*$.

Adapting the recurrence in Corollary 3.2.14 to give an algorithm for computing the Kazhdan-Lusztig polynomials $P_{y,w}$ is straightforward, and we include the following pseudocode for completeness. A more efficient and involved algorithm for computing the Kazhdan-Lusztig polynomials is described in du Cloux's paper [32].

Algorithm for computing the polynomials $(P_{y,w})_{y,w\in W}$.

- Begin by initializing every $P_{y,w} := 0$.
- Fix a total ordering $\prec$ of $W \times W$ such that $(y,w) \preceq (y',w')$ implies $y \geq y'$ and $w \leq w'$. Iterate over pairs $(y,w) \in W \times W$ in increasing order with respect to $\prec$.
- For each (y,w):
  - If $y \not\leq w$ then continue (i.e., proceed to next iteration).
  - If $y = w$ then set $P_{y,w} := 1$.
  - If $y < w$ then choose $s \in \mathrm{Des}_L(w)$ and proceed as follows:
    * If $s \notin \mathrm{Des}_L(y)$ then set $P_{y,w} := P_{sy,w}$.
    * If $s \in \mathrm{Des}_L(y)$ then set

    $$P_{y,w} := P_{sy,sw} + qP_{y,sw} - \sum_{\substack{z\in W;\, sz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)} \cdot \mu(z,sw) \cdot P_{y,z},$$

    where the constants $\mu(z,sw)$ are computed by applying the definition (3.2.6) to the presently stored values of all polynomials.

- At the termination of the preceding loop, the values of $P_{y,w}$ for $y, w \in W$ will be the unique ones satisfying Theorem-Definition 3.1.1.

An algorithm for computing the polynomials $P^\sigma_{y,w}$ likewise derives from Corollary 3.2.10, the twisted analog of our recurrence for $P_{y,w}$. Translating this result into an algorithm involves a little subtlety, as some terms on the right hand side of our recurrence for $P^\sigma_{y,w}$ can depend on terms on the left. There is actually only one such term: the summand indexed by $z = y$ when $sw = ws^*$ and $\ell(w) - \ell(y)$ is odd. In this case, however, Corollary 3.2.10(b) assumes the form

$$(q+1)P^\sigma_{y,w} = f + q^{(\ell(w)-\ell(y)+1)/2}\mu^\sigma(y,w)$$

where $f \in \mathbb{Z}[q]$ is determined by polynomials which we can assumed to be already known by induction. Given, f is straightforward to extract $P^\sigma_{y,w}$, as desired in the following pseudocode. Lusztig and Vogan describe a similar algorithm in [72, Section 4.5] in the case that (W,S) is crystallographic and $* = 1$ (but their statements actually hold more generally by results in [70].)

Algorithm for computing the polynomials $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*}$.

- Begin by initializing every $P^\sigma_{y,w} := 0$.
- Fix a total ordering $\prec$ of $\mathbf{I}_* \times \mathbf{I}_*$ such that $(y,w) \preceq (y',w')$ implies $y \geq y'$ and $w \leq w'$. Iterate over pairs $(y,w) \in \mathbf{I}_* \times \mathbf{I}_*$ in increasing order with respect to $\prec$.
- For each (y,w):
  - If $y \not\leq w$ then continue (i.e., proceed to next iteration).
  - If $y = w$ then set $P^\sigma_{y,w} := 1$.
  - If $y < w$ then choose $s \in \mathrm{Des}_L(w)$, set $w' := s \ltimes w$ and $c := \delta_{sw,ws^*}$ and $d := \delta_{sy,ys^*}$, and compute

    $$f := (q+1)^d P^\sigma_{s\ltimes y,w'} + q(q-d)P^\sigma_{y,w'} - \sum_{\substack{z\in\mathbf{I}_*;\, sz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)+c} \cdot m^\sigma(z \xrightarrow{s} w') \cdot P^\sigma_{y,z}.$$

    (In particular, compute the constants $m^\sigma(z \xrightarrow{s} w')$ by applying the definition (3.2.4) to the presently stored values of all polynomials.) Then proceed as follows:
    - ∗ If $s \notin \mathrm{Des}_L(y)$ then set $P^\sigma_{y,w} := P^\sigma_{s\ltimes y,w}$.
    - ∗ If $s \in \mathrm{Des}_L(y)$ and either $sw \neq ws^*$ or $\ell(w) - \ell(y)$ is even, then set

      $$P^\sigma_{y,w} := (q+1)^{-c} f.$$

    - ∗ If $s \in \mathrm{Des}_L(y)$ and $sw = ws^*$ and $\ell(w) - \ell(y)$ is odd, then set

      $$P^\sigma_{y,w} := a_0 + a_1 q + \cdots + a_n q^n$$

where $n := \frac{1}{2}(\ell(w) - \ell(y) - 1)$ and $a_0, a_1, \dots, a_n$ are the integers such that

$$f =: a_0 + (a_0 + a_1)q + (a_1 + a_2)q^2 + \cdots + (a_{n-1} + a_n)q^n.$$

- At the termination of the preceding loop, the values of $P^{\sigma}_{y,w}$ for $y, w \in \mathbf{I}_*$ will be the unique ones satisfying Theorem-Definition 3.1.2.

Next, a simple algorithm (which Du Cloux outlines in [33, Section 2.2]) for computing the structure constants $(h_{x,y;z})_{x,y,z \in W}$ in the Kazhdan-Lusztig basis arises from the multiplication formula Theorem 3.2.13. In particular, Theorem 3.2.13 gives precisely the values of the polynomials $h_{s,y;z}$ for $s \in S$ and $y, z \in W$, and if $x \in W$ such that $s \in \mathrm{Des}_L(x)$, then we have the formula

$$c_x = c_s c_{sx} - \sum_{\substack{x' \in W \\ sx' < x' < x}} \mu(x', sx) c_{x'}. \tag{3.4.3}$$

Substituting this for c_x in the product $c_x c_y$ allows use to compute $h_{x,y;z}$ inductively in terms of the polynomials $h_{x',y;z}$ for $x' < x$. This idea leaves us with the following simple procedure:

Algorithm for computing the structure constants $(h_{x,y;z})_{x,y,z \in W}$.

- Compute and store the values of $\mu(y, w)$ for all $y, w \in W$.
- Initialize $h_{s,y;z}$ for $s \in S$ and $y, z \in W$ according to Theorem 3.2.13, so that

$$h_{s,y;z} := \begin{cases} v + v^{-1} & \text{if } s \in \mathrm{Des}_L(y) \text{ and } y = z \\ 1 & \text{if } s \notin \mathrm{Des}_L(y) \text{ and } z = sy \\ \mu(z, y) & \text{if } s \in \mathrm{Des}_L(z) \setminus \mathrm{Des}_L(y) \text{ and } z < y \\ 0 & \text{otherwise.} \end{cases}$$

- Iterate over $y \in W$ in any order.
- For each y:
  - Initialize $h_{1,y;z} := \delta_{y,z}$ for $z \in W$.
  - Iterate over $x \in W \setminus \{1\}$ in order of increasing length.
  - For each x, choose $s \in \mathrm{Des}_L(x)$ and compute

$$h_{x,y;z} := \sum_{z' \in W} h_{sx,y;z'} h_{s,z';z} - \sum_{\substack{x' \in W \\ sx' < x' < x}} \mu(x', sx) h_{x',y;z}.$$

- At the termination of the preceding loop, the values of $h_{x,y;z}$ for $x, y, z \in W$ will be the unique Laurent polynomials satisfying $c_x c_y = \sum_{z \in W} h_{x,y;z} c_z$.

Once the array $(h_{x,y;z})_{x,y,z\in W}$ has been computed, one can compute the Laurent polynomials $\widetilde{h}_{x,y;z}$ via the identity

$$\widetilde{h}_{x,y;z} = \sum_{z'\in W} h_{x,y;z'} h_{z',(x^*)^{-1},z} \qquad \text{for } x \in W \text{ and } y,z \in \mathbf{I}_*.$$

Implementing this simple formula presents its own challenges in cases when the array $(h_{x,y;z})_{x,y,z\in W}$ is very large, in particular when (W,S) is of type H_4. One way to deal with this is to compute and store the two-dimensional arrays $(h_{x,y;z})_{x,z\in W}$ for each $y \in \mathbf{I}_*$; then one can compute all $\widetilde{h}_{x,y;z}$'s by computing (without saving) the array $\left(h_{z',(x^*)^{-1};z}\right)_{z,z'\in W}$ for each $x \in W$.

The formula (3.4.3) (with each $c_\bullet$ replaced by $C_\bullet$) in conjunction with Theorem 3.2.9 gives likewise a simple algorithm for computing the structure constants $h^\sigma_{x,y;z}$. In fact, the following pseudocode is almost identical to the algorithm for computing $h_{x,y;z}$.

Algorithm for computing the structure contants $\left(h^\sigma_{x,y;z}\right)_{x\in W;\, y,z\in\mathbf{I}_*}$.

- Compute and store the values of $m^\sigma(y \xrightarrow{s} w)$ for all $y,w \in \mathbf{I}_*$ and $s \in S$.
- Initialize $h^\sigma_{s,y;z}$ for $s \in S$ and $y,z \in \mathbf{I}_*$ according to Theorem 3.2.9, so that

$$h^\sigma_{s,y;z} := \begin{cases} v^2 + v^{-2} & \text{if } s \in \mathrm{Des}_L(y) \text{ and } y = z \\ v + v^{-1} & \text{if } s \notin \mathrm{Des}_L(y) \text{ and } z = sy \\ 1 & \text{if } s \notin \mathrm{Des}_L(y) \text{ and } z = sys^* \\ m^\sigma(z \xrightarrow{s} y) & \text{if } s \in \mathrm{Des}_L(z) \setminus \mathrm{Des}_L(y) \text{ and } z < s \ltimes y \\ 0 & \text{otherwise.} \end{cases}$$

- Iterate over $y \in \mathbf{I}_*$ in any order.
- For each y:
  - Initialize $h^\sigma_{1,y;z} := \delta_{y,z}$ for $z \in \mathbf{I}_*$.
  - Iterate over $x \in W \setminus \{1\}$ in order of increasing length.
  - For each x, choose $s \in \mathrm{Des}_L(x)$ and compute

$$h^\sigma_{x,y;z} := \sum_{z'\in\mathbf{I}_*} h^\sigma_{sx,y;z'} h^\sigma_{s,z';z} - \sum_{\substack{x'\in W \\ sx'<x'<x}} \mu(x',sx) h^\sigma_{x',y;z}.$$

- At the termination of the preceding loop, the values of $h^\sigma_{x,y;z}$ for $x \in W$ and $y,z \in \mathbf{I}_*$ will be the unique Laurent polynomials satisfying $C_x A_y = \sum_{z\in\mathbf{I}_*} h^\sigma_{x,y;z} A_z$.

3.4.4 Computations and conclusions

Du Cloux implemented efficient algorithms for computing the Kazhdan-Lusztig polynomials $P_{y,w}$ and the structure constants $h_{x,y;z}$ in his C++ program `Coxeter` [34]. Other implementations for calculating these quantities exist (e.g., in [41]), but at the time of writing `Coxeter` appears still to be the only program capable of computing all of Kazhdan-Lusztig structure constants in type H_4 in a reasonable amount of time (though, still on the order of days).

Du Cloux's final version of `Coxeter` includes instructions for ways in which to extend the program. We have made use of this ability to implement the algorithms in the previous section for computing the polynomials $P^\sigma_{y,w}$, $\widetilde{h}_{x,y;z}$, $h^\sigma_{x,y;z}$ for a finite Coxeter system with involution [83]. Extensive comments are included in this C++ code [83] with the details of the implementation. Using these extensions to Coxeter, we have computed the polynomials

$$\left(P^\pm_{y,w}\right)_{y,w\in W} \qquad \text{and} \qquad \left(h^\pm_{x,y;z}\right)_{x\in W;\, y,z\in \mathbf{I}_*}$$

for all finite irreducible Coxeter systems (W,S) with involution in ranks three, four, and five (see Table A.16). Of the cases considered, type H_4 is by far the most computationally intensive, requiring for the calculation of the polynomials $\left(h^\pm_{x,y;z}\right)_{x\in W;\, y,z\in \mathbf{I}_*}$ around one week's computing time (on a 2.26 GHz MacBook Pro) and around 100 GB of memory to store all (highly uncompressed) output files. (Even in this type verifying Conjectures A′ and B′ only takes a few minutes, however.) Tables A.17, A.18, A.19, A.20 show the least and greatest nonzero coefficients of the polynomials thus computed.

The outcome of this computation is that Conjectures A′, B′, and C′ hold whenever (W,S) is an irreducible Coxeter system of rank three, four, or five. Combining this with Corollary 3.4.3 and Theorem 3.4.8. gives the following theorem.

Theorem 3.4.10. Let (W,S) be a finite Coxeter system with an S-preserving involution $* \in \mathrm{Aut}(W)$.

(a) Conjectures A′ and B′ hold if all irreducible factors of (W,S) have rank at most 5.

(b) Conjecture C′ holds if all irreducible factors of (W,S) have rank 3, 4, or 5.

As one corollary to this result, we have this second theorem.

Theorem 3.4.11. Let (W,S) be any finite Coxeter system. If $* \in \mathrm{Aut}(W)$ is an S-preserving involution such that for each irreducible factor (W',S') of (W,S), it holds that $s^* \notin S' \setminus \{s\}$ for all $s \in S'$, then Conjectures A′ holds.

Proof. The condition in the theorem means that $*$ acts on any irreducible factor of (W,S) either as the identity or by interchanging it with another factor. Lusztig and Vogan prove that Conjecture A′ holds whenever (W,S) is a Weyl group and $*$ is the identity automorphism (see [72, §3.2 and §5.1]). As the only finite irreducible Coxeter systems which are not Weyl groups have rank two, three, or four, the theorem follows in light of Proposition 3.2.12, Lemma 3.4.1, and Theorem 3.4.10. □

Restating the previous theorem in the case that $*$ is trivial gives this final corollary.

Corollary 3.4.12. Conjecture A$'$ holds whenever (W, S) is a finite Coxeter system and $*$ is the identity automorphism.

Appendix A

Tables

Table A.1: Automorphisms of exceptional complex reflection groups; see Section 1.7.1

Exceptional Group G_i	Automorphism
G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}	$(s,t) \mapsto (s^{-1}, t^{-1})$
G_7, G_{11}, G_{19}	$(s,t,u) \mapsto (s, t^{-1}, su^{-1}s)$
G_{12}, G_{22}, G_{24}, G_{25}	$(s,t,u) \mapsto (u^{-1}, t^{-1}, s^{-1})$
G_{13}	$(s,t,u) \mapsto (s,u,t)$
G_{15}	$(s,t,u) \mapsto (s,t,tu^{-1}t)$
G_{23}, G_{28}, G_{30}, G_{35}, G_{36}, G_{37}	Identity automorphism
G_{26}	$(s,t,u) \mapsto (s^{-1}, t^{-1}, u^{-1})$
G_{27}, G_{29}, G_{34}	No such τ exists
G_{31}	$(s,t,u,v,w) \mapsto (u,t,s,w,v)$
G_{32}	$(s,t,u,v) \mapsto (v^{-1}, u^{-1}, t^{-1}, s^{-1})$
G_{33}	$(s,t,u,v,w) \mapsto (v,u,t,s,w)$

The automorphisms in the right column are specified in terms of their action on the generators for each exceptional group, in the notation of the corresponding presentation in [21]. Each automorphism $\tau \in \mathrm{Aut}(G)$ satisfies the conditions of Theorem 1.2.1 in the sense that $\tau^2 = 1$ and $\epsilon_\tau(\psi) = 1$ for all $\psi \in \mathrm{Irr}(G)$.

Table A.2: Irreducible multiplicities of χ_W in type E_6; see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
1	$\phi_{1,0}$	1_p	1	1	1	$\mathbb{1}$
1	$\phi_{6,1}$	6_p	1	1	1	$\mathbb{1}$
1	$\phi_{20,2}$	20_p	1	1	1	$\mathbb{1}$
1	$\phi_{64,4}$	64_p	1	1	1	$\mathbb{1}$
1	$\phi_{60,5}$	60_p	1	1	1	$\mathbb{1}$
1	$\phi_{81,6}$	81_p	1	1	1	$\mathbb{1}$
1	$\phi_{24,6}$	24_p	1	1	1	$\mathbb{1}$
1	$\phi_{81,10}$	$81_p'$	1	1	1	$\mathbb{1}$
1	$\phi_{60,11}$	$60_p'$	1	1	1	$\mathbb{1}$
1	$\phi_{24,12}$	$24_p'$	1	1	1	$\mathbb{1}$
1	$\phi_{64,13}$	$64_p'$	1	1	1	$\mathbb{1}$
1	$\phi_{20,20}$	$20_p'$	1	1	1	$\mathbb{1}$
1	$\phi_{6,25}$	$6_p'$	1	1	1	$\mathbb{1}$
1	$\phi_{1,36}$	$1_p'$	1	1	1	$\mathbb{1}$
2	$\phi_{30,3}$	30_p	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{15,4}$	15_q		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{15,5}$	15_p		1	S_2	sgn
2	$\phi_{30,15}$	$30_p'$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{15,16}$	$15_q'$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{15,17}$	$15_p'$		1	S_2	sgn
2	$\phi_{80,7}$	80_s	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{60,8}$	60_s		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{90,8}$	90_s		1	S_3	$\ydiagram{2,1}$
1	$\phi_{10,9}$	10_s		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
0	$\phi_{20,10}$	20_s		1	S_3	sgn

Table A.3: Irreducible multiplicities of χ_W in type E_7 (1 of 2); see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
1	$\phi_{1,0}$	1_a	1	1	1	$\mathbb{1}$
1	$\phi_{7,1}$	$7'_a$	1	1	1	$\mathbb{1}$
1	$\phi_{27,2}$	27_a	1	1	1	$\mathbb{1}$
1	$\phi_{21,3}$	$21'_b$	1	1	1	$\mathbb{1}$
1	$\phi_{189,5}$	$189'_b$	1	1	1	$\mathbb{1}$
1	$\phi_{210,6}$	210_a	1	1	1	$\mathbb{1}$
1	$\phi_{105,6}$	105_b	1	1	1	$\mathbb{1}$
1	$\phi_{168,6}$	168_a	1	1	1	$\mathbb{1}$
1	$\phi_{189,7}$	$189'_c$	1	1	1	$\mathbb{1}$
1	$\phi_{378,9}$	$378'_a$	1	1	1	$\mathbb{1}$
1	$\phi_{210,10}$	210_b	1	1	1	$\mathbb{1}$
1	$\phi_{105,12}$	105_c	1	1	1	$\mathbb{1}$
1	$\phi_{210,13}$	$210'_b$	1	1	1	$\mathbb{1}$
1	$\phi_{378,14}$	378_a	1	1	1	$\mathbb{1}$
1	$\phi_{105,15}$	$105'_c$	1	1	1	$\mathbb{1}$
1	$\phi_{189,20}$	189_c	1	1	1	$\mathbb{1}$
1	$\phi_{210,21}$	$210'_a$	1	1	1	$\mathbb{1}$
1	$\phi_{105,21}$	$105'_b$	1	1	1	$\mathbb{1}$
1	$\phi_{168,21}$	$168'_a$	1	1	1	$\mathbb{1}$
1	$\phi_{189,22}$	189_b	1	1	1	$\mathbb{1}$
1	$\phi_{21,36}$	21_b	1	1	1	$\mathbb{1}$
1	$\phi_{27,37}$	$27'_a$	1	1	1	$\mathbb{1}$
1	$\phi_{7,46}$	7_a	1	1	1	$\mathbb{1}$
1	$\phi_{1,63}$	$1'_a$	1	1	1	$\mathbb{1}$
1	$\phi_{512,11}$	$512'_a$	S_2	1	S_2	$\mathbb{1}$
1	$\phi_{512,12}$	512_a		1	S_2	sgn

Table A.4: Irreducible multiplicities of χ_W in type E_7 (2 of 2); see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
2	$\phi_{56,3}$	$56'_a$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{35,4}$	35_b		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{21,6}$	21_a		1	S_2	sgn
2	$\phi_{120,4}$	120_a	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{105,5}$	$105'_a$		1	S_2	sgn
0	$\phi_{15,7}$	$15'_a$		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{405,8}$	405_a	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{216,9}$	$216'_a$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{189,10}$	189_a		1	S_2	sgn
2	$\phi_{420,10}$	420_a	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{336,11}$	$336'_a$		1	S_2	sgn
0	$\phi_{84,12}$	84_a		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{420,13}$	$420'_a$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{336,14}$	336_a		1	S_2	sgn
0	$\phi_{84,15}$	$84'_a$		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{405,15}$	$405'_a$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{216,16}$	216_a		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{189,17}$	$189'_a$		1	S_2	sgn
2	$\phi_{120,25}$	$120'_a$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{105,26}$	105_a		1	S_2	sgn
0	$\phi_{15,28}$	15_a		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{56,30}$	56_a	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{35,31}$	$35'_b$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{21,33}$	$21'_a$		1	S_2	sgn
2	$\phi_{315,7}$	$315'_a$	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{280,8}$	280_b		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{70,9}$	$70'_a$		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
1	$\phi_{280,9}$	$280'_a$		1	S_3	$\yng(2,1)$
0	$\phi_{35,13}$	$35'_a$		1	S_3	sgn
2	$\phi_{315,16}$	315_a	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{280,17}$	$280'_b$		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{70,18}$	70_a		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
1	$\phi_{280,18}$	280_a		1	S_3	$\yng(2,1)$
0	$\phi_{35,22}$	35_a		1	S_3	sgn

Table A.5: Irreducible multiplicities of χ_W in type E_8 (1 of 3); see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
1	$\phi_{1,0}$	1_x	1	1	1	$\mathbb{1}$
1	$\phi_{8,1}$	8_z	1	1	1	$\mathbb{1}$
1	$\phi_{35,2}$	35_x	1	1	1	$\mathbb{1}$
1	$\phi_{560,5}$	560_z	1	1	1	$\mathbb{1}$
1	$\phi_{567,6}$	567_x	1	1	1	$\mathbb{1}$
1	$\phi_{3240,9}$	3240_z	1	1	1	$\mathbb{1}$
1	$\phi_{525,12}$	525_x	1	1	1	$\mathbb{1}$
1	$\phi_{4536,13}$	4536_z	1	1	1	$\mathbb{1}$
1	$\phi_{2835,14}$	2835_x	1	1	1	$\mathbb{1}$
1	$\phi_{6075,14}$	6075_x	1	1	1	$\mathbb{1}$
1	$\phi_{4200,15}$	4200_z	1	1	1	$\mathbb{1}$
1	$\phi_{2100,20}$	2100_y	1	1	1	$\mathbb{1}$
1	$\phi_{4200,21}$	$4200'_z$	1	1	1	$\mathbb{1}$
1	$\phi_{2835,22}$	$2835'_x$	1	1	1	$\mathbb{1}$
1	$\phi_{6075,22}$	$6075'_x$	1	1	1	$\mathbb{1}$
1	$\phi_{4536,23}$	$4536'_z$	1	1	1	$\mathbb{1}$
1	$\phi_{3240,31}$	$3240'_z$	1	1	1	$\mathbb{1}$
1	$\phi_{525,36}$	$525'_x$	1	1	1	$\mathbb{1}$
1	$\phi_{567,46}$	$567'_x$	1	1	1	$\mathbb{1}$
1	$\phi_{560,47}$	$560'_z$	1	1	1	$\mathbb{1}$
1	$\phi_{35,74}$	$35'_x$	1	1	1	$\mathbb{1}$
1	$\phi_{8,91}$	$8'_z$	1	1	1	$\mathbb{1}$
1	$\phi_{1,120}$	$1'_x$	1	1	1	$\mathbb{1}$
1	$\phi_{4096,11}$	4096_z	S_2	1	S_2	$\mathbb{1}$
1	$\phi_{4096,12}$	4096_x		1	S_2	sgn
1	$\phi_{4096,26}$	$4096'_x$	S_2	1	S_2	$\mathbb{1}$
1	$\phi_{4096,27}$	$4096'_z$		1	S_2	sgn
2	$\phi_{112,3}$	112_z	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{84,4}$	84_x		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{28,8}$	28_x		1	S_2	sgn
2	$\phi_{210,4}$	210_x	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{160,7}$	160_z		1	S_2	sgn
0	$\phi_{50,8}$	50_x		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{700,6}$	700_x	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{400,7}$	400_z		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{300,8}$	300_x		1	S_2	sgn

Table A.6: Irreducible multiplicities of χ_W in type E_8 (2 of 3); see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
2	$\phi_{2268,10}$	2268_x	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{972,12}$	972_x		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{1296,13}$	1296_z		1	S_2	sgn
2	$\phi_{2240,10}$	2240_x	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{1400,11}$	1400_{zz}		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{840,13}$	840_z		1	S_2	sgn
2	$\phi_{4200,12}$	4200_x	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{3360,13}$	3360_z		1	S_2	sgn
0	$\phi_{840,14}$	840_x		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{2800,13}$	2800_z	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{700,16}$	700_{xx}		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{2100,16}$	2100_x		1	S_2	sgn
2	$\phi_{5600,15}$	5600_z	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{3200,16}$	3200_x		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{2400,17}$	2400_z		1	S_2	sgn
2	$\phi_{5600,21}$	$5600'_z$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{3200,22}$	$3200'_x$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{2400,23}$	$2400'_z$		1	S_2	sgn
2	$\phi_{4200,24}$	$4200'_x$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{3360,25}$	$3360'_z$		1	S_2	sgn
0	$\phi_{840,26}$	$840'_x$		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{2800,25}$	$2800'_z$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{700,28}$	$700'_{xx}$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{2100,28}$	$2100'_x$		1	S_2	sgn
2	$\phi_{2240,28}$	$2240'_x$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{1400,29}$	$1400'_{zz}$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{840,31}$	$840'_z$		1	S_2	sgn
2	$\phi_{2268,30}$	$2268'_x$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{972,32}$	$972'_x$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{1296,33}$	$1296'_z$		1	S_2	sgn
2	$\phi_{700,42}$	$700'_x$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{400,43}$	$400'_z$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{300,44}$	$300'_x$		1	S_2	sgn
2	$\phi_{210,52}$	$210'_x$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{160,55}$	$160'_z$		1	S_2	sgn
0	$\phi_{50,56}$	$50'_x$		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{112,63}$	$112'_z$	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{84,64}$	$84'_x$		$(1,2)$	S_2	$\mathbb{1}$
0	$\phi_{28,68}$	$28'_x$		1	S_2	sgn

Table A.7: Irreducible multiplicities of χ_W in type E_8 (3 of 3); see Section 2.4.1

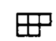
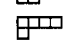
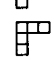
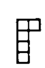
Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
2	$\phi_{1400,7}$	1400_z	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{1344,8}$	1344_x		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{448,9}$	448_z		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
1	$\phi_{1008,9}$	1008_z		1	S_3	
0	$\phi_{56,19}$	56_z		1	S_3	sgn
2	$\phi_{1400,8}$	1400_x	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{1050,10}$	1050_x		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{1575,10}$	1575_x		1	S_3	
1	$\phi_{175,12}$	175_x		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
0	$\phi_{350,14}$	350_x		1	S_3	sgn
2	$\phi_{1400,32}$	$1400'_x$	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{1050,34}$	$1050'_x$		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{1575,34}$	$1575'_x$		1	S_3	
1	$\phi_{175,36}$	$175'_x$		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
0	$\phi_{350,38}$	$350'_x$		1	S_3	sgn
2	$\phi_{1400,37}$	$1400'_z$	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{1344,38}$	$1344'_x$		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{448,39}$	$448'_z$		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
1	$\phi_{1008,39}$	$1008'_z$		1	S_3	
0	$\phi_{56,49}$	$56'_z$		1	S_3	sgn
3	$\phi_{4480,16}$	4480_y	S_5	1	S_5	$\mathbb{1}$
0	$\phi_{7168,17}$	7168_w		$(1,2)$	$S_2 \times S_3$	$\mathbb{1}$
2	$\phi_{3150,18}$	3150_y		$(1,2,3)$	$\mathbb{Z}_3 \times S_2$	$\mathbb{1}$
1	$\phi_{4200,18}$	4200_y		$(1,2)(3,4)$	Dih_8	$\mathbb{1}$
2	$\phi_{4536,18}$	4536_y		1	S_5	
2	$\phi_{5670,18}$	5670_y		1	S_5	
0	$\phi_{1344,19}$	1344_w		$(1,2,3,4)$	$\mathbb{Z}_4$	$\mathbb{1}$
0	$\phi_{2016,19}$	2016_w		$(1,2,3)(4,5)$	$\mathbb{Z}_3 \times S_2$	$\mathbb{1}$
0	$\phi_{5600,19}$	5600_w		$(1,2)$	$S_2 \times S_3$	$\mathbb{1} \otimes$ 
0	$\phi_{2688,20}$	2688_y		$(1,2)(3,4)$	Dih_8	ε''
1	$\phi_{420,20}$	420_y		$(1,2,3,4,5)$	$\mathbb{Z}_5$	$\mathbb{1}$
0	$\phi_{1134,20}$	1134_y		$(1,2,3)$	$\mathbb{Z}_3 \times S_2$	$\mathbb{1} \otimes \mathrm{sgn}$
1	$\phi_{1400,20}$	1400_y		1	S_5	
0	$\phi_{1680,22}$	1680_y		1	S_5	
0	$\phi_{168,24}$	168_y		$(1,2)(3,4)$	Dih_8	ε'
0	$\phi_{448,25}$	448_w		$(1,2)$	$S_2 \times S_3$	$\mathbb{1} \otimes \mathrm{sgn}$
0	$\phi_{70,32}$	70_y		1	S_5	

Table A.8: Irreducible multiplicities of χ_W in type F_4; see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
1	$\phi_{1,0}$	1_1	1	1	1	$\mathbb{1}$
1	$\phi_{9,2}$	9_1	1	1	1	$\mathbb{1}$
1	$\phi_{8,3}$	8_3	1	1	1	$\mathbb{1}$
1	$\phi'_{8,3}$	8_1	1	1	1	$\mathbb{1}$
1	$\phi_{8,9}$	8_2	1	1	1	$\mathbb{1}$
1	$\phi'_{8,9}$	8_4	1	1	1	$\mathbb{1}$
1	$\phi_{9,10}$	9_4	1	1	1	$\mathbb{1}$
1	$\phi_{1,24}$	1_4	1	1	1	$\mathbb{1}$
2	$\phi_{4,1}$	4_2	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{2,4}$	2_3		1	S_2	sgn
0	$\phi'_{2,4}$	2_1		$(1,2)$	S_2	$\mathbb{1}$
2	$\phi_{4,13}$	4_5	S_2	1	S_2	$\mathbb{1}$
0	$\phi_{2,16}$	2_2		1	S_2	sgn
0	$\phi'_{2,16}$	2_4		$(1,2)$	S_2	$\mathbb{1}$
3	$\phi_{12,4}$	12_1	S_4	1	S_4	$\mathbb{1}$
0	$\phi_{16,5}$	16_1		$(1,2)$	$S_2 \times S_2$	$\mathbb{1}$
1	$\phi_{6,6}$	6_1		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
2	$\phi'_{6,6}$	6_2		1	S_4	$\ydiagram{2,2}$
1	$\phi_{9,6}$	9_3		1	S_4	$\ydiagram{3,1}$
1	$\phi'_{9,6}$	9_2		$(1,2)(3,4)$	Dih_8	$\mathbb{1}$
0	$\phi_{4,7}$	4_4		$(1,2)$	$S_2 \times S_2$	$\mathbb{1} \otimes \mathrm{sgn}$
0	$\phi'_{4,7}$	4_3		$(1,2,3,4)$	$\mathbb{Z}_4$	$\mathbb{1}$
0	$\phi_{4,8}$	4_1		$(1,2)(3,4)$	Dih_8	ε''
0	$\phi_{1,12}$	1_3		1	S_4	$\ydiagram{2,1,1}$
0	$\phi'_{1,12}$	1_2		$(1,2)(3,4)$	Dih_8	ε'

Table A.9: Irreducible multiplicities of χ_W in type G_2; see Section 2.4.1

Multiplicity	Name in [30]	Name in [63]	Γ	x	$C_\Gamma(x)$	σ
1	$\phi_{1,0}$	Unit	1	1	1	$\mathbb{1}$
2	$\phi_{2,1}$	V	S_3	1	S_3	$\mathbb{1}$
0	$\phi_{2,2}$	V'		$(1,2)$	S_2	$\mathbb{1}$
1	$\phi_{1,3}$	ε_1		1	S_3	$\ydiagram{2,1}$
1	$\phi'_{1,3}$	ε_2		$(1,2,3)$	$\mathbb{Z}_3$	$\mathbb{1}$
1	$\phi_{1,6}$	Sign	1	1	1	$\mathbb{1}$

Table A.10: Irreducible multiplicities of χ_W in type H_3; see Section 2.4.1

Multiplicity	Name from §2.4.1	Name in [62]	Fourier transform $\mathbf{M}$	Index in $\mathbf{M}$
1	$\phi_{1,0}$	1	1	
1	$\phi_{1,15}$	$1'$	1	
1	$\phi_{5,2}$	5	1	
1	$\phi_{5,5}$	$5'$	1	
1	$\phi_{4,3}$	4	$\mathbf{M}_{S_2}$	$(1, \mathbb{1})$
1	$\phi_{4,4}$	$4'$		$(1, \mathrm{sgn})$
1	$\phi_{3,6}$	$3_b'$	$\mathbf{D}_5$	$(0, 1)$
1	$\phi_{3,8}$	$3_a'$		$(0, 2)$
1	$\phi_{3,1}$	3_b	$\mathbf{D}_5$	$(0, 1)$
1	$\phi_{3,3}$	3_a		$(0, 2)$

Table A.11: Irreducible multiplicities of χ_W in type H_4; see Section 2.4.1

Multiplicity	Name from §2.4.1	Name in [4, 5, 44]	Fourier transform $\mathbf{M}$	Index in $\mathbf{M}$
1	$\phi_{1,0}$	χ_1	1	
1	$\phi_{1,60}$	χ_2	1	
1	$\phi_{25,4}$	χ_{27}	1	
1	$\phi_{25,16}$	χ_{28}	1	
1	$\phi_{36,5}$	χ_{31}	1	
1	$\phi_{36,15}$	χ_{32}	1	
1	$\phi_{4,1}$	χ_3	$\mathbf{D}_5$	$(0,1)$
1	$\phi_{4,7}$	χ_5		$(0,2)$
1	$\phi_{4,31}$	χ_4	$\mathbf{D}_5$	$(0,1)$
1	$\phi_{4,37}$	χ_6		$(0,2)$
1	$\phi_{9,2}$	χ_{11}	$\mathbf{D}_5$	$(0,1)$
1	$\phi_{9,6}$	χ_{13}		$(0,2)$
1	$\phi_{9,22}$	χ_{12}	$\mathbf{D}_5$	$(0,1)$
1	$\phi_{9,26}$	χ_{14}		$(0,2)$
1	$\phi_{16,3}$	χ_{18}	$\mathbf{M}_{S_2}$	$(1,\mathbb{1})$
1	$\phi_{16,6}$	χ_{20}		$(1,\mathrm{sgn})$
1	$\phi_{16,18}$	χ_{21}	$\mathbf{M}_{S_2}$	$(1,\mathbb{1})$
1	$\phi_{16,21}$	χ_{19}		$(1,\mathrm{sgn})$
2	$\phi_{24,6}$	χ_{26}	(see §2.5.3)	
0	$\phi_{6,12}$	χ_7		
0	$\phi_{6,20}$	χ_8		
0	$\phi_{8,12}$	χ_9		
0	$\phi_{8,13}$	χ_{10}		
0	$\phi_{10,12}$	χ_{15}		
0	$\phi_{16,13}$	χ_{17}		
0	$\phi_{16,11}$	χ_{16}		
2	$\phi_{18,10}$	χ_{22}		
0	$\phi_{24,11}$	χ_{23}		
0	$\phi_{24,7}$	χ_{24}		
2	$\phi_{24,12}$	χ_{25}		
2	$\phi_{30,10,12}$	χ_{29}		
2	$\phi_{30,10,14}$	χ_{30}		
2	$\phi_{40,8}$	χ_{33}		
0	$\phi_{48,9}$	χ_{34}		

Table A.12: Irreducible multiplicities of $\chi_{W,\sigma}$ and χ_W in type $I_2(m)$; see Section 2.4.4

m odd

Character	$\chi_{W,1}$	$\chi_{W,r}$	χ_W
$\phi_{1,0}$	1	0	1
$\phi_{1,m}$	0	1	1
$\phi_{2,k}$ $(0 < k < \frac{m}{2})$	0	1	1

$m \equiv 2 \pmod 4$

Character	$\chi_{W,1}$	χ_{W,w_0}	$\chi_{W,r}$	$\chi_{W,s}$	χ_W
$\phi_{1,0}$	1	0	0	0	1
$\phi_{1,m}$	0	1	0	0	1
$\phi_{2,k}$ (k odd)	0	0	1	1	2
$\phi_{2,k}$ (k even)	0	0	0	0	0
$\phi'_{1,m/2}$	0	0	0	1	1
$\phi''_{1,m/2}$	0	0	1	0	1

$m \equiv 0 \pmod 4$

Character	$\chi_{W,1}$	χ_{W,w_0}	$\chi_{W,r}$	$\chi_{W,s}$	χ_W
$\phi_{1,0}$	1	0	0	0	1
$\phi_{1,m}$	0	1	0	0	1
$\phi_{2,k}$ (k odd)	0	0	1	1	2
$\phi_{2,k}$ (k even)	0	0	0	0	0
$\phi'_{1,m/2}$	0	0	0	0	0
$\phi''_{1,m/2}$	0	0	0	0	0

Table A.13: Left cells and conjugacy classes of involutions in type H_3; see Section 2.6.1

Left cell	Cell size	Cell character	1	$(abc)^5$	a	ac
I_i $(1 \leq i \leq 4)$	8	$\phi_{4,3} + \phi_{4,4}$	0	0	1	1
J_i $(1 \leq i \leq 5)$	5	$\phi_{5,2}$	0	0	0	1
J_i^* $(1 \leq i \leq 5)$	5	$\phi_{5,5}$	0	0	1	0
K_i $(1 \leq i \leq 3)$	6	$\phi_{3,1} + \phi_{3,3}$	0	0	2	0
K_i^* $(1 \leq i \leq 3)$	6	$\phi_{3,6} + \phi_{3,8}$	0	0	0	2
L	1	$\phi_{1,0}$	1	0	0	0
L^*	1	$\phi_{1,15}$	0	1	0	0

The last four columns are labeled by involutions in W. The numbers in these columns are the sizes of the intersections of the conjugacy class of the column label with the left cell labeling a given row.

Table A.14: Left cells and conjugacy classes of involutions in type H_4; see Section 2.6.2

Left cell	Cell size	Cell character	1	$(abcd)^{15}$	a	$(abc)^5$	ac
A_i $(1 \leq i \leq 8)$	326	$\phi_{24,6} + \phi_{24,12} + \phi_{24,7} + \phi_{24,11} + \phi_{8,12} + \phi_{8,13} + \phi_{18,10} + \phi_{30,10,12} + \phi_{30,10,14} + \phi_{40,8} + 2\phi_{48,9}$	0	0	0	0	12
A_i $(9 \leq i \leq 18)$	392	$\phi_{24,6} + \phi_{24,12} + \phi_{24,7} + \phi_{24,11} + \phi_{10,12} + \phi_{16,11} + \phi_{16,13} + \phi_{18,10} + \phi_{30,10,12} + \phi_{30,10,14} + 2\phi_{40,8} + 2\phi_{48,9}$	0	0	0	0	14
A_i $(19 \leq i \leq 24)$	436	$\phi_{24,6} + \phi_{24,12} + \phi_{24,7} + \phi_{24,11} + \phi_{6,12} + \phi_{6,20} + \phi_{16,11} + \phi_{16,13} + 2\phi_{30,10,12} + 2\phi_{30,10,14} + 2\phi_{40,8} + 2\phi_{48,9}$	0	0	0	0	16
B_i $(1 \leq i \leq 36)$	36	$\phi_{36,5}$	0	0	1	0	0
B_i^* $(1 \leq i \leq 36)$	36	$\phi_{36,15}$	0	0	0	1	0
C_i $(1 \leq i \leq 25)$	36	$\phi_{25,4}$	0	0	0	0	1
C_i^* $(1 \leq i \leq 25)$	36	$\phi_{25,16}$	0	0	0	0	1
D_i $(1 \leq i \leq 16)$	32	$\phi_{16,3} + \phi_{16,6}$	0	0	1	0	1
D_i^* $(1 \leq i \leq 16)$	32	$\phi_{16,18} + \phi_{16,21}$	0	0	0	1	1
E_i $(1 \leq i \leq 9)$	18	$\phi_{9,2} + \phi_{9,6}$	0	0	0	0	2
E_i^* $(1 \leq i \leq 9)$	18	$\phi_{9,22} + \phi_{9,26}$	0	0	0	0	2
F_i $(1 \leq i \leq 4)$	8	$\phi_{4,1} + \phi_{4,7}$	0	0	2	0	0
F_i^* $(1 \leq i \leq 4)$	8	$\phi_{4,31} + \phi_{4,37}$	0	0	0	2	0
G_1	1	$\phi_{1,0}$	1	0	0	0	0
G_1^*	1	$\phi_{1,60}$	0	1	0	0	0

The last four columns are labeled by involutions in W. The numbers in these columns are the sizes of the intersections of the conjugacy class of the column label with the left cell labeling a given row.

Table A.15: Left cells and conjugacy classes of involutions in type $I_2(m)$; see Section 2.6.3

m odd

Left cell	Cell size	Cell character	1	r
X	1	$\phi_{1,1}$	1	0
X^*	1	$\phi_{1,m}$	0	1
Y	$m-1$	$\sum_{0<k<\frac{m}{2}} \phi_{2,k}$	0	$(m-1)/2$
Y^*	$m-1$	$\sum_{0<k<\frac{m}{2}} \phi_{2,k}$	0	$(m-1)/2$

$m \equiv 2 \pmod 4$

Left cell	Cell size	Cell character	1	w_0	r	s
X	1	$\phi_{1,1}$	1	0	0	0
X^*	1	$\phi_{1,m}$	0	1	0	0
Y	$m-1$	$\phi'_{1,m/2} + \sum_{0<k<\frac{m}{2}} \phi_{2,k}$	0	0	$(m-2)/4$	$(m+2)/4$
Y^*	$m-1$	$\phi''_{1,m/2} + \sum_{0<k<\frac{m}{2}} \phi_{2,k}$	0	0	$(m+2)/4$	$(m-2)/4$

$m \equiv 0 \pmod 4$

Left cell	Cell size	Cell character	1	w_0	r	s
X	1	$\phi_{1,1}$	1	0	0	0
X^*	1	$\phi_{1,m}$	0	1	0	0
Y	$m-1$	$\phi'_{1,m/2} + \sum_{0<k<\frac{m}{2}} \phi_{2,k}$	0	0	$m/4$	$m/4$
Y^*	$m-1$	$\phi''_{1,m/2} + \sum_{0<k<\frac{m}{2}} \phi_{2,k}$	0	0	$m/4$	$m/4$

The last four columns are labeled by involutions in W. The numbers in these columns are the sizes of the intersections of the conjugacy class of the column label with the left cell labeling a given row.

Table A.16: Irreducible finite Coxeter systems with involution; see Section 3.4.1

Name	Dynkin diagram for (W, S)	Involution $* \in \mathrm{Aut}(W)$
A_n $(n \geq 1)$ ${}^2A_n$ $(n \geq 2)$	$s_1 - s_2 - \cdots - s_n$	Identity Diagram $s_i \mapsto s_{n+1-i}$
BC_n $(n \geq 3)$	$s_1 \overset{4}{-} s_2 - \cdots - s_n$	Identity
D_n $(n \geq 4)$ ${}^2D_n$ $(n \geq 4)$	$\begin{array}{l} s_1 \\ \quad > s_3 - \cdots - s_n \\ s_2 \end{array}$	Identity Diagram $\begin{cases} s_1 \leftrightarrow s_2 \\ s_i \mapsto s_i \ (i \geq 3) \end{cases}$
E_6 ${}^2E_6$	$\begin{array}{c} s_2 \\ \vert \\ s_1 - s_3 - s_4 - s_5 - s_6 \end{array}$	Identity Diagram $\begin{cases} s_1 \leftrightarrow s_6 \\ s_3 \leftrightarrow s_5 \\ s_i \mapsto s_i \ (i = 2, 4) \end{cases}$
E_7	$\begin{array}{c} s_2 \\ \vert \\ s_1 - s_3 - s_4 - s_5 - s_6 - s_7 \end{array}$	Identity
E_8	$\begin{array}{c} s_2 \\ \vert \\ s_1 - s_3 - s_4 - s_5 - s_6 - s_7 - s_8 \end{array}$	Identity
F_4 ${}^2F_4$	$s_1 - s_2 \overset{4}{-} s_3 - s_4$	Identity Diagram $s_i \mapsto s_{5-i}$
H_3	$s_1 \overset{5}{-} s_2 - s_3$	Identity
H_4	$s_1 \overset{5}{-} s_2 - s_3 - s_4$	Identity
$I_2(m)$ $(m \geq 4)$ ${}^2I_2(m)$ $(m \geq 4)$	$s_1 \overset{m}{-} s_2$	Identity Diagram $s_i \mapsto s_{3-i}$

All Dynkin diagrams are labeled to coincide with the indexing conventions in `Coxeter` [34]. The types BC_2, ${}^2BC_2$, G_2, ${}^2G_2$ are omitted since they coincide with types $I_2(m)$, ${}^2I_2(m)$ for $m = 4, 6$.

Table A.17: Minimum nonzero coefficients in KL-type polynomials; see Section 3.4.4

Type	$P_{y,w}$ $(y,w \in \mathbf{I}_*)$	$P^{\sigma}_{y,w}$	$P^{+}_{y,w}$	$P^{-}_{y,w}$
A_3	1	1	1	(all polynomials are zero)
${}^2A_3$	1	-1	1	1
BC_3	1	-1	1	1
H_3	1	-1	1	1
A_4	1	1	1	(all polynomials are zero)
${}^2A_4$	1	-1	1	1
BC_4	1	-1	1	1
D_4	1	-2	1	1
${}^2D_4$	1	-1	1	1
F_4	1	-2	1	1
${}^2F_4$	1	-1	1	1
H_4	1	-9	1	1
A_5	1	1	1	1
${}^2A_5$	1	-1	1	1
BC_5	1	-3	1	1
D_5	1	-3	1	1
${}^2D_5$	1	-2	1	1

Table A.18: Maximum nonzero coefficients in KL-type polynomials; see Section 3.4.4

Type	$P_{y,w}$ $(y,w \in \mathbf{I}_*)$	$P^{\sigma}_{y,w}$	$P^{+}_{y,w}$	$P^{-}_{y,w}$
A_3	1	1	1	(all polynomials are zero)
${}^2A_3$	1	1	1	1
BC_3	1	1	1	1
H_3	3	1	2	1
A_4	2	2	2	(all polynomials are zero)
${}^2A_4$	2	1	1	1
BC_4	5	3	4	1
D_4	4	3	3	2
${}^2D_4$	10	8	7	2
F_4	12	8	9	5
${}^2F_4$	12	2	6	6
H_4	5,116	213	2,651	2,465
A_5	4	4	4	1
${}^2A_5$	4	2	3	2
BC_5	35	10	21	14
D_5	17	8	11	6
${}^2D_5$	17	4	10	7

Table A.19: Minimum nonzero coefficients in KL-type structure constants; see Section 3.4.4

Type	$\widetilde{h}_{x,y;z}$ ($x \in W$; $y, z \in \mathbf{I}_*$)	$h^{\sigma}_{x,y;z}$	$h^{+}_{x,y;z}$	$h^{-}_{x,y;z}$
A_3	1	1	1	1
${}^2A_3$	1	−3	1	1
BC_3	1	−8	1	1
H_3	1	−49	1	1
A_4	1	1	1	1
${}^2A_4$	1	−10	1	1
BC_4	1	−156	1	1
D_4	1	−85	1	1
${}^2D_4$	1	−30	1	1
F_4	1	−2,007	1	1
${}^2F_4$	1	−86	1	1
H_4	1	−60,353,800	1	1
A_5	1	1	1	1
${}^2A_5$	1	−162	1	1
BC_5	1	−9,924	1	1
D_5	1	−3,319	1	1
${}^2D_5$	1	−1,538	1	1

Table A.20: Maximum nonzero coefficients in KL-type structure constants; see Section 3.4.4

Type	$\widetilde{h}_{x,y;z}$ ($x \in W$; $y, z \in \mathbf{I}_*$)	$h^{\sigma}_{x,y;z}$	$h^{+}_{x,y;z}$	$h^{-}_{x,y;z}$
A_3	132	10	66	66
${}^2A_3$	132	7	66	66
BC_3	905	28	451	454
H_3	15,676	106	7,870	7,806
A_4	3,748	61	1,892	1,856
${}^2A_4$	4,698	36	2,358	2,340
BC_4	397,846	767	199,042	198,804
D_4	42,384	246	21,226	21,225
${}^2D_4$	42,384	116	21,225	21,159
F_4	108,380,588	8,995	54,192,072	54,188,516
${}^2F_4$	108,380,588	2,600	54,191,594	54,188,994
H_4	59,133,414,193,112,056	467,325,554	29,566,707,126,594,414	29,566,707,066,517,642
A_5	922,740	912	461,826	460,914
${}^2A_5$	922,740	506	461,404	461,336
BC_5	1,319,190,596	42,248	659,608,306	659,582,290
D_5	89,307,651	11,123	44,652,166	44,655,485
${}^2D_5$	89,307,651	4,748	44,655,112	44,652,539

Appendix B

Figures

Figure B-1: Labeled tree illustrating part (a) of Proposition 3.3.9.

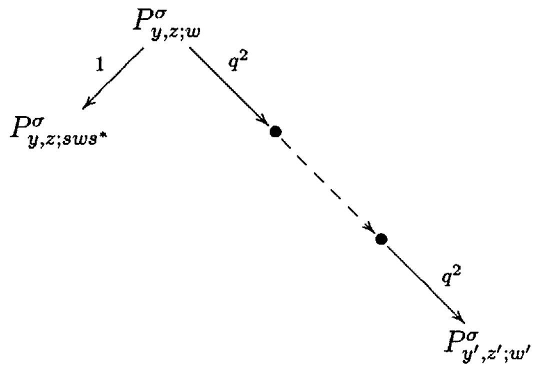


Figure B-2: Labeled tree illustrating part (b) of Proposition 3.3.9.

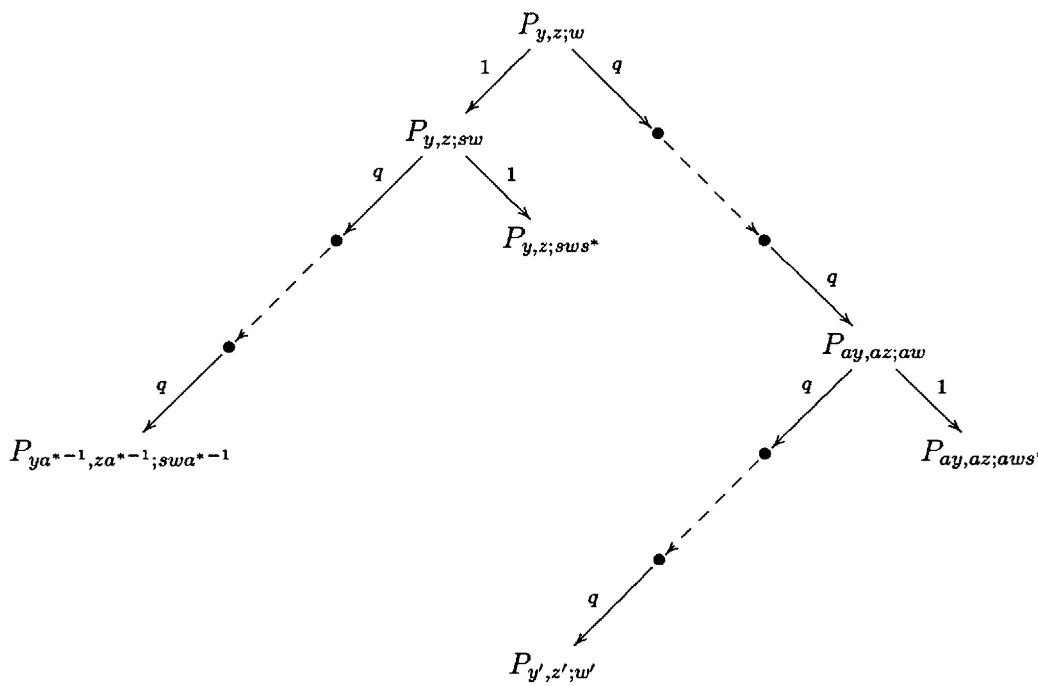


Figure B-3: Labeled tree illustrating part (a) of Proposition 3.3.10

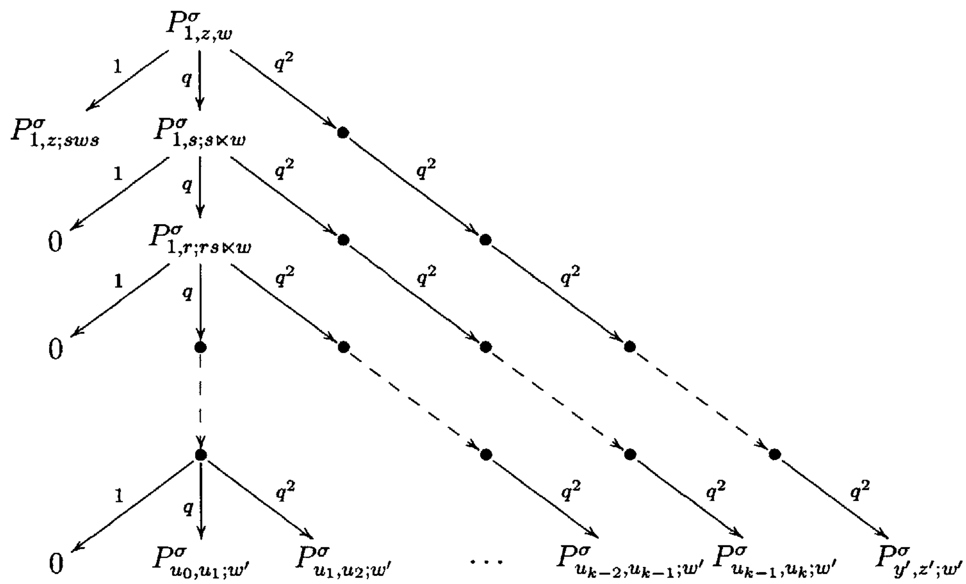


Figure B-4: Labeled tree illustrating part (b) of Proposition 3.3.10

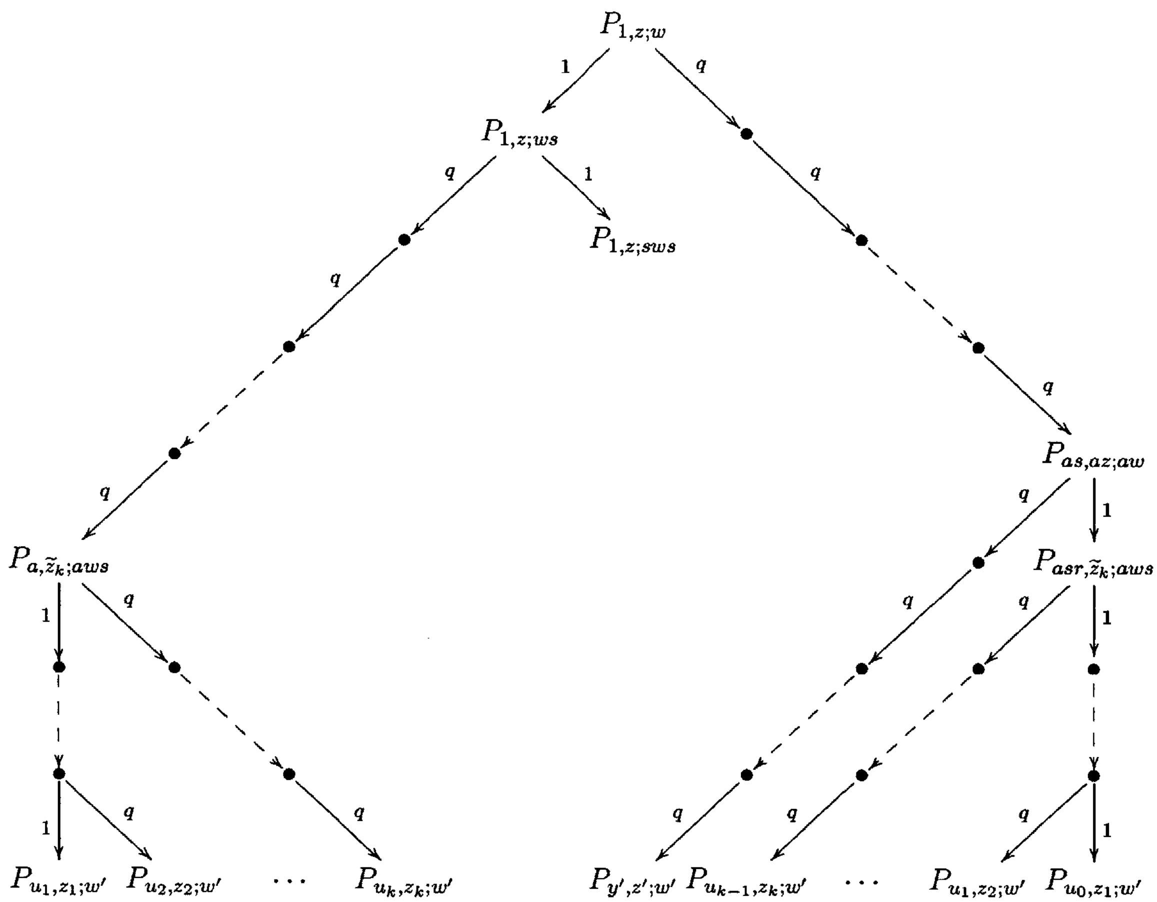


Figure B-5: Labeled tree illustrating part (a) of Proposition 3.3.11

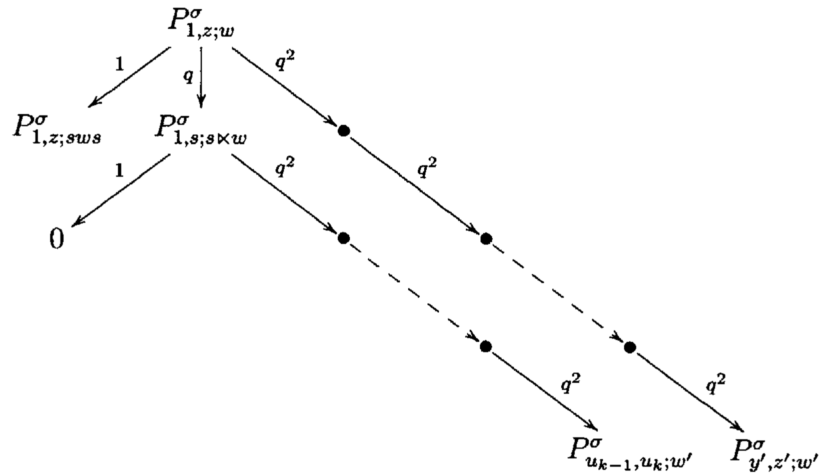
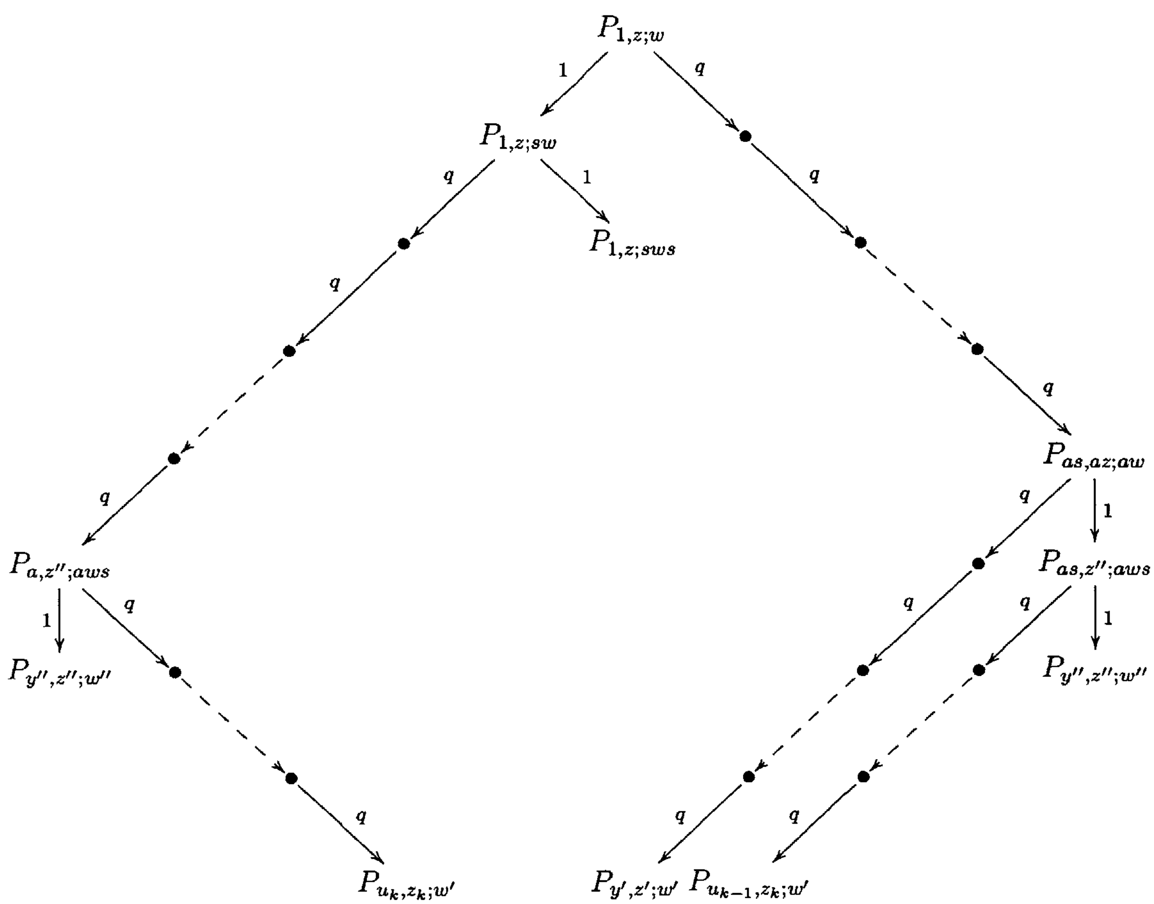


Figure B-6: Labeled tree illustrating part (b) of Proposition 3.3.11 (when $k > 1$)



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