14.471 Public Economics I
Fall 2007

Solutions for Problem Set #4 (14.471)

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November 30, 2007

1 Retirement Saving

1.1 Part (a)

The individual solves:

$$\begin{aligned} \max_{C_1,C_2} \quad & \log(C_1)+\log(C_2) \\ s.t. \;\; & C_1+\tfrac{C_2}{1+r(1-\tau)}=W_1 \end{aligned}$$

$$\begin{aligned} \max_{C_1,C_2} \quad & \beta_1\log(C_1)+\beta_2\log(C_2) \\ s.t. \;\; & p_1C_1+p_2C_2=W_1 \end{aligned}$$

can be written as

$$C_1^*=\frac{\frac{\beta_1}{\beta_1+\beta_2}W_1}{p_1};\quad C_2^*=\frac{\frac{\beta_2}{\beta_1+\beta_2}W_1}{p_2}.$$

Using this with $r=2$ and $\tau=\frac{1}{2}$, we find the solution with no income tax is $C_1^*=\frac{1}{2},\;\; C_2^*=\frac{3}{2}$, and the solution with income tax is $C_1^*=\frac{1}{2},\;\; C_2^*=1$.

1.2 Part (b)

The consumer's budget constraints are depicted in the following figure.

1.3 Part (c)

With the retirement saving program, the consumer's budget constraint may be written as

$$\begin{aligned} C_1+\frac{1}{3}C_2=1 &\quad \text{if } .8<C_1\le 1 \\ C_1+\frac{1}{2}C_2=1.1 &\quad \text{if } 0\le C_1\le .8. \end{aligned}$$

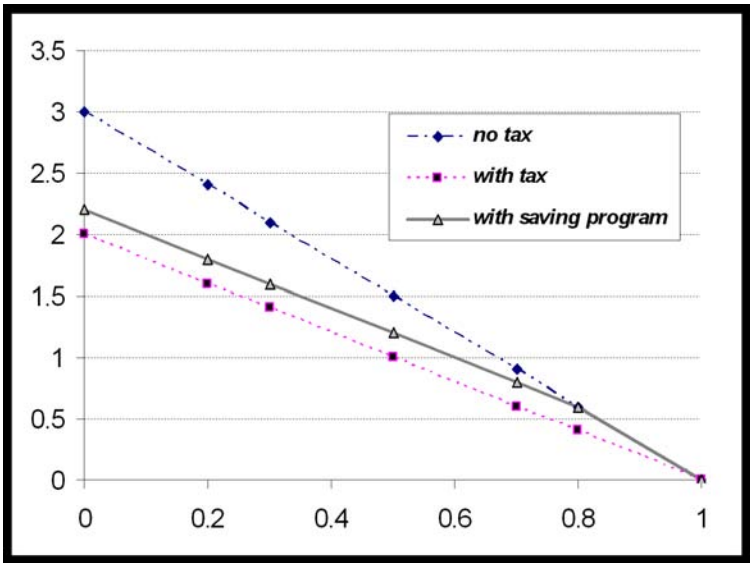

Figure 1: Budget Set I

Using the budget constraint over the first segment yields $C_1^* = \frac{1}{2}$ which is infeasible. Over the second segment, we obtain $C_1^* = \frac{11}{20}$ and $C_2^* = \frac{11}{10}$, which is a valid solution. This means that first period saving declines relative to part (a). This is apparent from considering the relevant portion of the new budget constraint, where we see that the new program is equivalent to a pure increase in income. This implies greater consumption in both periods (due to preferences) and hence lower savings.

1.4 Part (d)

The new budget set is drawn below. The budget constraint may be represented as

$$C_1 + \frac{1}{2}C_2 = 1 \quad \text{if } .5 \le C_1 \le 1$$
$$C_1 + \frac{1}{3}C_2 = \frac{5}{6} \quad \text{if } 0 \le C_1 < .5.$$

Over the first segment, we obtain $C_1^* = \frac{1}{2}$ and $C_2^* = 1$, and on the second segment, we have $C_1^* = \frac{5}{12}$ and $C_2^* = \frac{5}{4}$. Note that due to the nonconvexity of the budget constraint both are valid solutions. However, $U(\frac{1}{2}, 1) < U(\frac{5}{12}, \frac{5}{4})$, so the solution is $C_1^* = \frac{5}{12}$ and $C_2^* = \frac{5}{4}$.

The policy considered in (b) and (c) alters relative prices for relatively small values of savings, but otherwise acts as an upward parallel shift in the budget set, leaving only income effects. The second policy, on the other hand, provides benefits to those saving a relatively large amount. In particular, it is designed with the kink at exactly the point where people are saving in the solution to (a) and offers a greater return to saving beyond this point, which must weakly increase savings. (This is essentially a pure substitution effect.)

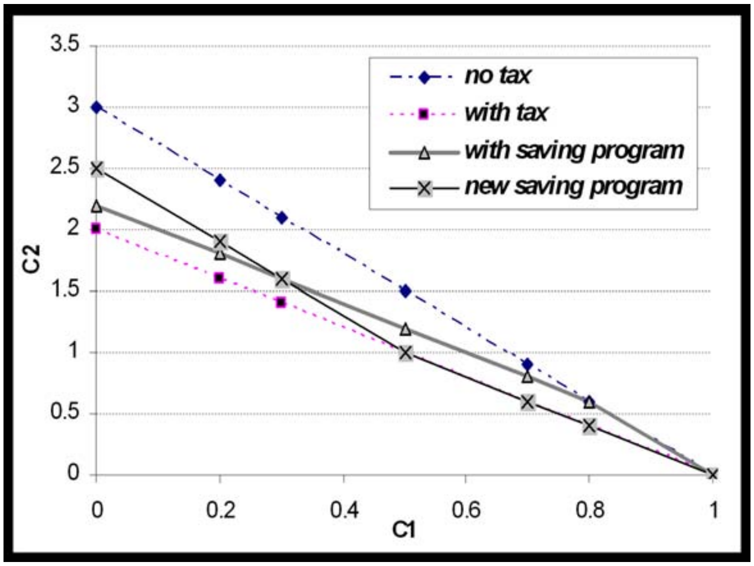

Figure 2: Budget Set II

1.5 Part (e)

Under this policy, the budget set looks almost the same as the one depicted in part (d) and can be represented as

$$C_1 + \frac{1}{2}C_2 = 1 \quad \text{if } .49 \leq C_1 \leq 1$$
$$C_1 + \frac{1}{3}C_2 = \frac{83}{100} \quad \text{if } 0 \leq C_1 < .49.$$

Using this budget constaint, we find that the utility maximizing solution is to set $C_1^* = \frac{83}{200}$ and $C_2^* = \frac{249}{200}$. This yields $S = \frac{117}{200} > \frac{7}{12}$ which is the savings from part (d), so this policy increases personal savings relative to part (d). Government saving also increases since the revenue cost of the "retirement saving program" is reduced when the threshold for tax-exempt saving is raised from 0.50 to 0.51. Therefore, national saving increases.

2 Optimal Saving Distortions and Welfare

2.1 Part (a)

With the CRRA specification, the inverse Euler equation is

$$c_0^\gamma = \frac{q_a}{\beta}\mathbb{E}c_1^\gamma$$

and hence

$$1 = \frac{q_a}{\beta}\mathbb{E}\left(\frac{c_1}{c_0}\right)^\gamma = \frac{q_a}{\beta}\mathbb{E}\exp\left[\gamma\log\left(\frac{c_1}{c_0}\right)\right] = \frac{q_a}{\beta}\exp(\gamma\mu + \gamma^2\sigma^2/2)$$

by the fact that $\log(c_1/c_0) \sim N(\mu, \sigma^2)$. This implies

$$q_a = \beta\exp(-\gamma\mu - \gamma^2\sigma^2/2).$$

Similarly, the Euler equation implies

$$q_b = \beta\mathbb{E}\left(\frac{c_1}{c_0}\right)^{-\gamma} = \beta\exp(-\gamma\mu + \gamma^2\sigma^2/2).$$

Hence, we obtain

$$\frac{q_b}{q_a} = \exp(\gamma^2\sigma^2) \approx 1 + \gamma^2\sigma^2 > 1,$$

which implies a positive implicit tax $\gamma^2\sigma^2$ on the return to saving. The wedge increases with both γ and σ.

2.2 Part (b)

In the log-case, we have from (a)

$$q_b = \beta\exp(\sigma^2/2 - \mu).$$

Let us first compute the allocation that leaves incentives and total expected utility from consumption unchanged and minimizes expected cost. To do that, start from the original allocation with $u_0 = \log(c_0)$ and $u_1(\theta_1) = \log(c_1(\theta_1))$ and distort it to $u_0^* = u_0 - \beta\Delta$ and $u_1^*(\theta_1) = u_1(\theta_1) + \Delta$. Clearly, this leaves incentives and total expected utility from consumption unaffected. Then we solve

$$\begin{aligned}\min_{\Delta} \quad & \exp(u_0 - \beta\Delta) + q_b \mathbb{E} \exp(u_1 + \Delta) \\ = \quad & \exp(\log(c_0) - \beta\Delta) + q_b \mathbb{E} \exp(\log(c_1) + \Delta) \\ = \quad & c_0 \left[\exp(-\beta\Delta) + \beta \exp(\sigma^2/2 - \mu) \exp(\Delta) \mathbb{E} \exp(\log(c_1/c_0))\right] \\ = \quad & c_0[\exp(-\beta\Delta) + \beta \exp(\Delta + \sigma^2)].\end{aligned}$$

From the FOC, we obtain

$$\Delta = -\frac{\sigma^2}{1+\beta}$$

and hence

$$c_0^* = c_0 \exp\left(\frac{\beta\sigma^2}{1+\beta}\right) > c_0$$

and

$$c_1^* = c_1 \exp\left(-\frac{\sigma^2}{1+\beta}\right) < c_1.$$

This allows us to compute the welfare gains from the optimal savings distortion in terms of the cost ratio

$$\frac{c_0 + q_b \mathbb{E} c_1}{c_0^* + q_b \mathbb{E} c_1^*} = \exp\left(-\frac{\beta\sigma^2}{1+\beta}\right) \frac{1 + \beta \exp(\sigma^2)}{1+\beta} \tag{1}$$

after some simplifications.

Let us next find the allocation that leaves incentives and expected cost unchanged and maximizes expected utility from consumption. To do so, let us again start from the original allocation $u_0 = \log(c_0)$ and $u_1(\theta_1) = \log(c_1(\theta_1))$ and move to $\tilde{u}_0 = u_0 + \Delta_0$ and $\tilde{u}_1(\theta_1) = u_1(\theta_1) + \Delta_1$ such that

$$\exp(\tilde{u}_0) + q_b \mathbb{E} \exp(\tilde{u}_1(\theta_1)) = c_0 + q_b \mathbb{E} c_1(\theta_1). \tag{2}$$

Clearly, this move leaves incentives and expected cost unaffected. Then we solve

$$\max_{\Delta_0, \Delta_1} \Delta_0 + \beta\Delta_1$$

subject to (2). Let us use (2) to solve for Δ_0 as a function of Δ_1. Substituting the definitions of $\tilde{u}_0$ and $\tilde{u}_1$ in (2) yields

$$c_0 \exp(\Delta_0) + q_b \exp(\Delta_1) \mathbb{E} c_1(\theta_1) = c_0 + q_b \mathbb{E} c_1(\theta_1)$$

and thus

$$\exp(\Delta_0) + q_b \exp(\Delta_1) \exp(\mu + \sigma^2/2) = 1 + q_b \exp(\mu + \sigma^2/2).$$

Using $q_b = \beta \exp(\sigma^2/2 - \mu)$, this simplifies to

$$\Delta_0 = \log[1 + \beta \exp(\sigma^2)(1 - \exp(\Delta_1))]$$

after few manipulations. Hence, we just solve

$$\max_{\Delta_1} \log[1 + \beta \exp(\sigma^2)(1 - \exp(\Delta_1))] + \beta \Delta_1$$

with the FOC

$$\exp(\sigma^2 + \Delta_1) = 1 + \beta \exp(\sigma^2)(1 - \exp(\Delta_1)).$$

This yields

$$\Delta_1 = \log \left[\frac{1 + \beta \exp(\sigma^2)}{(1 + \beta) \exp(\sigma^2)} \right]$$

and hence

$$\Delta_0 = \log \left[\frac{1 + \beta \exp(\sigma^2)}{1 + \beta} \right].$$

The optimal distorted allocation is

$$\tilde{c}_0 = c_0 \frac{1 + \beta \exp(\sigma^2)}{1 + \beta}$$

and

$$\tilde{c}_1(\theta_1) = c_1(\theta_1) \frac{1 + \beta \exp(\sigma^2)}{(1 + \beta) \exp(\sigma^2)}.$$

Now let us compute λ such that

$$\log((1 + \lambda)c_0) + \beta \mathbb{E} \log((1 + \lambda)c_1) = \log(\tilde{c}_0) + \beta \mathbb{E} \log(\tilde{c}_1).$$

Substituting $\tilde{c}_0$ and $\tilde{c}_1$ and simplifying, this becomes

$$(1 + \beta) \log(1 + \lambda) = \log \left[\frac{1 + \beta \exp(\sigma^2)}{1 + \beta} \right] + \beta \log \left[\frac{1 + \beta \exp(\sigma^2)}{(1 + \beta) \exp(\sigma^2)} \right]$$

and, after some manipulations,

$$1 + \lambda = \exp\left(-\frac{\beta \sigma^2}{1 + \beta} \right) \frac{1 + \beta \exp(\sigma^2)}{1 + \beta},$$

which of course exactly coincides with the result we found in (1).

Finally, let us compute q^* such that the agent's Euler equation is satisfied for $\{c_0^*, c_1^*(\theta_1)\}$. It is given by

$$q^* = \beta \mathbb{E}\left(\frac{c_1^*}{c_0^*}\right)^{-1} = \beta \exp(\sigma^2)\mathbb{E}\left(\frac{c_1}{c_0}\right)^{-1} = \beta \exp(3\sigma^2/2 - \mu).$$

The wedge is thus given by

$$\frac{q^*}{q_b} = \exp(\sigma^2) \approx 1 + \sigma^2,$$

which confirms the result from (a) for the special case $\gamma = 1$. Similarly, we can compute $\tilde{q}$ such that

$$\tilde{q} = \beta \mathbb{E}\left(\frac{\tilde{c}_1}{\tilde{c}_0}\right)^{-1} = \beta \exp(3\sigma^2/2 - \mu) = q^*$$

and hence we obtain the same implicit tax on the return to savings

$$\tau = \frac{q^*}{q_b} - 1 \approx \sigma^2.$$

3 Part (c)

The planning problem with 3 periods becomes

$$\max \log(c_0) + \beta \mathbb{E}\left[\log(c_1(\theta_1)) - v\left(\frac{y(\theta_1)}{\theta_1}\right) + \beta \log(c_2(\theta_1))\right]$$

subject to

$$c_0 + q_b \mathbb{E}[c_1(\theta_1) - y(\theta_1) + q_b c_2(\theta_1)] = 0.$$

Combining the FOCs for $c_1(\theta_1)$ and $c_2(\theta_1)$ immediately implies

$$c_2(\theta_1) = \beta q_b c_1(\theta_1) \quad \forall \theta_1.$$

To calibrate the model, note that a period here is about 30 years long, so if the annual discount factor is .97, the discount factor in our model is roughly $\beta = .4$. Also, let's assume an annual consumption growth rate of about 5%, so $\mu = 30 * .05 = 1.5$. Finally, we need a number for the variance of consumption growth, which is hard to determine (see Farhi and Werning (2007) for some discussion). A reasonable number from the literature (e.g. Blundell, Pistaferri and Preston (2004)) may be about 1% for the variance of annual consumption growth. The variance for our 30-year period may therefore be approximated by

$\sigma^2 \approx 30*.01 = .3$. Based on these estimates, we obtain $\lambda \approx .96\%$ and $q^*/q_b = \exp(\sigma^2) \approx 1.35$. This, however, is a wedge for a 30 year period, so the corresponding annual value would be $1.35^{1/30} \approx 1.01$.

4 Implementation of Optimal Saving Distortion

The social planner's problem is to maximize the agent's expected utility subject to both budget constraint and IC constraint.

$$\max u(c_1) + v(y_1) + \beta\{(1-\pi)[u(c_2) + v(y_2)] + \pi u(c_d)\}$$

s.t.

$$c_1 + \frac{(1-\pi)c_2 + \pi c_d}{r} = y_1 + \frac{(1-\pi)y_2}{r}$$

$$u(c_2) + v(y_2) \geq u(c_d)$$

Note that here I have defined r as one plus interest rate, and $r\beta = 1$.

Let λ and μ denote the Lagrange multipliers of the budget constraint and IC constraint, respectively. We have the following FOCs:

$$u'(c_1^*) = \lambda \tag{1}$$

$$v'(y_1^*) = -\lambda \tag{2}$$

$$\beta(1-\pi)u'(c_2^*) + \mu u'(c_2^*) = \beta(1-\pi)\lambda \tag{3}$$

$$\beta(1-\pi)v'(y_2^*) + \mu v'(y_2^*) = -\beta(1-\pi)\lambda \tag{4}$$

$$\beta\pi u'(c_d^*) - \mu u'(c_d^*) = \beta\pi\lambda \tag{5}$$

Taking the ratio of (1) to (2) and the ratio of (3) to (4), we have

$$\frac{u'(c_1^*)}{v'(y_1^*)} = -1, \frac{u'(c_2^*)}{v'(y_2^*)} = -1 \tag{6}$$

Since the agent supplies one unit of labor to produce one unit of consumption goods, (6) suggests that there should be no distortion on the leisure-consumption margin, that is, the social planner should not impose labor taxes in either period. Now let's look at the intertemporal consumption margin. If we add (3) and (5), we have

$$\beta(1-\pi)u'(c_2^*) + \beta\pi u'(c_d^*) + \mu\left[u'(c_2^*) - u'(c_d^*)\right] = \beta\lambda = \beta u'(c_1^*)$$

or equivalently,

$$(1-\pi)u'(c_2^*) + \pi u'(c_d^*) + \frac{\mu}{\beta}\left[u'(c_2^*) - u'(c_d^*)\right] = u'(c_1^*)$$

Since we know that $c_2^* > c_1^* > c_d^*$, it has to be true that $u'(c_2^*) - u'(c_d^*) < 0$. Because the Lagrange multiplier μ is positive, we have

$$(1-\pi)u'(c_2^*) + \pi u'(c_d^*) > u'(c_1^*) \tag{7}$$

In the case where there is no tax on savings, the agent should equate his marginal utility of consumption in the two periods. Equation (7) suggests that to satisfy the IC constraint, the social planner would like to let the agent have a higher marginal utility (i.e. lower consumption) in the second period and a lower marginal utility (i.e. higher consumption) in the first period. This means that it is optimal for the social planner to discourage savings. $k^* = y_1^* - c_1^*$.

4.1 Part (a)

If the social planner imposes a uniform linear tax τ on savings in the second period on both able and disabled agents, the agent solves the following optimization problem:

$$\max u(c_1) + v(y_1) + \beta\left[(1-\pi)(u(c_2) + v(y_2)) + \pi u(c_d)\right]$$

s.t.

$$c_1 + k = y_1$$

$$c_2 = y_2 + r(1-\tau)k$$

$$c_d = r(1-\tau)k$$

(Note that I've defined the tax τ here in a non-standard way as a tax on the gross return instead of the net return as usual to simplify the following expressions. The analysis with a tax on the net return would be completely analogous, and there is always a one-to-one mapping between the two.) Once we write down all the FOCs, we will find the following Euler equation holds:

$$u'(c_1) = (1-\tau)\left[(1-\pi)u'(c_2) + \pi u'(c_d)\right]$$

where we used the fact that $\gamma\beta = 1$. To implement the optimal allocation $\{(c_1^*, y_1^*), (c_2^*, y_2^*), (c_d^*, 0), k^*\}$, the social planner has to pick τ such that

$$u'(c_1^*) = (1-\tau)\left[(1-\pi)u'(c_2^*) + \pi u'(c_d^*)\right] < (1-\tau)u'(c_d^*) \tag{8}$$

The last step is due to the fact that $c_2^* > c_1^* > c_d^*$. If (8) is true, we will find that the agent wants to double-deviate. Double deviation here means that when the agent says he is disabled no matter what his true type is, she can do better than consuming (c_1^*, y_1^*) in the first period and $(c_d^*, 0)$ in the second period. Suppose the agent saves a little bit more in the first period and claims to be disabled in the second period. His utility becomes

$$\begin{aligned}
\tilde{U} &= u(c_1^* - \varepsilon) + v(y_1^*) + \beta u(c_d^* + r(1-\tau)\varepsilon) \\
&\approx u(c_1^*) + v(y_1^*) + \beta u(c_d^*) + \varepsilon\{(1-\tau)u'(c_d^*) - u'(c_1^*)\} \\
&= U + \varepsilon\{(1-\tau)u'(c_d^*) - u'(c_1^*)\} \\
&> U
\end{aligned}$$

where U represents the utility level the agent gets by consuming (c_1^*, y_1^*) in the first period and $(c_d^*, 0)$ in the second period. The last step follows from inequality (8). Therefore, a uniform linear tax on savings cannot achieve the optimal allocation. Of course, the optimal allocation can be implemented if non-linear taxes are available. You tell the agent to choose (c_2^*, y_2^*) if he says he is able and $(c_d^*, 0)$ if he says he is disabled. He would be punished if he chooses anything else. However, the optimal allocations are no longer implementable with a uniform linear tax. The reason is that $\{c_1^*, y_1^*, c_d^*, 0\}$ is not in the argmax set of the agent who faces the uniform linear tax and wants to always claim to be disabled in the second period.

4.2 Part (b)

When the social planner taxes the able and disabled agents differently on their savings, the following Euler equation holds:

$$u'(c_1) = (1-\tau_2)(1-\pi)u'(c_2) + (1-\tau_d)\pi u'(c_d) \tag{9}$$

If the agent double deviates, he gets a utility of

$$\tilde{U} = U + \varepsilon\{(1-\tau_d)u'(c_d^*) - u'(c_1^*)\}$$

In order to prevent the double-deviation, the social planner needs to make sure that $\tilde{U} = U$, or equivalently,

$$\tau_d = 1 - \frac{u'(c_1^*)}{u'(c_d^*)} \tag{10}$$

4.3 Part (c)

From (c) we know that when the social planner taxes able agents at τ_2 and disabled agents at τ_d, the agent is going to consume and save such that (9) is satisfied. When the optimal allocation is implemented, we will have

$$u'(c_1^*) = (1-\tau_2)(1-\pi)u'(c_2^*) + (1-\tau_d)\pi u'(c_d^*)$$

If we substitute (10) into the equation above, we can solve for τ_2 and get

$$\tau_2 = 1 - \frac{u'(c_1^*)}{u'(c_2^*)}$$

Since we know that $c_2^* > c_1^* > c_d^*$, we know that $\tau_2 < 0$ (subsidy) and $\tau_d > 0$ (tax).

4.4 Part (d)

The total amount of revenue collected is

$$\begin{aligned} R &= (1-\pi)\tau_2 rk^* + \pi\tau_d rk^* \\ &= rk^* \left\{ (1-\pi)\left[1 - \frac{u'(c_1^*)}{u'(c_2^*)}\right] + \pi\left[1 - \frac{u'(c_1^*)}{u'(c_d^*)}\right] \right\} \\ &= rk^* u'(c_1^*) \left\{ \frac{1}{u'(c_1^*)} - (1-\pi)\frac{1}{u'(c_2^*)} - \pi\frac{1}{u'(c_d^*)} \right\} \\ &= 0 \end{aligned}$$

The last step comes from equations (1), (3) and (5). By taxing savings when the agent is disabled and his marginal utility is high and subsidizing savings when the agent is able to work and his marginal utility is low, the tax effectively reduces the marginal return of savings (i.e. discourages savings) even though on average it collects no revenue. The reason why we need to discourage savings is because we have the IC constraint. By taxing savings in the bad state (disabled), we make sure that the able agents would not want to pretent to be disabled agents and double-deviate.