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Solutions for Problem Set #2 (14.471)

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1 Question 1

1.1 Part (a)

Suppose consumers face an initial budget set B_0. Individual j's utility maximization problem is

$$\max_{c,Y} U^j(G^c(c), G^Y(Y))$$

subject to

$$(c, Y) \in B_0,$$

where $c = (c_1, ..., c_N)$ and $Y = (Y_1, ..., Y_M)$. Due to the particular form of the utility function, we can decompose the maximization into two stages:

- First, consumer j solves
$$\max_{g^c, g^Y} U^j(g^c, g^Y)$$
subject to
$$(g^c, g^Y) \in b_0$$
with $b_0 = \{(g^c, g^Y) | \exists (c, Y) \text{ with } G^c(c) = g^c, G^Y(Y) = g^Y \text{ and } (c, Y) \in B_0\}$. Denote the maximizers obtained from solving this problem by (g^{cj}, g^{Yj}).

- Given (g^{cj}, g^{Yj}) from the first stage, consumer j chooses some $(c, Y) \in B_0$ s.t. $G^c(c) = g^{cj}$ and $G^Y(Y) = g^{Yj}$.

Define the following two functions:

$$e^{G^c}(g^c) = \min_c \sum_{i=1}^{N} c_i \text{ s.t. } G^c(c) = g^c$$

and

$$e^{G^Y}(g^Y) = \max_Y \sum_{i=1}^{M} Y_i \text{ s.t. } G^Y(Y) = g^Y.$$

Then consider the move to the new budget set B_1 given by

$$B_1 = \left\{ (c, Y) \middle| \sum_{i=1}^{N} c_i \leq e^{G^c}(g^c),\ \sum_{i=1}^{M} Y_i \geq e^{G^Y}(g^Y) \text{ and } (g^c, g^Y) \in b_0 \right\}.$$

This reform does not affect stage 1 of the consumers' utility maximization problem since they can still pick any tuple (g^c, g^Y) from the original set b_0. Hence, they continue to choose (g^{cj}, g^{Yj}) and their utilities are therefore unchanged. However, consumers may choose different bundles (c, Y) in the second stage when faced with the new budget set B_1. By construction of the functions $e^{G^c}(g^c)$ and $e^{G^Y}(g^Y)$, this change is such that the economy's resource constraint is at least weakly relaxed after introducing the reform since consumers now choose (c, Y) efficiently. Therefore, any optimal tax system must generate a budget set of the form of B_1. Defining

$$\hat{b} = \left\{ \left(\sum_{i=1}^{N} c_i, \sum_{i=1}^{M} Y_i \right) \middle| \sum_{i=1}^{N} c_i \leq e^{G^c}(g^c),\ \sum_{i=1}^{M} Y_i \geq e^{G^Y}(g^Y) \text{ and } (g^c, g^Y) \in b_0 \right\},$$

we can express B_1 as

$$B_1 = \left\{ (c, Y) \middle| \left(\sum_{i=1}^{N} c_i, \sum_{i=1}^{M} Y_i \right) \in \hat{b} \right\}$$

and hence

$$B_1 = \left\{ (c, Y) \middle| \sum_{i=1}^{N} c_i \leq T\left(\sum_{i=1}^{M} Y_i \right) \right\}$$

for some function $T(.)$.

1.2 Part (b)

Here we can apply a generalized version of the uniform commodity taxation result from the lecture. Using the primal approach, we solve the Pareto problem

$$\max_{c^j, Y^j} \sum_j \lambda^j U^j(G^c(c^j), G^Y(Y^j))$$

(where λ^j are the Pareto weights) subject to the resource constraint

$$\sum_j \left(\sum_i c_i^j - \sum_i Y_i^j \right) = 0$$

and the implementability constraints

$$\sum_i U_1^j G_i^c c_i^j + \sum_i U_2^j G_i^Y Y_i^j = 0 \quad \forall j.$$

With μ^j as the multiplier of the implementability constraint of individual j and γ for the resource constraint, the FOC for c_k^j is

$$\lambda^j U_1^j G_k^c + \mu^j U_1^j G_k^c + \mu^j \sum_i \left(U_{11}^j G_i^j G_k^j c_i^j + U_1^j G_{ik}^c c_i^j \right) = \gamma.$$

Deviding by the FOC for c_0^j, we obtain the optimality condition

$$\frac{U_1^j G_k^c \left[\lambda^j + \mu^j + \mu^j \sum_i \frac{U_{11}^j G_i^j G_k^j c_i^j + U_1^j G_{ik}^c c_i^j}{U_c^j G_k^c} \right]}{U_1^j G_0^c \left[\lambda^j + \mu^j + \mu^j \sum_i \frac{U_{11}^j G_i^j G_0^j c_i^j + U_1^j G_{i0}^c c_i^j}{U_c^j G_0^c} \right]} = 1.$$

If G^c is homogeneous of degree 1, we have

$$\sum_i U_{11}^j G_i^j G_k^j c_i^j = U_{11}^j G_k^j \sum_i G_i^j c_i^j = U_{11}^j G_k^j G^j$$

and

$$\sum_i U_1^j G_{ik}^c c_i^j = U_1^j \sum_i G_{ki}^c c_i^j = 0$$

by Euler's theorem (note that G_k^c is homogeneous of degree 0). Hence, the above optimality condition reduces to

$$\frac{G_k^c}{G_0^c} = 1 \quad \forall k,$$

which is independent of j. Hence, all consumption goods should be taxed at the same rate. By completely symmetric arguments, all incomes should be taxed at the same rate if G^Y is homogeneous of degree 1. Since we can normalize one tax rate to 0, this implies that under these conditions it is optimal to tax all incomes at the same rate.

1.3 Part (c)

The result in (a) implies that gender based taxation is not efficient if households have the assumed preferences, even if elasticities vary across household members (which can be captured by the G^Y-function). If arbitrary non-linear taxes are not available, we need more restrictive assumptions to rule out the efficiency of gender-based taxation (see (b)).

2 Question 2

2.1 Part (a)

Recall the general condition for the Pareto efficiency of an income tax schedule derived in the lecture:

$$-\hat{\hat{\mu}}'(Y) - \hat{\hat{\mu}}(Y)\frac{\partial \text{MRS}(c, Y)}{\partial c} \leq h(Y) \tag{1}$$

with

$$\hat{\hat{\mu}}(Y) = \frac{T'(Y)}{1 - T'(Y)}h(Y)\frac{1}{\frac{1}{\varepsilon_w^*(Y)Y} + \frac{T''(Y)}{1-T'(Y)}}. \tag{2}$$

This can be simplified using the assumptions given in the problem, which imply that

$$\frac{\partial \text{MRS}(c, Y)}{\partial c} = 0$$

since we assume no income effects on labor supply, $\varepsilon_w^*(Y) = \varepsilon_w^*$ independent of Y since we assume that the compensated labor supply elasticity does not vary with income, and finally $T'(Y) = \tau$ and $T''(Y) = 0$ independent of Y due to the linearity of the tax schedule. Hence, we obtain

$$\hat{\hat{\mu}}(Y) = \frac{\tau}{1 - \tau}\varepsilon_w^* Y h(Y)$$

and, using the Pareto distribution for income $h(Y) = kY^{-k-1}\underline{Y}^k$,

$$\hat{\hat{\mu}}(Y) = k\frac{\tau}{1 - \tau}\varepsilon_w^* Y^{-k}\underline{Y}^k.$$

Substituting

$$\hat{\hat{\mu}}'(Y) = -k^2\frac{\tau}{1 - \tau}\varepsilon_w^* Y^{-k-1}\underline{Y}^k$$

into (1) yields

$$k^2\frac{\tau}{1 - \tau}\varepsilon_w^* Y^{-k-1}\underline{Y}^k \leq kY^{-k-1}\underline{Y}^k$$

and, after cancelling terms,

$$\frac{\tau}{1-\tau}\varepsilon_w^* k \leq 1 \tag{3}$$

as the condition for the Pareto efficiency of the linear tax schedule. For instance, setting $\tau = .3$, $\varepsilon_w^* = .5$ and $k = 3$, which are empirically plausible values, the condition is clearly satisfied. However, when we increase the marginal tax rate, the elasticity of labor supply or the parameter k (which means that the distribution of income becomes more concentrated at the bottom), the inequality becomes less likely to hold.

2.2 Part (b)

The analysis becomes more complicated if τ and ε^* depend on Y since $\hat{\hat{\mu}}'(Y)$ is a more complicated object in this case (see equation (2)). Intuitively, if $\varepsilon^*(Y)$ is increasing in Y, then the condition for Pareto optimality is more likely to be violated at high income levels. The same holds for a progressive tax schedule, where $\tau/(1-\tau)$ is increasing in Y.

3 Question 3

3.1 Part (a)

Given the quasilinear utility function

$$U(c, Y, \theta) = c - \frac{1}{2}\left(\frac{Y}{\theta}\right)^2$$

and the tax function

$$T(Y) = \tau Y,$$

an individual of type θ solves

$$\max_Y \, (1-\tau)Y - \frac{1}{2}\left(\frac{Y}{\theta}\right)^2$$

with the FOC

$$Y = (1-\tau)\theta^2.$$

We can use this relationship to compute the density of the skill distribution from the density of the income distribution, which is given by

$$h(Y) = kY^{-k-1}\underline{Y}^k.$$

Noting that the Jacobian of the transformation is

$$\frac{dY}{d\theta} = 2(1-\tau)\theta,$$

we obtain

$$\begin{aligned} f(\theta) &= k((1-\tau)\theta^2)^{-k-1}\underline{Y}^k \times 2(1-\tau)\theta \\ &= 2k(1-\tau)^{-k}\theta^{-2k-1}\underline{Y}^k \end{aligned}$$

and

$$F(\theta) = 1 - (1-\tau)^{-k}\theta^{-2k}\underline{Y}^k.$$

3.2 Part (b)

Using the notation from recitation, we consider the optimal income taxation problem

$$\max_{v(\theta),Y(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} W[v(\theta)]f(\theta)d\theta$$

subject to the incentive constraints

$$v'(\theta) = U_\theta[e(v(\theta), Y(\theta), \theta), Y(\theta), \theta] \quad \forall\, \theta \tag{4}$$

and the resource constraint

$$\int_{\underline{\theta}}^{\overline{\theta}} [Y(\theta) - e(v(\theta), Y(\theta), \theta)]f(\theta)d\theta = R. \tag{5}$$

We form the Hamiltonian

$$\begin{aligned} \mathcal{H} &= W[v(\theta)]f(\theta) + \mu(\theta)U_\theta[e(v(\theta), Y(\theta), \theta), Y(\theta), \theta] \\ &\quad + \gamma[Y(\theta) - e(v(\theta), Y(\theta), \theta)]f(\theta), \end{aligned}$$

where $\mu(\theta)$ is the costate variable associated with the law of motion of the state variable $v(\theta)$ given by (4) and γ is the multiplier associated with the (isoperimetric) resource constraint. By the principle of optimality, the optimality conditions for the control $Y(\theta)$ and the state $v(\theta)$ are, respectively,

$$\frac{\partial \mathcal{H}}{\partial Y} = \mu(U_{\theta c}e_Y + U_{\theta Y}) + \gamma f(1 - e_Y) = 0 \tag{6}$$

and

$$\frac{\partial \mathcal{H}}{\partial v} = W'f + \mu U_{\theta c} e_v - \gamma f e_v = -\mu', \tag{7}$$

where I dropped the arguments of functions to reduce notation. In addition, we have the transversality conditions $\mu(\underline{\theta}) = \mu(\overline{\theta}) = 0$.

Using the specification given in the problem set (see also Saez (2001), section 5 and appendix) with $U[c, Y, \theta] = c - 1/2 \times (Y/\theta)^2$ and $W(v) = \log(v)$, we can simplify these FOCs considerably. Note that

$$U_\theta = \frac{Y^2}{\theta^3}, \quad U_{\theta c} = 0, \quad U_{\theta Y} = 2\frac{Y}{\theta^3}, \quad e_v = 1, \quad e_Y = \frac{Y}{\theta^2} \text{ and } W' = \frac{1}{v}.$$

Substituting these expressions in (6), we are able to solve for $Y(\theta)$ as follows:

$$Y(\theta) = \frac{\gamma f(\theta)\theta^3}{\gamma f(\theta)\theta - 2\mu(\theta)}. \tag{8}$$

Also, (7) reduces to

$$\mu'(\theta) = \left(\gamma - \frac{1}{v(\theta)}\right) f(\theta). \tag{9}$$

Finally, using (8), (4) becomes

$$v'(\theta) = \frac{Y(\theta)^2}{\theta^3} = \left(\frac{\gamma f(\theta)}{\gamma f(\theta)\theta - 2\mu(\theta)}\right)^2 \theta^3. \tag{10}$$

Equations (9) and (10) form a two-dimensional system of ordinary differential equations in $\mu(\theta)$ and $v(\theta)$. Using the computational procedure discussed in recitation, it can be solved numerically given the density function $f(\theta)$ derived in part (a) and the two boundary conditions $\mu(\underline{\theta}) = \mu(\overline{\theta}) = 0$. Refer to the MATLAB-code posted on the website for details and results. Due to our different distributional assumptions, we obtain an inversely U-shaped pattern of marginal tax rates in contrast to Saez (2001).

4 Question 4

4.1 Part (a)

Note that household 2 is at a kink point, so we use households 1, 3, and 4 to estimate the labor supply function.

Household	Pre-tax Wage	After-tax Wage	Virtual Income
1	3	3	0
3	10	5	35
4	10	5	40

We have 3 unknowns and 3 equations, so we can estimate the 3 parameters of the labor supply function exactly. We solve the system:

$$\begin{pmatrix} 15.15 \\ 8.25 \\ 7.25 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 35 & 5 \\ 1 & 40 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

This gives us the parameter values:

$$\alpha = 15; \quad \beta = -0.2; \quad \gamma = 0.05$$

4.2 Part (b)

Total revenue is $T(y_3) + T(y_4) = 2 * 0.5 * (92.5 - 60) = 32.5$. If the government were to raise this amount of revenue by levying a single lump sum tax on all 4 individuals, the lump sum tax would be $32.5/4 = 8.125$. Recall that the equivalent variation is the amount of money that if given to an individual would be equivalent to the policy change (i.e. getting this amount of money would give the individual the same utility as he would have after the policy change). In the case of household 1, EV is just the lump sum tax, 8.125. (Another way to think about this is that EV is the amount of money you would be willing to give up in order to avoid the policy change under consideration. In this case, household 1 should be indifferent between giving up 8.125 and paying a lump sum tax of 8.125.)

5 Question 5

5.1 Part (a)

We first translate the tax schedule into budget sets and set up the problem. Since the bottom tax bracket is 25%, each individual's after-tax non-labor income is effectively $y_0^{virt} = \frac{3}{4}$. At the kink point, $\bar{l} = \frac{1}{2w}$ and the virtual income for the second tax bracket is $y_1^{virt} =$

$y_0^{virt} + \bar{l}w(\tau_1 - \tau_0) = \frac{3}{4} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{7}{8}$. Note that jello and coconut should have the same price because of the production technology.

$$\max_{J,C,L} \quad \log(J) + \log(C) + \log(1-L)$$

$$s.t.\ J + C = \frac{3}{4} + \frac{3}{4}wL \quad if \quad L \leq \frac{1}{2w}$$
$$J + C = \frac{7}{8} + \frac{1}{2}wL \quad if \quad L > \frac{1}{2w}$$

First, let's consider segment 1 (25% tax rate) of the non-linear budget set. The FOCs of this problem are

$$\frac{1}{J} = \frac{1}{C} = \lambda; \quad \frac{1}{1-L} = \frac{3}{4}w\lambda$$

Plugging these FOCs in the budget constraint, we get

$$J = C = \frac{1}{\lambda} = \frac{1}{4} + \frac{1}{4}w \tag{11}$$
$$L = 1 - \frac{4}{3w\lambda} = \frac{2}{3} - \frac{1}{3w} \tag{12}$$

For the low type worker whose wage is 1 jello packet per unit of labor supplied, $w = 1$ and $\bar{l} = \frac{1}{2}$. Equations (11) and (12) suggest that

$$J_l = C_l = \frac{1}{2}$$
$$L_l = \frac{1}{3},$$

which is consistent with $L_l \leq \bar{l}$. For the high type worker whose wage is 2 jello packets per unit of labor supplied, $w = 2$ and $\bar{l} = \frac{1}{4}$. Equation (1.2) suggests that $L_h = \frac{1}{2}$, which is inconsistent with $L_h \leq \bar{l}$. Now we need to consider segment 2 (50% tax rate) of the non-linear budget set. The FOCs of this problem are

$$\frac{1}{J} = \frac{1}{C} = \lambda; \quad \frac{1}{1-L} = \frac{1}{2}w\lambda$$

Plugging these FOCs in the budget constraint, we get

$$J = C = \frac{1}{\lambda} = \frac{7}{24} + \frac{1}{6}w \tag{13}$$
$$L = 1 - \frac{2}{w\lambda} = \frac{2}{3} - \frac{7}{12w}. \tag{14}$$

Again, the high type has $w = 2$ and $\bar{l} = \frac{1}{4}$. Equations (13) and (14) suggest that

$$\begin{aligned} J_h &= C_h = \frac{5}{8} \\ L_h &= \frac{3}{8}, \end{aligned}$$

which is consistent with $L_h > \bar{l}$.

5.2 Part (b)

Per capita tax revenue is

$$\begin{aligned} R &= \frac{1}{2}\left(\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{4}\times\frac{3}{2}+\frac{1}{2}\left(1+2\times\frac{3}{8}-\frac{3}{2}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right)=\frac{5}{12} \end{aligned}$$

5.3 Part (c)

Now coconut expenditures may be deducted from consumers' taxable incomes. The new tax schedule is as follows.

$$\begin{aligned} \tau &= .25 \quad if \quad 1+wL-C \leq \frac{3}{2} \\ &= .50 \quad if \quad 1+wL-C > \frac{3}{2} \end{aligned}$$

To find the change in collected revenue resulting from the change in the tax code, we need to solve for the new demands of the two types of consumer. The price of coconuts has now fallen from 1 to $1-\tau$. Each consumer will now solve the following constrained utility maximization problem.

$$\max_{J,C,L} \quad \log(J)+\log(C)+\log(1-L)$$

$$s.t.\ J+\frac{3}{4}C=\frac{3}{4}+\frac{3}{4}wL \quad if \quad L \leq \frac{1}{2w}+\frac{C}{w}$$

$$J+\frac{3}{4}C=\frac{7}{8}+\frac{1}{2}wL \quad if \quad L > \frac{1}{2w}+\frac{C}{w}$$

First, let's consider segment 1 (25% tax rate) of the non-linear budget set. FOCs of this problem are

$$\frac{1}{J} = \lambda; \quad \frac{1}{C} = \frac{3}{4}\lambda; \quad \frac{1}{1-L} = \frac{3}{4}w\lambda$$

Plugging these FOCs in the budget constraint, we get

$$J = \frac{1}{\lambda} = \frac{1}{4} + \frac{1}{4}w \tag{15}$$

$$C = \frac{4}{3\lambda} = \frac{1}{3} + \frac{1}{3}w \tag{16}$$

$$L = 1 - \frac{4}{3w\lambda} = \frac{2}{3} - \frac{1}{3w} \tag{17}$$

For the low type worker, we have

$$\begin{aligned} J_l &= \frac{1}{2} \\ C_l &= \frac{2}{3} \\ L_l &= \frac{1}{3}, \end{aligned}$$

which is consistent with $L \leq \frac{1}{2w} + \frac{C}{w}$. For the high type worker, we have

$$\begin{aligned} J_h &= \frac{3}{4} \\ C_h &= 1 \\ L_h &= \frac{1}{2}, \end{aligned}$$

which is also consistent with $L \leq \frac{1}{2w} + \frac{C}{w}$. Therefore, if coconut purchases are tax deductible, both types are on the first segment of the non-linear budget set (25% tax rate).

Per capita tax revenue in this case would become

$$\begin{aligned} R' &= \frac{1}{2}\left(\frac{1}{4}\left(1 + \frac{1}{3} - \frac{2}{3}\right) + \frac{1}{4}\left(1 + 2 \times \frac{1}{2} - 1\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{6} + \frac{1}{4}\right) = \frac{5}{24} \end{aligned}$$

The change in per capita revenue is

$$\Delta R = R' - R = \frac{5}{24} - \frac{5}{12} = -\frac{5}{24}$$

Each of the following factors play a role in the revenue change:

- Spending on coconut is deductible, which mechanically reduces tax revenue.
- Because coconut price is effectively lower than before, workers would like to consume more coconuts now, which further reduces tax revenue.
- Because of the coconut deduction, the high type worker now faces the low marginal tax rate (25%) instead of the high marginal tax rate (50%). This pushes tax revenue down even more.
- Lower marginal tax rate will cause the high type worker to work more, which reduces the revenue cost of the coconut tax deduction (i.e. increases revenue).

5.4 Part (d)

If we use the static revenue estimation to calculate the per capita revenue cost of the proposal, we have

$$RC_{static} = \frac{1}{2}\left[\left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{5}{8} \times \frac{1}{2}\right)\right] = \frac{7}{32} = 0.219$$

The actual per capita revenue cost of the proposal should be

$$RC_{actual} = \frac{5}{12} - \frac{5}{24} = \frac{5}{24} = 0.208$$

We can see that the two are not very different from each other. In this special case, static revenue estimation is a relatively accurate and fast way of calculating the effects of a tax change. However, it is possible that in general models the new equilibrium that results from a policy change will look drastically different from the old equilibrium, making static revenue estimation less accurate.