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Summary: An Earth-attached and thus rotating reference frame is almost always used for the analysis of geophysical flows. The equation of motion transformed into a steadily rotating reference frame includes two terms that involve the rotation vector; a centrifugal term and a Coriolis term. In the special case of an Earth-attached reference frame, the centrifugal term is exactly canceled by gravitational mass attraction and drops out of the equation of motion. When we solve for the acceleration seen from an Earth-attached frame, the Coriolis term is interpreted as a force. The rotating frame perspective gives up the properties of global momentum conservation and invariance to Galilean transformation. Nevertheless, it leads to a greatly simplified analysis of geophysical flows since only the comparatively small relative velocity, i.e., winds and currents, need be considered.

The Coriolis force has a simple mathematical form, $-2\boldsymbol{\Omega} \times \boldsymbol{V}'M$, where $\boldsymbol{\Omega}$ is Earth's rotation vector, $\boldsymbol{V}'$ is the velocity observed from the rotating frame and M is the particle mass. The Coriolis force is perpendicular to the velocity and can do no work. It tends to cause a deflection of velocity, and gives rise to two important modes of motion: (1) If the Coriolis force is the only force acting on a moving particle, then the velocity vector of the particle will be continually deflected and rotate clockwise in the northern hemisphere and anticlockwise in the southern hemisphere. These so-called inertial oscillations are a first approximation of the upper ocean currents that are generated by a transient wind event. (2) If the Coriolis force is balanced by a steady force, say a pressure gradient, then the resulting wind or current is also steady and is perpendicular to the force. An approximate geostrophic momentum balance of this kind is the defining characteristic of the large scale, extra-tropical circulation of the atmosphere and oceans.

Contents

1 The defining characteristic of large-scale, geophysical flows.

The large-scale, horizontal flows of Earth's atmosphere and ocean take the form of circulations around centers of high or low pressure. Global-scale circulations include the atmospheric jet stream that encircles the mid-latitudes in both hemispheres (Fig. 1), and the oceanic circumpolar current that encircles the Antarctic continent. Smaller scale circulations often dominate the weather. Hurricanes and mid-latitude storms, for example, have a more or less circular flow around a low pressure center, and many regions of the ocean are filled with slowly revolving eddies having high or low pressure anomalies. The pressure anomaly that is associated with each of these circulations can be understood as the direct consequence of mass excess or deficit in the overlying fluid.

What is at first surprising is that large scale mass and pressure anomalies persist for many days or weeks even in the absence of an external energy source. The flow of mass that would be expected to accelerate down the pressure gradient and disperse the mass and pressure anomaly does not occur. Instead, large-scale winds and currents are observed to flow in a direction that almost parallel to lines of constant pressure and from this we can infer that the pressure gradient force, which is normal to lines of constant pressure, must be nearly balanced by a second force, the Coriolis force,[1, 2] that tends to deflect winds and currents to the right in the northern hemisphere and to the left in the southern hemisphere.[3] A momentum balance between a pressure gradient and the deflecting Coriolis force is called a geostrophic balance, and is perhaps the defining characteristic of large scale atmospheric and oceanic flows outside of equatorial regions.

We attribute quite profound physical consequences to the Coriolis force, and yet we cannot point to a physical interaction as the cause of the Coriolis force in the direct and simple way that we can relate pressure anomalies to the mass field. Rather, the Coriolis force arises from motion itself, and specifically from our

[1]Footnotes provide references, extensions or qualifications of material discussed in the main text, and homework assignments; they may be skipped on first reading.

[2]After the French physicist and engineer, Gaspard G. de Coriolis, 1792-1843, whose seminal contributions include the systematic derivation of the rotating frame equation of motion and the development of the gyroscope. An informative history of the Coriolis force is by A. Persson, 'How do we understand the Coriolis force?', *Bull. Am. Met. Soc.*, **79**(7), 1373-1385 (1998).

[3]'What's it do right on the equator?' (S. Adams, *It's Obvious You Won't Survive by Your Wits Alone*, p. 107, Andrews and McNeil Press, Kansas City, Kansas, 1995). By symmetry we would expect that the Coriolis deflection does not occur on the equator, and the contrast between equatorial and mid-latitude circulation patterns (i.e., the pressure and velocity fields) is therefore of great interest here. How would you characterize the difference between mid-latitude and tropical regions of Fig. 1? You might also want to visit the web site of Fleet Numerical Meteorology and Oceanography Center, https://www.fnmoc.navy.mil and then follow links to publicly accessible data, MyWxmap, and then choose an equatorial region and map of interest.

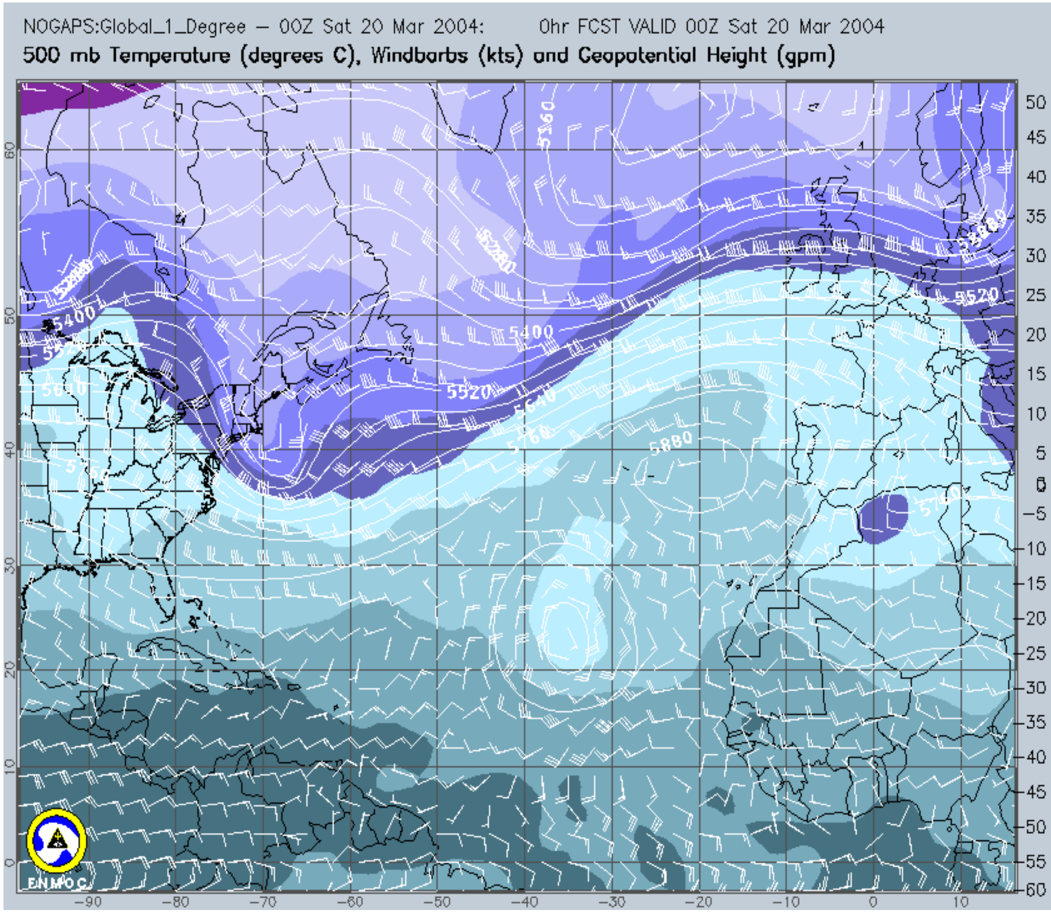

Figure 1: A weather map showing conditions at 500 mb over the North Atlantic on 20 March, 2004, produced by the Fleet Numerical Meteorology and Oceanography Center. [3] Variables are temperature (colors, scale at right is degrees C), the height of the 500 mb pressure surface (white contours, values are approximately meters above sea level) and the wind vector (as 'barbs' at the rear of the vector, one thin barb = 10 knots ≈ 5 m s^{-1}, one heavy barb = 50 knots). Note that the wind vectors appear to be nearly parallel to the contours of constant height everywhere poleward of about 10^o latitude, indicative of an approximate geostrophic balance. Three things to note: (1) This weather map depicts the pressure field by showing the height of a constant pressure surface, rather than the pressure at a constant height. Elevated height of a pressure surface is consistent with high pressure at the same level, and a height gradient is equivalent to a pressure gradient. (2) The winds plotted here are a combination of observed winds and model-predicted winds. The models are not constrained to yield a geostrophic balance, though that usually appears to hold closely outside of equatorial regions. (3) The dominant feature on the 500 mb surface (a middle level of the atmosphere) and at mid-latitudes is almost always the jet stream. On long-term average, the jet stream winds blow from west to east and the 500 mb pressure surface within the jet stream slopes upwards toward lower latitude. Any particular realization of the jet stream is likely to exhibit pronounced north-south undulations. On this day the jet stream winds were southwesterly over the North Atlantic in the latitude range between about 35 N and 55 N. This fairly common pattern transports relatively warm, moist air toward Northern Europe, and has been argued to be a significant part of the reason that Northern Europe enjoys relatively benign winters (Seager, R., D. S. Battisti, J. Yin, N. Gordon, N. Naik, A. C. Clement and M. Cane, 'Is the Gulf Stream responsible for Europe's mild winters?', Q. J. R. Meteorol. Soc., **128**, pp. 1-24, 2002.)

common practice to analyze the atmosphere and ocean using an Earth-attached and thus rotating and noninertial reference frame. This makes the Coriolis force distinct from other important forces in ways and with consequences that are the theme of this essay.

1.1 Classical mechanics on a rotating Earth

For the purpose of studying the Coriolis force we need consider the motion and dynamics of only a single particle, or the equivalent for a fluid, a single parcel. If the parcel is observed from an inertial reference frame, then the classical (Newtonian) equation of motion is just

$$\frac{d(M\boldsymbol{V})}{dt} = \boldsymbol{F} + \boldsymbol{g}_* M,$$

where d/dt is an ordinary time derivative, $\boldsymbol{V}$ is the velocity in a three-dimensional space, and M is the parcel's mass. The mass will be presumed constant in all that follows, and the equation of motion rewritten as

$$\frac{d\boldsymbol{V}}{dt} M = \boldsymbol{F} + \boldsymbol{g}_* M. \tag{1}$$

$\boldsymbol{F}$ is the sum of the forces that we can specify *a priori* given the complete knowledge of the environment, e.g., a pressure gradient, or frictional drag with the ground or adjacent parcels, and $\boldsymbol{g}_*$ is gravitational mass attraction. These are said to be central forces insofar as they are effectively instantaneous, they act in a radial direction between parcels (and in the case of gravitational mass attraction, between parcels and the Earth) and hence they occur as action-reaction force pairs. For the purpose at hand it is appropriate to make two strong simplifications of $\boldsymbol{F}$: 1) we will specify $\boldsymbol{F}$ independently of the motion of surrounding parcels, and, 2) we will make no allowance for the continuity of volume of a fluid flow, i.e., that two parcels can not occupy the same point in space. As a result, our solutions will apply strictly only to a very special class of fluid flows — those that are spatially homogeneous.

This inertial reference frame[4] equation of motion has two fundamental properties that we note here because we are about to give them up:

Global conservation. For each of the central forces acting on the parcel there will be a corresponding reaction force acting on the part of the environment that sets up the force. Thus the global time rate of change of momentum (global means parcel plus the environment) due to the sum of all of the forces $\boldsymbol{F} + \boldsymbol{g}_* M$ is zero, i.e., global momentum is conserved. Usually our attention is focused on the local problem, i.e., the parcel only, with global conservation taken for granted and not analyzed explicitly.

[4]It is worthwhile to define 'inertial', an adjective that arises over and over again in this essay, as does 'reference frame'. Inertia has Latin roots *in+artis* meaning without art or skill and secondarily, resistant to change. Since Newton's *Principia* physics usage has emphasized the latter, a parcel having inertia will remain at rest, or if in motion, continue without change unless subjected to an external force. By reference frame we mean a coordinate system that serves to arithmetize the position of parcels, a clock to tell the time, and an observer who makes an objective record of positions and times. A reference frame may or may not be attached to a physical object. In this essay we suppose purely classical physics so that measurements of length and of time are identical in all reference frames. This common sense view of space and time begins to fail when velocities approach the speed of light, which is not an issue here. An inertial reference frame is one in which all parcels have the property of inertia and in which the total momentum is conserved, i.e., all forces occur as action-reaction force pairs. How this plays out in the presence of gravity will be discussed briefly in Section 3.1.

Invariance to Galilean transformation. Eq. (1) should be invariant to a steady (linear) translation of the reference frame, often called a Galilean transformation. A constant velocity added to $\boldsymbol{V}$ will cause no change in the time derivative, and if added to the environment should as well cause no change in the forces $\boldsymbol{F}$ or $\boldsymbol{g} * M$. Like the global balance just noted, this property is not invoked frequently, but is a powerful guide to the appropriate forms of the forces $\boldsymbol{F}$. For example, a frictional force that satisfies Galilean invariance should depend upon the spatial difference of the velocity with respect to a surface or adjacent parcels, and not the parcel velocity only.

When it comes to the practical analysis of the atmosphere or ocean we invariably use a reference frame that is attached to the Earth — true (literal) inertial reference frames are simply not accessible. Some of the reasons for this are discussed in a later section, 4.3; for now we are concerned with the consequence that, because of the Earth's rotation, an Earth-attached reference frame is significantly noninertial for the large-scale motions of the atmosphere and ocean. The equation of motion (1) transformed into an Earth-attached reference frame (examined in detail in Sections 2 and 4.1) is

$$\frac{d\boldsymbol{V}'}{dt}M = -2\boldsymbol{\Omega}\times\boldsymbol{V}'M + \boldsymbol{F}' + \boldsymbol{g}M, \tag{2}$$

where the prime on a vector indicates that it is observed from the rotating frame, $\boldsymbol{\Omega}$ is Earth's rotation vector and $\boldsymbol{g}M$ is the time-independent inertial force, gravitational mass attraction plus the centrifugal force associated with Earth's rotation. This combined inertial force will be called 'gravity' and discussed further in Section 4.1. Our main interest is the term, $2\boldsymbol{\Omega}\times\boldsymbol{V}'M$, commonly called the Coriolis force in geophysics. The Coriolis force has a very simple mathematical form; it is always perpendicular to the parcel velocity and will thus act to change the direction of the velocity unless it is balanced by another force, e.g., very often a pressure gradient as noted in the opening paragraph.

1.2 The goal and the plan

Eq. (2) applied to geophysical flows is not controversial, and if our intentions were strictly practical we could just accept it, as we do a few fundamental concepts of classical mechanics, e.g., mass and gravitational mass attraction, and move on to applications. However, the Coriolis force is not a fundamental concept of that kind and yet for many students (and more) it has a certain similar, mysterious quality. The goal and the plan of this essay is to take a rather slow and careful journey from Eq. (1) to (2) so that at the end we should be able to explain:[5]

1) The origin of the term $2\boldsymbol{\Omega}\times\boldsymbol{V}'M$, and in what respect it is appropriate to call it the Coriolis 'force'. We have already hinted that the Coriolis term represents an inertial force (reviewed in Section 2.1) that arises from the rotation of a reference frame. The origin is thus mainly kinematic, i.e., more mathematical than physical, and we begin in Section 2.2 with the transformation of the inertial frame equation of motion Eq. (1) into a rotating reference frame and Eq. (2). What we should call the Coriolis term is less clear than is Eq. (2) itself; in the classical dynamics literature the same term is called an acceleration, a pseudo force, a virtual force, an apparent force, an inertial force — our choice when we do not have to be

[5] 'Explanation is indeed a virtue; but still, less a virtue than an anthropocentric pleasure.' B. van Frassen, 'The pragmatics of explanation', in *The Philosophy of Science*, Ed. by R. Boyd, P. Gasper and J. D. Trout. (The MIT Press, Cambridge Ma, 1999).

concise — a *non*inertial force, or, most equivocal of all, a fictitious correction force. [6] We will stick with just plain 'Coriolis force' on the basis of what the Coriolis term does, considered in Sections 3, 4 and 5, and summarized on closing in Section 6.

2) The global conservation and Galilean transformation properties of Eq. (2). Two simple applications of the rotating frame equation of motion are considered in Section 3. These illustrate the often marked difference between inertial and rotating frame descriptions of the same motion, and they also show that the rotating frame equation of motion does *not* retain these fundamental properties.

3) The relationship between the Coriolis and centrifugal forces, and the absence of the latter in Eq. (2). The special and especially important case of an Earth-attached reference frame is discussed in Section 4. As we will see, Eq. (2) applies on a rotating planet, say, where the centrifugal force is exactly canceled, and does not obtain for a general rotating reference frame.

4) The new modes of motion in Eq. (2) compared with Eq. (1), and the geostrophic balance commonly observed to hold in the atmosphere and ocean. A very simple problem that illustrates some consequences of the Coriolis force is treated in Section 5. Eq. (2) admits two modes of motion dependent upon the Coriolis force; a free oscillation, usually called an inertial oscillation, and forced, steady motion, called a geostrophic wind or current when the force $\boldsymbol{F}'$ is a pressure gradient.[7]

1.3 About this essay

This essay is pedagogical in aim and style. It has been written for students who are beginning a quantitative study of Earth science and geophysical fluid dynamics and who have some background of classical mechanics and applied mathematics. Rotating reference frames and the Coriolis force are discussed in many classical mechanics texts[8] and in most fluid mechanics textbooks that treat geophysical flows. [9] There is nothing

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[6]The latter is by by J. D. Marion, *Classical Mechanics of Particles and Systems* (Academic Press, NY, 1965), who describes the plight of a rotating observer as follows (the double quotes are his): '... the observer must postulate an additional force - the centrifugal force. But the "requirement" is an artificial one; it arises solely from an attempt to extend the form of Newton's equations to a non inertial system and this may be done only by introducing a fictitious "correction force". The same comments apply for the Coriolis force; this "force" arises when attempt is made to describe motion relative to the rotating body.' We are inclined to be more inclusive regarding reference frames and observers, probably because we anticipate being rotating observers. Our position will be that all reference frames and observers are equally valid. Noninertial reference frames do indeed incur inertial forces that are not found in otherwise comparable inertial frames, but these inertial forces are not *ad hoc* corrections as Marion's quote (taken out of context) might seem to imply.

[7]By now you may be thinking that all this talk of 'forces, forces, forces' is tedious, and even a archaic. Modern dynamics is more likely to be developed around the concepts of energy, action and minimization principles, which are very useful in some special classes of fluid flow. However, it remains that the vast majority of fluid mechanics proceeds along the path of Eq. (1) laid down by Newton. In part this is because mechanical energy is not conserved in most real fluid flows and in part because the interaction between a fluid parcel and its surroundings is often described in terms of forces, as we will continue to do here.

[8]In order of increasing level: A. P. French, *Newtonian Mechanics* (W. W. Norton Co., 1971); A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw-Hill, NY, 1990); C. Lanczos, *The Variational Principles of Mechanics* (Dover Pub., NY, 1949). A clear treatment by variational methods is by L. D. Landau and E. M. Lifshitz *Mechanics*, (Pergamon, Oxford, 1960).

[9]Textbooks on geophysical fluid dynamics emphasize mainly the consequences of Earth's rotation; excellent introductions at about the level of this essay are by J. R. Holton, *An Introduction to Dynamic Meteorology, 3rd Ed.* (Academic Press, San Diego, 1992), and

fundamental and new added here, but the hope is that this essay will make a useful supplement to these and other sources[10] by providing somewhat greater mathematical detail than do most fluid dynamics texts (in Section 2), while emphasizing relevant geophysical phenomena that are missed in most physics texts (in Sections 4 and 5).[11]

This text is meant to be accompanied by five Matlab scripts that allow for a wide range of experimentation and better graphical presentation than is possible in a hard copy. [12] This text and the Matlab scripts may be freely copied and distributed for personal, educational purposes. The essay may be cited as an unpublished manuscript available from the author's web page. Comments and questions are encouraged and may be addressed to the author at jprice@whoi.edu.

2 Noninertial reference frames and inertial forces.

The first step toward understanding the origin of the Coriolis force is to describe the origin of inertial forces in the simplest possible context, a pair of reference frames that are represented by displaced coordinate axes, Fig. (2). Frame one is labeled X and Z and frame two is labeled X' and Z'. Only relative motion is significant, but there is no harm in assuming that frame one is stationary and that frame two is displaced by a time-dependent vector, $\boldsymbol{X_o}(t)$. The measurements of position, velocity, etc. of a given parcel will thus be different in frame two vs. frame one. Just how the measurements differ is a matter of pure kinematics; there is no physics involved until we use the accelerations to write an equation of motion, e.g., Eq. (2).

2.1 Kinematics of a translating reference frame

If the position vector of a given parcel is $\boldsymbol{X}$ when observed from frame one, then from within frame two the same parcel will be observed at the position

$$\boldsymbol{X}' = \boldsymbol{X} - \boldsymbol{X_o}.$$

a particularly thorough account of the Coriolis force is by B. Cushman-Roisin, *Introduction to Geophysical Fluid Dynamics* (Prentice Hall, Englewood Cliffs, New Jersey, 1994). Somewhat more advanced is A. E. Gill, *Atmosphere-Ocean Dynamics* (Academic Press, NY, 1982).

[10]There are several essays or articles that, like this one, aim to clarify the Coriolis force. A fine treatment in great depth is by H. M. Stommel and D. W. Moore, *An Introduction to the Coriolis Force* (Columbia Univ. Press, 1989); the present Section 4.1 owes a great deal to their work. A detailed analysis of particle motion including the still unresolved matter of the apparent southerly deflection of dropped particles is by M. S. Tiersten and H. Soodak, 'Dropped objects and other motions relative to a noninertial earth', *Am. J. Phys.*, **68**(2), 129–142 (2000). An excellent web page is http://met.no/english/topics/nomek˙2005/coriolis.pdf and for general science students, http://www.ems.psu.edu/%7Efraser/Bad/BadFAQ/BadCoriolisFAQ.html

[11]The Coriolis force is exploited to measure the angular velocity required for vehicle control systems, http://www.siliconsensing.com, and to measure mass transport in fluid flow, http://www.micromotion.com.

[12]The Matlab scripts — rotation.m, Coriolis.m, Coriolis-forced.m, partslope.m and geoadjPE.m — can be recovered from the Mathworks File Exchange archive, http://www.mathworks.com/matlabcentral/fileexchange/loadCategory.do in the 'Earth Sciences' category where the file name is Coriolis, or from the author's web page, http://www.whoi.edu/science/PO/people/jprice/class/Coriolis.zip, where the most recent draft of this manuscript may also be found, http://www.whoi.edu/science/PO/people/jprice/class/aCt.pdf

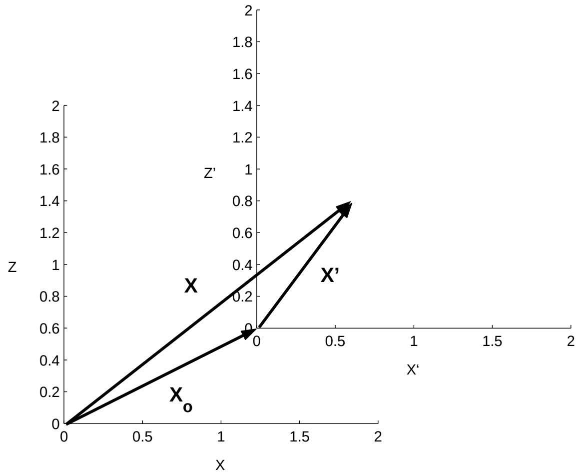

Figure 2: Two reference frames are represented by coordinate axes that are displaced by the vector $\boldsymbol{X_o}$ that is time-dependent. In this Section 2.1 we consider only a relative translation, so that frame two maintains a fixed orientation with respect to frame one. The rotation of frame two will be considered beginning in Section 2.2.

The position vector of a parcel thus depends upon the reference frame. Suppose that frame two is translated and possibly accelerated with respect to frame one, while maintaining a constant orientation (rotation will be considered shortly). If the velocity of a parcel observed in frame one is $d\boldsymbol{X}/dt$, then in frame two the same parcel will be observed to have velocity

$$\frac{d\boldsymbol{X}'}{dt} = \frac{d\boldsymbol{X}}{dt} - \frac{d\boldsymbol{X_o}}{dt}.$$

The accelerations are similarly $d^2\boldsymbol{X}/dt^2$ and

$$\frac{d^2\boldsymbol{X}'}{dt^2} = \frac{d^2\boldsymbol{X}}{dt^2} - \frac{d^2\boldsymbol{X_o}}{dt^2}. \tag{3}$$

We are going to assume that frame one is an inertial reference frame, i.e., that parcels observed in frame one have the property of inertia so that their momentum changes only in response to a force, $\boldsymbol{F}$, i.e., Eq. (1). From Eq. (1) and from Eq. (3) we can easily write down the equation of motion for the parcel as it would be observed frame two:

$$\frac{d^2\boldsymbol{X}'}{dt^2}M = -\frac{d^2\boldsymbol{X_o}}{dt^2}M + \boldsymbol{F} + \boldsymbol{g}_*M. \tag{4}$$

Terms of the sort $-(d^2\boldsymbol{X_o}/dt^2)M$ appearing in the frame two equation of motion (4) will be called 'inertial forces', and when these terms are nonzero, frame two is said to be 'noninertial'. As an example, suppose that frame two is subject to a constant acceleration, $d^2\boldsymbol{X_o}/dt^2 = \boldsymbol{e_x}a_x + \boldsymbol{e_z}a_z$ where $a_x, a_z > 0$ so that the acceleration of frame two relative to frame one is upward and to the right in Fig. (2). All parcels observed from within frame two would then appear to be subject to an inertial force, $-(\boldsymbol{e_x}a_x + \boldsymbol{e_z}a_z)M$, directed downward and to the left, and exactly opposite the acceleration of frame two with respect to frame one. This inertial force is exactly proportional to the mass of the parcel, regardless of what the mass is, and so evidently it is an acceleration that is imposed, and not a force *per se*. In this important regard, these inertial forces are indistinguishable from gravitational mass attraction. If the inertial forces are dependent only upon position, as is gravitational mass atraction, then they might as well be added with $\boldsymbol{g}_*$ to make a single acceleration field, usually termed gravity and denoted by just plain $\boldsymbol{g}$. Indeed, it is only the gravity field, $\boldsymbol{g}$, that can be observed directly (more in Section 4.1). But unlike gravitational mass attraction, there is no physical interaction involved in producing an inertial force, and hence there is no action-reaction force pair. Global momentum conservation thus does not obtain in the presence of inertial forces. There is indeed something equivocal about these so-called inertial forces, and is not unwarranted that many authors [6] deem these terms to be 'virtual' or 'fictitious correction' forces.

Whether an inertial force is problematic or not depends entirely upon whether $d^2\boldsymbol{X_o}/dt^2$ is known or not. If it should happen that the acceleration of frame two is not known, then all bets are off. For example, imagine observing the motion of a pendulum within an enclosed trailer that was moving along in stop-and-go traffic. The pendulum would be observed to lurch forward and backward as if the local gravitational acceleration was changing randomly with time, and we would soon conclude that dynamics in such a noninertial reference frame was going to be a very difficult endeavor. We could at least infer that an inertial force was to blame if it was observed that all of the parcels in the trailer, observers included, experienced exactly the same unaccounted acceleration. Very often we do know the relevant inertial forces well enough to use noninertial reference frames with great precision, e.g., Earth's gravity field is well-known from extensive and ongoing survey and the Coriolis force can be readily calculated.

In the specific example of reference frame translation considered here we could just as well transform the observations made from frame two back into the inertial frame one, use the inertial frame equation of motion to make a calculation, and then transform back to frame two if required. By that tactic we could avoid altogether the seeming delusion of an inertial force. However, when it comes to the observation and analysis of Earth's atmosphere and ocean, there is really no choice but to use an Earth-attached and thus rotating and noninertial reference (discussed in Section 4.3). That being so, we have to contend with the Coriolis force, an inertial force that arises from the rotation of an Earth-attached frame. The kinematics of rotation add a small complication (next section), but if you followed the development of Eq. (4), then it is fair to say that you already understand the essential origin of the Coriolis force.

2.2 Kinematics of a rotating reference frame

The second step toward understanding the origin of the Coriolis force is to learn the equivalent of Eq. (3) for the case of a steadily rotating (rather than translating) reference frame. The tactic will be to develop the component-wise form of the equations of motion and then construct the corresponding geometric, vector equation. Reference frame one will again be assumed to be stationary and defined by a triad of orthogonal unit vectors, $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, $\boldsymbol{e_3}$, that are time-independent (Fig. 3). A parcel P can then be located by a position vector $\boldsymbol{X}$

$$\boldsymbol{X} = \boldsymbol{e_1}x_1 + \boldsymbol{e_2}x_2 + \boldsymbol{e_3}x_3, \tag{5}$$

where the Cartesian (rectangular) components, x_i, are the projection of $\boldsymbol{X}$ onto each of the unit vectors in turn. It is useful to rewrite Eq. (3) using matrix notation. The unit vectors are made the elements of a row matrix,

$$\mathbb{E} = [\boldsymbol{e_1}\ \boldsymbol{e_2}\ \boldsymbol{e_3}], \tag{6}$$

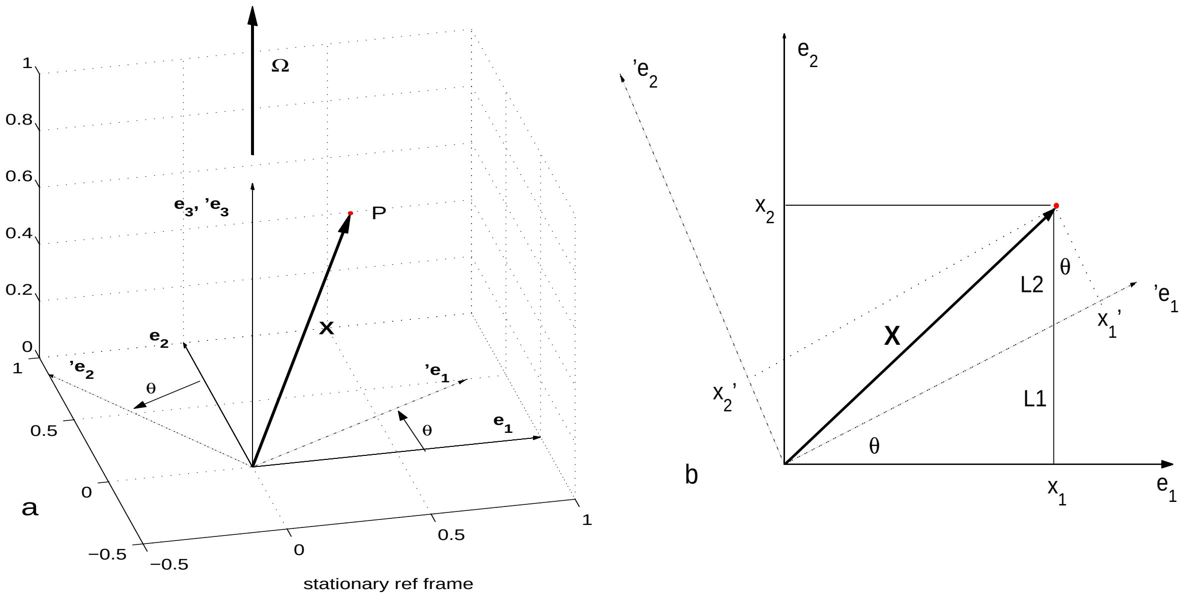

Figure 3: (a) A parcel P is located by the tip of a position vector, $\boldsymbol{X}$. The stationary reference frame has solid unit vectors that are presumed to be time-independent, and a second, rotated reference frame has dashed unit vectors that are labeled $`\boldsymbol{e_i}$. The reference frames have a common origin, and rotation is about the $\boldsymbol{e_3}$ axis. The unit vector $\boldsymbol{e_3}$ is thus unchanged by this rotation and so $`\boldsymbol{e_3} = \boldsymbol{e_3}$. This holds also for $\boldsymbol{\Omega}' = \boldsymbol{\Omega}$, and so we will use $\boldsymbol{\Omega}$ exclusively. The angle θ is counted positive when the rotation is counterclockwise. (b) The components of $\boldsymbol{X}$ in the stationary reference frame are x_1, x_2, x_3, and in the rotated reference frame they are x'_1, x'_2, x'_3.

and the components x_i are taken to be the elements of a column matrix,

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{7}$$

Eq. (3) may then be written in a way that conforms with the usual matrix multiplication rules as

$$\boldsymbol{X} = \mathbb{E}\mathbb{X}. \tag{8}$$

The vector $\boldsymbol{X}$ and its time derivatives are presumed to have an objective existence, i.e., they represent something physical that is unaffected by our arbitrary choice of a reference frame. Nevertheless, the way these vectors appear does depend upon the reference frame, and for our purpose it is essential to know how the position, velocity and acceleration vectors appear when they are observed from a steadily rotating reference frame. In a later part of this section we will identify the rotating reference frame as an Earth-attached reference frame and the stationary frame as one aligned on the distant fixed stars. It is assumed that the motion of the rotating frame can be represented by a single rotation vector, $\boldsymbol{\Omega}$, and that the $\boldsymbol{e_3}$ unit vector can be aligned with $\boldsymbol{\Omega}$ with no loss of generality, Fig. (3a). We can go a step further and align the origins of the stationary and rotating reference frames because the Coriolis force is independent of position (Section 2.2).

2.2.1 Transforming the position, velocity and acceleration vectors

Position: Back to the question at hand: how does this position vector look when viewed from a second reference frame that is rotated through an angle θ with respect to the first frame? The answer is that the vector 'looks' like the components appropriate to the rotated reference frame, and so we need to find the projection of $\boldsymbol{X}$ onto the unit vectors that define the rotated frame. The details are shown in Fig. (3b); notice that $x_2 = L1 + L2$, $L1 = x_1 \tan\theta$, and $x_2' = L2\cos\theta$. From this it follows that $x_2' = (x_2 - x_1\tan\theta)\cos\theta = -x_1\sin\theta + x_2\cos\theta$. By a similar calculation we can find that $x_1' = x_1\cos(\theta) + x_2\sin(\theta)$. The component x_3' that is aligned with the axis of the rotation vector is unchanged, $x_3' = x_3$, and so the set of equations for the primed components may be written as a column vector

$$\mathbb{X}' = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \\ x_3 \end{bmatrix}. \tag{9}$$

By inspection this can be factored into the product

$$\mathbb{X}' = \mathbb{R}\mathbb{X}, \tag{10}$$

where $\mathbb{X}$ is the matrix of stationary frame components and $\mathbb{R}$ is the rotation matrix,

$$\mathbb{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

The position vector observed from the rotated frame will be denoted by $\boldsymbol{X}'$; to construct $\boldsymbol{X}'$ we sum the rotated components, $\mathbb{X}'$, times a set of unit vectors that are fixed and thus

$$\boxed{\boldsymbol{X}' = \boldsymbol{e_1}x_1' + \boldsymbol{e_2}x_2' + \boldsymbol{e_3}x_3' = \mathbb{E}\mathbb{X}'} \tag{12}$$

For example, the position vector $\boldsymbol{X}$ of Fig. (3) is at an angle of about 45 degrees counterclockwise from the $\boldsymbol{e_1}$ unit vector and the rotated frame is at $\theta = 30$ degrees counterclockwise from the stationary frame one. That being so, the position vector viewed from the rotated reference frame, $\boldsymbol{X}'$, makes an angle of 45 - 30 = 15 degrees with respect to the $\boldsymbol{e_1}$ (fixed) unit vector seen within the rotated frame, Fig. (4). As a kind of verbal shorthand we might say that the position vector has been 'transformed' into the rotated frame by Eqs. (9) and (12). But since the vector has an objective existence, what we really mean is that the components of the position vector are transformed by Eq. (9) and then summed with fixed unit vectors to yield what should be regarded as a new vector, $\boldsymbol{X}'$, the position vector that we observe from the rotated (or rotating) reference frame.[13]

[13] If the somewhat formal-looking Eqs. (9) through (12) do not have an immediate and concrete meaning for you, then the remainder of this important section will probably be a loss. Some questions/assignments to help you along: 1) Verify Eqs. (9) and (12) by some direct experimentation, i.e., try them and see. 2) Show that the transformation of the vector components given by Eqs. (10) and (11) leaves the magnitude of the vector unchanged, i.e., $|\boldsymbol{X}'| = |\boldsymbol{X}|$. 3) Verify that $\boldsymbol{R}(\theta_1)\boldsymbol{R}(\theta_2) = \boldsymbol{R}(\theta_1 + \theta_2)$ and that $\boldsymbol{R}(\theta)^{-1} = \boldsymbol{R}(-\theta)$, where $\boldsymbol{R}^{-1}$ is the inverse (and also the transpose) of the rotation matrix. 4) Show that the unit vectors that define the rotated frame can be related to the unit vectors of the stationary frame by $\grave{\boldsymbol{E}} = \boldsymbol{E}\boldsymbol{R}^{-1}$ and hence the unit vectors observed from the stationary frame

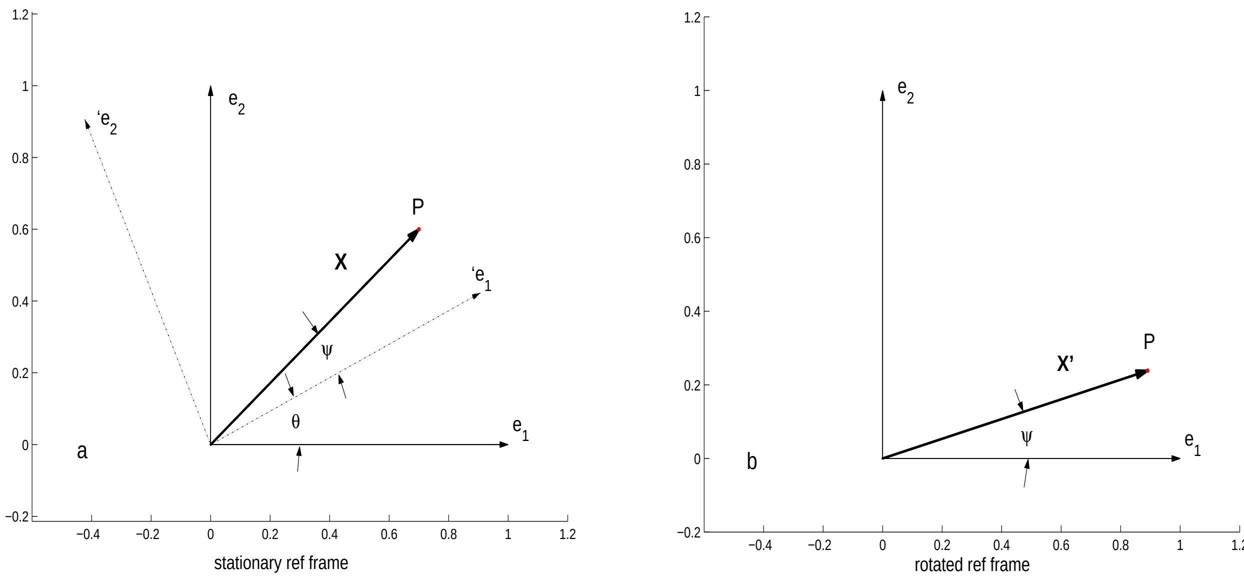

Figure 4: (a) The position vector $\boldsymbol{X}$ seen from the stationary reference frame. (b) The position vector as seen from the rotated frame, denoted by $\boldsymbol{X}'$. Note that in the rotated reference frame the unit vectors are labeled $\boldsymbol{e_i}$ since they are fixed; when these unit vectors are seen from the stationary frame, as on the left, they are labeled $\grave{\boldsymbol{e}}_{\boldsymbol{i}}$. If the position vector is stationary in the stationary frame, then $\theta + \psi = constant$. The angle ψ then changes as $d\psi/dt = -d\theta/dt = -\Omega$, and thus the vector $\boldsymbol{X}'$ appears to rotate at the same rate but in the opposite sense as does the rotating reference frame.

Velocity: The velocity of parcel P seen in the stationary frame is just the time rate of change of the position vector seen in that frame,

$$\frac{d\boldsymbol{X}}{dt} = \frac{d}{dt}\mathbb{E}\mathbb{X} = \mathbb{E}\frac{d\mathbb{X}}{dt},$$

since $\mathbb{E}$ is time-independent. The velocity of parcel P as seen from the rotating reference frame is similarly

$$\frac{d\boldsymbol{X}'}{dt} = \frac{d}{dt}\mathbb{E}\mathbb{X}' = \mathbb{E}\frac{d\mathbb{X}'}{dt},$$

which indicates that the time derivatives of the rotated components are going to be very important in what follows. For the first derivative we find

$$\frac{d\mathbb{X}'}{dt} = \frac{d(\mathbb{R}\mathbb{X})}{dt} = \frac{d\mathbb{R}}{dt}\mathbb{X} + \mathbb{R}\frac{d\mathbb{X}}{dt}. \tag{13}$$

The second term on the right side of Eq. (13) represents velocity components from the stationary frame that have been transformed into the rotating frame, as in Eq. (10). If the rotation angle θ was constant so that $\mathbb{R}$

turn the opposite direction of the position vector observed from the rotating frame (and thus the reversed prime). What, then, can you make of $\grave{\boldsymbol{E}}\boldsymbol{X}'$? A concise and clear reference on matrix respresentations of coordinate transformations is by J. Pettofrezzo *Matrices and Transformations* (Dover Pub., New York, 1966). An excellent all-around reference for undergraduate-level applied mathematics including coordinate transformations is by M. L. Boas, *Mathematical Methods in the Physical Sciences, 2nd edition* (John Wiley and Sons, 1983). Note that some authors define the angle θ of the rotation matrix to be the angle turned by the vector rather than the angle turned by the second reference frame, as is presumed here.

was independent of time, then the first term on the right side would vanish and the velocity components would transform exactly as do the components of the position vector. In that case there would be no Coriolis force.

When the rotation angle is time-varying, as we intend it will be here, the first term on the right side of Eq. (13) is non-zero and represents a velocity component that is induced solely by the rotation of the reference frame. With Earth-attached reference frames in mind, we are going to take the angle θ to be

$$\theta = \theta_0 + \Omega t,$$

where Ω is Earth's rotation rate, a constant defined below (and θ_0 is unimportant). Though Ω is constant, the associated reference frame is nevertheless accelerating and is noninertial in the same way that circular motion at a steady speed is accelerating because the direction of the velocity vector is continually changing. Given this $\theta(t)$, the time-derivative of the rotation matrix is

$$\frac{d\mathbb{R}}{dt} = \Omega \begin{bmatrix} -\sin\theta(t) & \cos\theta(t) & 0 \\ -\cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

which, notice, this has all the elements of $\mathbb{R}$, but shuffled around. By inspection, this matrix can be factored into the product of a matrix $\mathbb{C}$ and $\mathbb{R}$ as

$$\frac{d\mathbb{R}}{dt} = \Omega\, \mathbb{C}\mathbb{R}(\theta(t)), \tag{15}$$

where the matrix $\mathbb{C}$ is

$$\mathbb{C} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi/2). \tag{16}$$

Multiplication by $\mathbb{C}$ acts to knock out the component $(\,)_3$ that is parallel to $\boldsymbol{\Omega}$ and causes a rotation of $\pi/2$ in the plane perpendicular to $\boldsymbol{\Omega}$. Substitution into Eq. (13) gives the velocity components appropriate to the rotating frame

$$\frac{d(\mathbb{R}\mathbb{X})}{dt} = \Omega\mathbb{C}\mathbb{R}\mathbb{X} + \mathbb{R}\frac{d\mathbb{X}}{dt}, \tag{17}$$

or using the prime notation $(\,)'$ to indicate multiplication by $\mathbb{R}$, then

$$\boxed{\frac{d\mathbb{X}'}{dt} = \Omega\mathbb{C}\mathbb{X}' + \left(\frac{d\mathbb{X}}{dt}\right)'} \tag{18}$$

The second term on the right side of Eq. (18) is just the rotated velocity components and is present even if Ω vanished (a rotated but not a rotating reference frame). The first term on the right side represents a velocity that is induced by the rotation rate of the rotating frame; this induced velocity is proprtional to Ω and makes an angle of $\pi/2$ radians to the right of the position vector in the rotating frame (assuming that $\Omega > 0$).

To calculate the vector form of this term we can assume that the parcel P is stationary in the stationary reference frame so that $d\boldsymbol{X}/dt = 0$. Viewed from the rotating frame, the parcel will appear to move clockwise at a rate that can be calculated from the geometry (Fig. 5). Let the rotation in a time interval δt be given by $\delta\psi = -\Omega\delta t$; in that time interval the tip of the vector will move a distance $|\boldsymbol{\delta X}'| = |\boldsymbol{X}'|\sin(\delta\psi) \approx |\,\boldsymbol{X}'|\delta\psi$, assuming the small angle approximation for $\sin(\delta\psi)$. The parcel will move

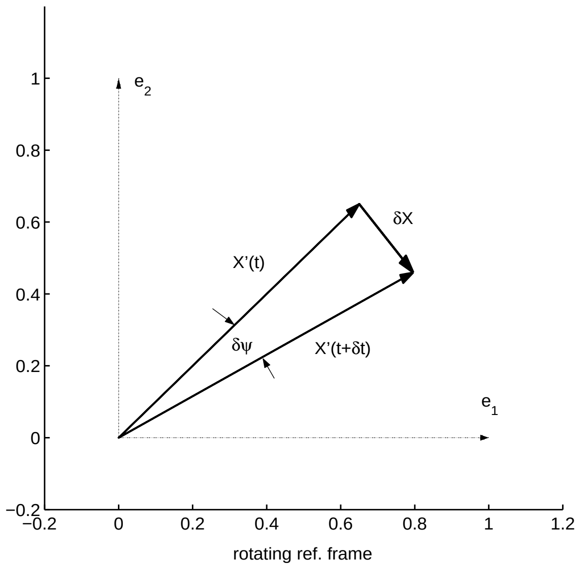

Figure 5: The position vector $\boldsymbol{X}'$ seen from the rotating reference frame. The unit vectors that define this frame, $\grave{}\boldsymbol{e_i}$, appear to be stationary when viewed from within this frame, and hence we label them with $\boldsymbol{e_i}$ (not primed). Assume that $\Omega > 0$ so that the rotating frame is turning counterclockwise with respect to the stationary frame, and assume that the parcel P is stationary in the stationary reference frame so that $d\boldsymbol{X}/dt = 0$. Parcel P as viewed from the rotating frame will then appear to move clockwise at a rate that can be calculated from the geometry.

in a direction that is perpendicular (instantaneously) to $\boldsymbol{X}'$. The velocity of parcel P as seen from the rotating frame and due solely to the coordinate system rotation is thus $\lim_{\delta t \to 0} \frac{\delta \boldsymbol{X}'}{\delta t} = -\boldsymbol{\Omega} \times \boldsymbol{X}'$, the vector equivalent of $\Omega \mathbb{C} \mathbb{X}'$, where $\boldsymbol{\Omega} \times \boldsymbol{X}'$ is the cross-product (Fig. 6). The vector equivalent of Eq. (18) is then

$$\boxed{\frac{d\boldsymbol{X}'}{dt} = -\boldsymbol{\Omega} \times \boldsymbol{X}' + \left(\frac{d\boldsymbol{X}}{dt}\right)'} \tag{19}$$

The relation between time derivatives given by Eq. (19) is general; it applies to all vectors, e.g., velocity vectors, and moreover, it applies for vectors defined at all points in space. [14] Hence the relationship between the time derivatives may be written as an operator equation,

$$\boxed{\frac{d(\ )'}{dt} = -\boldsymbol{\Omega} \times (\ )' + \left(\frac{d(\ )}{dt}\right)'} \tag{20}$$

that is valid for all vectors. From left to right the terms are: 1) the time rate of change of a vector as seen in the rotating reference frame, 2) the cross-product of the rotation vector with the vector and 3) the time rate change of the vector as seen in the stationary frame and then rotated into the rotating frame. One way to describe Eq. (20) is that the time rate of change and prime operators do not commute, the difference being the cross-product term which, notice, represents a time rate change in the *direction* of the vector, but not the magnitude. Term 1) is the time rate of change that we observe directly or that we seek to solve when we are working from the rotating frame.

[14] Imagine arrows taped to a turntable with random orientations. Once the turntable is set into (solid body) rotation, all of the arrows will necessarily rotate at the same rotation rate regardless of their position or orientation. The rotation will, of course, cause a translation of the arrows that depends upon their location, but the rotation rate is necessarily uniform. This is of some importance for our application to a rotating Earth, since Earth's motion includes a rotation about the polar axis, as well as an orbital motion around the Sun and yet we represent Earth's rotation by a single vector.

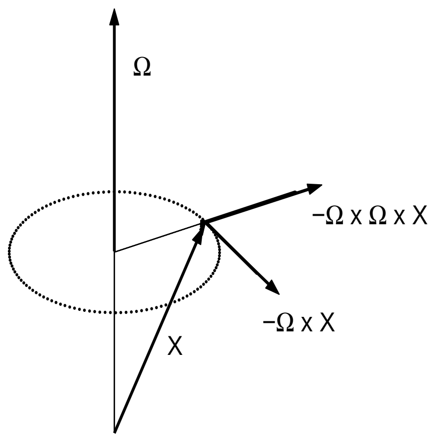

Figure 6: A schematic showing the relationship of a vector $\boldsymbol{X}$, and various cross-products with a second vector $\boldsymbol{\Omega}$ (note the signs). The vector $\boldsymbol{X}$ is shown with its tail perched on the axis of the vector $\boldsymbol{\Omega}$ as if it were a position vector. This helps us to visualize the direction of the cross-products, but it is important to note that the relationship among the vectors and vector products shown here holds for all vectors, regardless of where they are defined in space or the physical quantity, e.g., position or velocity, that they represent.

Acceleration: Our goal is to relate the accelerations seen in the two frames and so we differentiate Eq. (18) once more and after rearrangement of the kind used above find that the components satisfy

$$\boxed{\frac{d^2\mathbb{X}'}{dt^2} = 2\Omega\mathbb{C}\frac{d\mathbb{X}'}{dt} - \Omega^2\mathbb{C}^2\mathbb{X}' + \left(\frac{d^2\mathbb{X}}{dt^2}\right)'} \tag{21}$$

The middle term on the right includes multiplication by the matrix $\mathbb{C}^2 = \mathbb{C}\mathbb{C}$,

$$\mathbb{C}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi/2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi/2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi) = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that knocks out the component corresponding to the rotation vector $\boldsymbol{\Omega}$ and reverses the other two components; the vector equivalent of $-\Omega^2\mathbb{C}^2\mathbb{X}'$ is thus $-\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{X}'$ (Fig. 6). The vector equivalent of Eq. (21) is then [15]

$$\boxed{\frac{d^2\boldsymbol{X}'}{dt^2} = -2\boldsymbol{\Omega}\times\frac{d\boldsymbol{X}'}{dt} - \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{X}' + \left(\frac{d^2\boldsymbol{X}}{dt^2}\right)'} \tag{22}$$

Note the similarity with Eq. (3). From left to right the terms of this equation are 1) the acceleration as seen in the rotating frame, 2) the Coriolis term, 3) the centrifugal [16] term, and 4) the acceleration as seen in the stationary frame and then rotated into the rotating frame. As before, term 1) is the acceleration that we directly observe or analyze when we are working from the rotating reference frame.

[15] The relationship between the stationary and rotating frame velocity vectors given by Eqs. (18) and (19) is clear visually and becomes intuitive given just a little experience. It is not so easy to intuit the corresponding relationship between the accelerations given by Eqs. (22) and (21). Hence, to understand the transformation of acceleration there is no choice but to understand the mathematical steps (to be able to reproduce, be able to explain) going from Eq. (18) to Eq. (21) and/or from Eq. (19) to Eq. (22).

[16] 'Centrifugal' and 'centripetal' have Latin roots, *centri+fugere* and *centri+peter*, meaning center-fleeing and center-seeking, respectively. Taken literally they would indicate the sign of a radial force, for example. However, they are very often used to mean the specific term $\omega^2 r$, i.e., centrifugal force when it is on the right side of an equation of motion and centripetal acceleration when it is on the left side.

2.2.2 Stationary ⇒ Inertial; Rotating ⇒ Earth-Attached

The third and final step toward the origin of the Coriolis force is to specify exactly what we mean by an inertial reference frame, and so define departures from inertial for the rotating frame two. To make frame one inertial we presume that the unit vectors $\boldsymbol{e_i}$ could in principle be aligned on the distant, fixed stars. [17] The rotating frame two is presumed to be attached to Earth, and the rotation rate Ω is then given by the rate at which the same fixed stars are observed to rotate overhead, one revolution per sidereal day, 23 hrs, 56 min and 4.09 sec, or

$$\Omega = 7.2921 \times 10^{-5} \text{ rad sec}^{-1}.$$

Earth's rotation rate is very nearly constant, and the axis of rotation maintains a nearly steady bearing on a point on the celestial sphere that is close to the North Star, Polaris. [18]

[17] 'Fixed stars' serve as sign posts for the spatially-averaged mass of the universe on the hypothesis that inertia arises whenever there is an acceleration (linear or rotational) with respect to the mass of the universe as a whole. This grand idea was expressed most forcefully by the Austrian philosopher and physicist Ernst Mach, and is often termed Mach's Principle (see, e.g., J. Schwinger, *Einsteins Legacy* Dover Publications, 1986; M. Born, *Einstein's Theory of Relativity*, Dover Publications, 1962). Mach's Principle seems to be in accord with all empirical data, but is not, in and of itself, a mechanism of inertia. A new hypothesis takes the form of so-called vacuum stuff that is presumed to pervade all of space and provides a local mechanism for resistance to accelerated motion (see P. Davies, 'On the meaning of Mach's principle', http://www.padrak.com/ine/INERTIA.html). The debate between Newton and Leibniz over the reality of absolute space, which had seemed to go in favor of relative space, Leibniz and Mach's Principle, has been renewed in the search for a physical origin of inertia.

Observations on the fixed stars are an exquisitely precise means to define the rotation rate of an Earth-attached reference frame, and for example, the rotation rate sensors noted in footnote 11 read out the Earth's rotation rate with respect to the fixed stars as a kind of gage pressure, called 'Earth rate'. There is, evidently, a meaningful, absolute rotation rate. On the other hand, observations of the fixed stars can not, in general, be used in the same way to define the translation or (linear) acceleration of a reference frame. The only way to know if a reference frame that is aligned on the fixed stars is inertial is to carry out mechanics experiments and test whether Eq.(1) holds. If it does, the frame is inertial. This would be a vacuuous exercise except for the assertion that all subsequent mechanics experiments would also find this frame to be inertial.

[18] There are small but observable variations of Earth's rotation rate due mainly to changes in the atmosphere and ocean circulation and mass distribution within the cryosphere, see B. F. Chao and C. M. Cox, 'Detection of a large-scale mass redistribution in the terrestrial system since 1998,' Science, **297**, 831–833 (2002), and R. M. Ponte and D. Stammer, 'Role of ocean currents and bottom pressure variability on seasonal polar motion,' *J. Geophys. Res.*, **104**, 23393–23409 (1999). The inclination of the rotation vector with respect to the orbital plane also varies by a few degrees on a time scale of several tens of thousands of years and the direction of the rotation axis precesses on a similar time scale. These slow variations of Earth's orbital parameters (slow for our present purpose) may be an important element of climate, see e.g., J. A. Rial, 'Pacemaking the ice ages by frequency modulation of Earth's orbital eccentricity,' *Science*, **285**, 564–568 (1999).

As well, Earth's motion within the solar system and galaxy is much more complex than a simple spin around the polar axis. Among other things, the Earth orbits the Sun in a counterclockwise direction with a rotation rate of 1.9910×10^{-7} sec^{-1}, which is about 0.3% of the rotation rate Ω. Does this orbital motion enter into the Coriolis force, or otherwise affect the dynamics of the atmosphere and oceans? The short answer is no and yes. We have already accounted for the rotation of the Earth with respect to the fixed stars. Whether this rotation is due to a spin about an axis centered on the Earth or due to a solid body rotation about a displaced center is not relevant for the Coriolis force *per se*, as noted in the discussion of Eq. (20). However, since Earth's polar axis is tilted significantly from normal to the plane of the Earth's orbit, and since the polar axis remains nearly aligned on the North Star throughout an orbit, we can ascribe the rotation Ω to spin. The orbital motion about the Sun does give rise to tidal forces, which are small but important spatial variations of the centrifugal/gravitational balance that holds for the Earth-Sun and for the Earth-Moon as a whole (described particularly well by French [8]). A question for you: What is the rotation rate of the Moon? Hint, make a sketch of the Earth-Moon orbital system and consider what we observe of the Moon from Earth. What would the Coriolis and centrifugal forces be on the Moon?

Assume that the inertial frame equation of motion is

$$\frac{d^2\mathbb{X}}{dt^2}M = \mathbb{F} + \mathbb{G}_* M \quad \text{and} \quad \frac{d^2\boldsymbol{X}}{dt^2}M = \boldsymbol{F} + \boldsymbol{g}_* M \tag{23}$$

($\mathbb{G}_*$ is the component matrix of $\boldsymbol{g}*$). The acceleration and force can always be viewed from another reference frame that is rotated (but not rotating) with respect to the first frame,

$$\left(\frac{d^2\mathbb{X}}{dt^2}\right)' M = \mathbb{F}' + \mathbb{G}'_* M \quad \text{and} \quad \left(\frac{d^2\boldsymbol{X}}{dt^2}\right)' M = \boldsymbol{F}' + \boldsymbol{g}'_* M, \tag{24}$$

as if we had chosen a different set of fixed stars or multiplied both sides of Eq. (22) by the same rotation matrix. This equation of motion preserves the global conservation and Galilean transformation properties of Eq. (23). To find the rotating frame equation of motion, we use Eqs. (21) and (22) to eliminate the rotated acceleration from Eq. (24) and then solve for the acceleration seen in the rotating frame: the components are

$$\boxed{\frac{d^2\mathbb{X}'}{dt^2}M = 2\Omega\mathbb{C}\frac{d\mathbb{X}'}{dt}M - \Omega^2\mathbb{C}^2\mathbb{X}'M + \mathbb{F}' + \mathbb{G}'_* M} \tag{25}$$

and the vector equivalent is

$$\boxed{\frac{d^2\boldsymbol{X}'}{dt^2}M = -2\boldsymbol{\Omega}\times\frac{d\boldsymbol{X}'}{dt}M - \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{X}'M + \boldsymbol{F}' + \boldsymbol{g}_*'M} \tag{26}$$

Notice that Eq. (26) has the form of Eq. (4), the difference being that the noninertial reference frame is rotating rather than merely translating. If the origin of this noninertial reference frame was also accelerating, then we would have a third inertial force term, $-(d^2\boldsymbol{X_o}/dt^2)M$. Notice too that we are not yet at Eq. (2); in Section 4.1 we will indicate why the centrifugal force and gravitational mass attraction terms are combined into $\boldsymbol{g}$.

2.2.3 Remarks on the transformed equation of motion

Once we have in hand the transformation rule for accelerations, Eq.(22), the path to the rotating frame equation of motion is short and direct — if Eq. (24) holds in a given reference frame, say an inertial frame, then Eqs. (25) and (26) hold exactly in a frame that rotates at the constant rate and direction $\boldsymbol{\Omega}$ with respect to the first frame. The rotating frame equation of motion includes two terms that are dependent upon the rotation vector, the Coriolis term, $2\boldsymbol{\Omega}\times(d\boldsymbol{X}'/dt)$, and the centrifugal term, $\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{X}'$. Very often these terms are written on the left side of an equation of motion as if they were going to be regarded as part of the acceleration,

$$\frac{d^2\boldsymbol{X}'}{dt^2}M + 2\boldsymbol{\Omega}\times\frac{d\boldsymbol{X}'}{dt}M + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{X}'M = \boldsymbol{F}' + \boldsymbol{g}*'M. \tag{27}$$

If we compare the left side of Eq. (27) [19] with Eq. (22) it is evident that the rotated acceleration is equal to the rotated force,

$$\left(\frac{d^2\boldsymbol{X}}{dt^2}\right)' M = \boldsymbol{F}' + \boldsymbol{g}*'M,$$

[19]Recall that $\boldsymbol{\Omega} = \boldsymbol{\Omega}'$ and so we could put a prime on every vector in this equation. That being so, we would be better off to remove the visually distracting primes and simply note that the resulting equation holds in a steadily rotating reference frame. We will hang onto the primes for now, since we will be considering both inertial and rotating reference frames until Section 5.

which is well and true (and the same as Eq. 24). However, it is crucial to understand that the left side of Eq. (27) taken all at once is *not* the acceleration that we observe or seek to analyze when we use a rotating reference frame; the acceleration we observe in a rotating frame is $d^2\boldsymbol{X}'/dt^2$, the first term only. Once we solve for $d^2\boldsymbol{X}'/dt^2$, it follows that the Coriolis and centrifugal terms are, figuratively or literally, sent to the right side of the equation of motion where they are interpreted as if they were forces. Therein lies the fate of the Coriolis and centrifugal terms and there too is the seed of our possible confusion regarding these terms.

When the Coriolis and centrifugal terms are regarded as forces — as we intend they should be when we use a rotating reference frame — they have some peculiar properties. From Eq. (27) (and Eq. (4)) we can see that the centrifugal and Coriolis terms are inertial forces and are exactly proportional to the mass of the parcel observed, M, whatever that mass may be. The acceleration field for these inertial forces arises from the rotational acceleration of the reference frame, combined with relative velocity for the Coriolis force. They differ from central forces $\boldsymbol{F}$ and $\boldsymbol{g}*M$ in the respect that there is no physical interaction that causes the Coriolis or centrifugal force and hence there is no action-reaction force pair. As a consequence the rotating frame equation of motion does not retain the global conservation of momentum that is a fundamental property of the inertial frame equation of motion and central forces (an example of this nonconservation is described in Section 3.4). Similarly, we note here only that invariance to Galilean transformation is lost since the Coriolis force involves the velocity and not just the velocity derivatives. Thus $\boldsymbol{V}'$ is an absolute velocity in the rotating reference frame of the Earth. If we need to call attention to these special properties of the Coriolis force, then the usage Coriolis *inertial* force seems appropriate because it is free from the taint of unreality that goes with 'virtual force', 'fictitious correction force', etc., and because it gives at least a hint at the origin of the Coriolis force. It is important to be aware of these properties of the rotating frame equation of motion, and also to be assured that in most analysis of geophysical flows they are of no great practical consequence. What is important is that the rotating frame equation of motion offers a very significant gain in simplicity compared to the inertial frame equation of motion (discussed in Section 4.3).

The Coriolis and centrifugal forces taken individually have simple interpretations. From Eq. (26) it is evident that the Coriolis force is normal to the velocity, $d\boldsymbol{X}'/dt$, and to the rotation vector, $\boldsymbol{\Omega}$. The Coriolis force will thus tend to cause the velocity to change direction but not magnitude, and is appropriately termed a deflecting force as noted in Section 1. The centrifugal force is in a direction perpendicular to and directed away from the axis of rotation. Notice that the Coriolis force is independent of position, while the centrifugal force clearly is not. The centrifugal force is independent of time. How these forces effect dynamics in simplified conditions will be considered further in Sections 3, 4.3 and 5.

3 Inertial and noninertial descriptions of elementary motions.

To appreciate some of the properties of a noninertial reference frame we will analyze several examples of (truly) elementary motions whose inertial frame dynamics is very simple and familiar. The only issue will be how these motions appear from a noninertial reference frame. (Readers who find these examples just too elementary may skip the bulk of this section, but should take a look at the summary, 3.4.) Our specific aim is to see how the inertial forces — gravity, centrifugal and Coriolis — depend upon the reference frame. The analysis is very straightforward in that all we plan to do is evaluate the appropriate equation of motion, Eq. (23) or (26), in highly simplified conditions. In another respect it is slightly subtle insofar as the terms that

A characterization of the forces on geophysical flows.

	central	inertial	Galilean invariant
contact forces	yes	no	yes
grav. mass attraction	yes	yes	yes
Coriolis	no	yes	no
centrifugal	no	yes	yes

Table 1: Contact forces are pressure gradients and frictional forces. In this table we also ignore electromagnetic forces that are usually very small.

represent inertial forces will seem to change identity, as if by fiat. To understand that there is more to the analysis than relabeling and reinterpreting terms in an arbitrary way, it will be very helpful for you to make a sketch of each case and to pay close attention to the acceleration, especially.

As we will describe below, there is an important difference between what we will term the contact forces, $\boldsymbol{F}$, that act over the surface of the parcel, and the acceleration due of gravity, $\boldsymbol{g}M$, which is an inertial force that acts throughout the body of the parcel (and note that in this section we will not distinguish between g and $g*$). To measure the contact forces we could enclose the parcel in a wrap-around strain gage that measures and reads out the vector sum of the tangential and normal stress acting on the surface of the parcel. To measure gravity we could measure the direction of a plumb line, which we could then use to define vertical, and so align the $\boldsymbol{e_z}$ unit vector. The amplitude of the acceleration could then be measured by observing the period of oscillation of a simple pendulum.[20]

3.1 Switching sides

Consider a parcel of mass M that is at rest and in contact with the ground, say, in a reference frame where the acceleration of gravity is known from independent observations. The strain gauge will read out a contact force F_z, which, from the perspective of the parcel is upwards. The vertical component of the equation of motion is then

$$\frac{d^2z}{dt^2}M = F_z - gM.$$

If we evaluate the acceleration, $d^2z/dt^2 = 0$, then

$$0 = F_z - gM, \tag{28}$$

[20]A plumb line is nothing more than a weight, the plumb bob, that hangs from a string, the plumb line (and *plumbum* is Latin for lead, Pb). When the plumb bob is at rest, the plumb line is parallel to the local acceleration field. If the weight is displaced and released, it becomes a simple pendulum, and the period of oscillation, P, can be used to infer the amplitude of the acceleration, $g = L/(P/2\pi)^2$, where L is the length of the plumb line. If the reference frame is attached to the rotating Earth, then the measured inertial acceleration includes a contribution from the centrifugal force, discussed in Section 4.1. The motion of the pendulum will be effected also by the Coriolis force, and in this context a simple pendulum is often termed a Foucault pendulum, discussed further in a later footnote 32. In this section we consider gravity, rather than gravitational mass attraction and centrifugal force due to Earth's rotation separately. When centrifugal force arises here, it will be due to a very rapidly rotating platform noted explicitly.

which indicates a static force balance between the upward contact force, F_z, (equal in magnitude to the weight of the parcel) and the downward force due to gravity. Suppose that we observe the same parcel from a reference frame that is in free-fall and so is accelerating downwards at the rate $-g$.[21] When viewed from this reference frame the parcel appears to be accelerating upwards at the rate g that is just the complement of the acceleration of the free-falling frame. In this frame there is no gravitational force, and so the only force we recognize as acting on the parcel is the contact force, which is unchanged from the case before, $F_z = gM$. The equation of motion for the parcel observed from this free-falling reference frame is then

$$\frac{d^2z'}{dt^2}M = F_z,$$

or if we evaluate the acceleration, $d^2z'/dt^2 = g$,

$$gM = F_z. \tag{29}$$

Notice that in going from Eq. (28) to Eq. (29) the contact force is unchanged (invariant) while the term involving g has switched sides; g is an inertial force in the reference frame appropriate to Eq. (28) and is transformed into an acceleration in the free-falling reference frame described by Eq. (29). The equation of motion makes prefectly good sense either way. As we will see next, the same sort of thing happens with centrifugal and Coriolis inertial forces when we transform to or from a rotating reference frame.

Now consider the horizontal motion and dynamics of this parcel, so that gravity and the vertical component of the motion can be ignored. We will presume that $\boldsymbol{F} = 0$, and hence the inertial frame equation of motion expanded in polar coordinates (derived in Appendix A and repeated here for convenience),

$$\begin{aligned}\frac{d^2\boldsymbol{X}}{dt^2}M &= \left(\frac{d^2r}{dt^2} - r\omega^2\right)M\boldsymbol{e_r} + \left(2\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right)M\boldsymbol{e_\theta} \\ &= F_r\boldsymbol{e_r} + F_\theta\boldsymbol{e_\theta},\end{aligned} \tag{30}$$

vanishes term by term. Suppose that the same parcel is viewed from a steadily rotating reference frame and that it is at a distance r' from the origin of the rotating frame. Viewed from this frame the parcel will have a velocity $\boldsymbol{V}' = -\boldsymbol{\Omega} \times \boldsymbol{X}'$ and will appear to be moving around a circle of radius $r' = constant$ and in a direction opposite the rotation of the reference frame, $\omega' = -\Omega$, just as in Figure (5). The rotating frame equation of motion in polar coordinates is just

$$\begin{aligned}\frac{d^2\boldsymbol{X}'}{dt^2}M &= \left(\frac{d^2r'}{dt^2} - r'\omega'^2\right)M\boldsymbol{e}'_{\boldsymbol{r}} + \left(2\omega'\frac{dr'}{dt} + r'\frac{d\omega'}{dt}\right)M\boldsymbol{e}'_{\boldsymbol{\theta}} \\ &= \left(r'\Omega^2M + 2\Omega\omega'r'M + F'_r\right)\boldsymbol{e}'_{\boldsymbol{r}} + \left(-2\Omega\frac{dr'}{dt}M + F'_\theta\right)\boldsymbol{e}'_{\boldsymbol{\theta}}.\end{aligned} \tag{31}$$

[21] Gravitational mass attraction is an inertial force and a central force that has a very long range. Imagine two gravitating bodies and a reference frame attached to one of them, say parcel one, which will then be observed to be at rest. If parcel two is then found to accelerate towards parcel one, the total momentum of the system (parcel one plus parcel two) will not be conserved, i.e., in effect, gravity would not be recognized as a central force. A reference frame attached to one of the parcels is thus not inertial. To define an inertial reference frame in the presence of mutually gravitating bodies we can use the center of mass of the system, and then align on the fixed stars. This amounts to putting the entire system into free-fall with respect to any larger scale (external) gravitational mass attraction (for more on gravity and inertial reference frames see http://plato.stanford.edu/entries/spacetime-iframes/).

We presume that we can read the strain gage from this rotating frame as well, and note that $F'_r = F'_\theta = 0$. All of the other azimuthal component terms vanish individually, but three of the radial component terms are nonzero,

$$-r'\omega'^2 = r'\Omega^2 + 2\Omega\omega' r', \tag{32}$$

and indicate an interesting balance between the centripetal acceleration, $-r'\omega'^2$ (left hand side), and the sum of the centrifugal and Coriolis inertial forces $/M$ (right hand side, and note that $\omega' = -\Omega$).[22] Interesting perhaps, but disturbing as well; a parcel that is at rest in the inertial frame and subject to no (horizontal) forces whatever has acquired a rather complex momentum balance simply because it has been observed from a rotating reference frame. It is sorely tempting to deem the Coriolis and centrifugal terms of Eq. (32) to be 'virtual', or 'fictitious, correction' forces, as is often done, [6] and to be consistent we should do the same for the centripetal acceleration term. But labeling terms this way may only obscure the fundamental issue — that inertial forces are relative to a reference frame. As we found in the example of a free-falling reference frame, this applies just as much for gravitational mass attraction as it does for centrifugal and Coriolis forces.

3.2 To get a feel for the Coriolis force

The centrifugal force is something that we encounter in daily life. For example, a runner having $V = 5$ m sec^{-1} and making a moderately sharp turn, radius $R = 15$ m, will easily feel the centrifugal force, $(V^2/R)M \approx 0.15gM$, and will compensate instinctively by leaning toward the center of the turn. It would never occur to the runner to label the centrifugal force as anything other than real.

The Coriolis force associated with Earth's rotation is very subtle by comparison, only about $2\Omega VM \approx 10^{-4}gM$ for the same runner. To experience the Coriolis force in the same direct way that we can feel the centrifugal force, i.e., to feel it in your bones, thus will require a platform having a rotation rate that exceeds Earth's rotation rate by a factor of about 10^4. A typical merry-go-round having a rotation rate of about $\Omega = 2\pi/12$rad sec^{-1} = 0.5 rad sec^{-1} is ideal. We are going to calculate the forces that you will feel while sitting or walking about on a merry-go-round, and so will need to estimate your mass, say $M = 75$ kg (approximately the standard airline passenger before the era of super-sized meals and passengers).

3.2.1 Zero relative velocity

To start, let's presume that you are sitting quietly near the outside radius $r = 6$ m of a merry-go-round that it is rotating at a steady rate, Ω = 0.5 rad sec^{-1}. How does the momentum balance of your motion depend upon the reference frame, whether inertial or rotating, that is used to observe and describe your motion?

Viewed from an approximate **inertial frame** outside of the merry-go-round (fixed stars are not required given the rapid rotation rate), the polar coordinate momentum balance Eq. (30) with $\omega = \Omega$ and $dr/dt = d\omega/dt = F_\theta = 0$ reduces to a two term radial balance,

$$-r\Omega^2 M \quad = \quad F_r, \tag{33}$$

[22] Two problems for you: 1) Given the polar coordinate velocity, Eq. (70), show that Eq. (31) can be derived also from the vector form of the equation of motion, Eq. (26). 2) Sketch the balance of forces in Eq. (31) in a case where the rotation rate Ω is positive and then again where it is negative. Is this consistent with Eq. (26)?

in which a centripetal acceleration ($\times M$) is balanced by an inward-directed radial (contact) force, F_r. We can readily evaluate the former and find $-r\Omega^2 M = F_r = -112$ N, which is equal to the weight on a mass of $F_r/g = 11.5$ kg for a nominal g. This is just what the strain gauge (the one on the seat of your pants) would read out.

Viewed from the **rotating reference frame**, i.e., your seat on the merry-go-round, you are stationary and of course not accelerating. To evaluate the rotating frame momentum equation, Eq. 31, we thus set $\omega' = 0, r'$ = constant, and are left with a two term radial force balance,

$$0 = r'\Omega^2 M + F'_r. \tag{34}$$

The physical conditions are unchanged and thus the strain gage reads out exactly as before, and $F'_r = F_r = -112$ N. What has changed is that the term $r'\Omega^2 M$, an acceleration in the inertial frame, is now on the right side of the momentum equation and is the centrifugal force. Within the rotating frame, the centrifugal force is quite vivid; it appears that you are being pushed outwards, or centrifugally, by an inertial force that is opposed by the centripetal contact force F'_r. This is exactly the relationship between weight and a contact force described in Section 3.1. The centrifugal force produces a radial acceleration on every stationary object that depends only upon the radius, r'. For example, a plumb line makes an angle to the vertical of $arctan(r'\Omega^2/g)$, where the vertical direction and g are in the absence of rotation. The centrifugal force thus contributes to the direction and magnitude of the time-independent acceleration field observed in the rotating frame, an important point that we will return to in Section 4.1.

3.2.2 With relative velocity

Most merry-go-rounds have signs posted which caution riders to remain in their seats once the ride begins. This is a good and prudent rule, of course, but if your goal is to get a feel for the Coriolis force then you may decide to go for a (very cautious) walk on the merry-go-round. We will presume that the relative velocity, i.e., your walking velocity, is specified, and then calculate the contact force that must be exerted by the merry-go-round upon you as a consequence.

Azimuthal relative velocity: Let's assume that you walk azimuthally so that $r = 6$ m and constant. A reasonable walking pace under the circumstance is about $U_w = 1.5$ m s^{-1}, which corresponds to a relative rotation rate $\omega' = 0.25$ rad sec^{-1}, and recall that $\Omega = 0.5$ rad sec^{-1}. Let's also assume that you walk in the direction of the merry-go-round rotation so that $\omega = \Omega + \omega' = 0.75$ rad sec^{-1}.

From the **inertial frame** momentum equation (30) we can readily calculate that the centripetal force required to maintain $r = constant$ at this greater angular velocity is

$$-r\omega^2 M = -r(\Omega + \omega')^2 M = F_r \approx -253 \text{ N},$$

or roughly twice the force required when you were seated. If you then reverse direction and walk at the same speed against the rotation of the merry-go-round, F_r is reduced to about -28 N. This pronounced variation of F_r with ω' is a straightforward consequence of the quadratic dependence of centripetal acceleration upon the rotation rate, ω.

When this motion is viewed from the **rotating frame** of the merry-go-round, we distinguish between the rotation rate of the merry-go-round, Ω, and the relative rotation rate, ω', due to your walking speed. The radial component of the rotating frame momentum equation reduces to

$$-r'\omega'^2 M = (r'\Omega^2 + 2r'\Omega\omega')M + F'_r. \tag{35}$$

The term on the left is the comparatively small centripetal acceleration; the first term on the right is the usual centrifugal force, and the second term on the right, $2r'\Omega\omega'$, is the Coriolis force. The Coriolis force is substantial, $2r'\Omega\omega' M \pm 112$ N, with the sign determined by the direction of your motion relative to Ω. If $\Omega > 0$ and $\omega' > 0$ then the Coriolis force is positive and radial and to the right of and normal to the azimuthal relative velocity. Given what we found in the previous paragraph, it is tempting to identify the Coriolis force as the (relative)velocity-dependent part of the centrifugal force. This is, however, somewhat loose and approximate; loose because the centrifugal force is defined to be dependent upon rotation rate and position only and approximate because this ignores the small centripetal acceleration term.

Radial relative velocity: If you are still able, consider a (very cautious) walk in the radial direction. To isolate the effects of radial motion we will presume that your radial speed is constant at $dr'/dt = 1$ m s^{-1} and that you walk along a radial line so that your rotation rate also remains constant at $\omega = \Omega$. In practice this is very difficult to do for more than a few steps, but that will suffice. The resulting contact force $\boldsymbol{F}$ is then in the azimuthal direction, and its magnitude and sense can most easily be interpreted in terms of the balance of angular momentum, $A = \omega r'^2 M$. In this circumstance the rate of change of your angular momentum A has been fully specified,

$$\frac{dA}{dt} = 2\Omega r' \frac{dr'}{dt} M = r' F_\theta,$$

and must be accompanied by an azimuthal torque, $r' F_\theta$, exerted by the merry-go-round upon you.

Viewed from an **inertial frame**, the azimuthal component of the momentum balance, Eq. (30), reduces to

$$2\Omega \frac{dr}{dt} M = F_\theta, \tag{36}$$

where $F_\theta \approx -75$ N for the given data. The azimuthal contact force F_θ has the form of the Coriolis force, but remember that we are viewing the motion from an inertial frame so that there is no Coriolis force. If the radial motion is inward so that $dr/dt < 0$, then F_θ must be negative, or opposite the direction of the merry-go-round rotation, since your angular momentum is necessarily becoming less positive. (Be sure that these signs are clear before going on to consider this motion from the rotating frame.)

From within the **rotating frame**, the momentum equation reduces to an azimuthal force balance

$$0 = -2\Omega \frac{dr'}{dt} M + F'_\theta, \tag{37}$$

where $-2\Omega \frac{dr'}{dt} M$ is the Coriolis force and $F'_\theta = F_\theta$ as before. The contact force exerted by the merry-go-round, F'_θ, is balanced by an inertial force, the Coriolis force, in the direction opposed to F'_θ. For example, if your radial motion is inward, $\frac{dr'}{dt} \leq 0$, then the Coriolis force, $-2\Omega \frac{dr'}{dt} M \geq 0$, is to the right of and normal to your relative velocity, just as we would have expected from the vectorial Coriolis force. This interpretation of a Coriolis force is exactly parallel to the interpretation of centrifugal force in the example of steady, circular motion and Eq. (34): an acceleration seen from an inertial frame appears to be an inertial force when viewed from the rotating frame.

Be careful! If you have a chance to do this experiment some day you will learn from direct experience whether the Coriolis force is better described as real or as a fictitious correction force. Ask permission of the operator before you start walking around, and exercise genuine caution. The Coriolis force is an inertial force and so is distributed throughout your body. If the contact force between you and the merry-go-round acted only through the soles of your sneakers, say, then the result would be a significant force couple tending to tip you over. It is therefore essential that you maintain a secure hand grip at all times. The radial Coriolis force associated with azimuthal motion is much like an increase or slackening of the centrifugal force and so is not particularly difficult to compensate. However, the azimuthal Coriolis force associated with radial motion is quite surprising, even assuming that you are the complete master of this analysis. If you do not have access to a merry-go-round or if you feel that this experiment might be unwise, then see Stommel and Moore [10] for alternate ways to accomplish some of the same things.

3.3 An elementary projectile problem

A very simple projectile problem can reveal some other aspects of rotating frame dynamics. Assume that a projectile is launched at a speed U_0 and at an angle to the ground β from a location $[x \;\; y] = [0 \;\; y_0]$. The only force presumed to act on the projectile after launch is the downward force of gravity, $-gM\boldsymbol{e_3}$, which is the same in either reference frame.

3.3.1 From the inertial frame

The equations of motion and initial conditions in Cartesian components are linear and uncoupled;

$$\frac{d^2x}{dt^2} = 0; \quad x(0) = 0, \quad \frac{dx}{dt} = 0, \tag{38}$$

$$\frac{d^2y}{dt^2} = 0; \quad y(0) = y_0, \quad \frac{dy}{dt} = U_0 \cos\beta,$$

$$\frac{d^2z}{dt^2} = -g; \quad z(0) = 0, \quad \frac{dz}{dt} = U_0 \sin\beta,$$

where M has been divided out. The solution for $0 < t < \frac{2U_0 \sin\beta}{g}$

$$\begin{aligned} x(t) &= 0, \\ y(t) &= y_0 + tU_0 \cos\beta, \\ z(t) &= t(U_0 \sin\beta - \frac{1}{2}gt) \end{aligned} \tag{39}$$

is in Fig. (7).

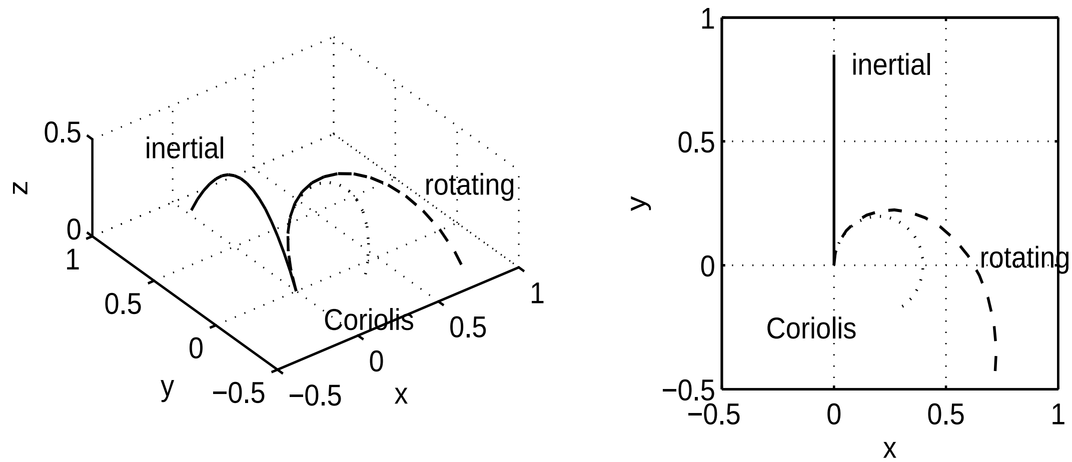

Figure 7: Trajectory of a parcel launched with a horizontal velocity in the positive y-direction as seen from an inertial reference frame (solid line, displaced in the y-direction only), and as seen from a rotating frame (dashed, curves lines). The left and right panels are 3-dimensional and plan views. The dotted curve is with the Coriolis force only (the motivation for this is in Section 4). This trajectory has the form of a circle, and if the projectile had not returned to the surface, $z = 0$, it would have made a complete loop back to the starting point.

3.3.2 From the rotating frame

How would this same motion look when viewed from a rotating reference frame? And, how could we compute the motion from within a rotating reference frame?

The first question can be answered most directly by rotating the trajectory, Eq. (39), via the rotation matrix, Eq. (12), $\mathbb{X}' = \mathbb{R}\mathbb{X}$ with $\theta = \Omega t$, and with the result

$$x'(t) = (y_0 + tU_0 \cos\beta) \sin(\Omega t), \tag{40}$$

$$y'(t) = (y_0 + tU_0 \cos\beta) \cos(\Omega t),$$

$$z'(t) = z \; = \; t(U_0 \sin\beta - \frac{1}{2} g t),$$

valid over the time interval as before. The z component is unchanged in going to the rotating reference frame since the rotation axis was aligned with z. This is quite general; motion that is parallel to the rotation vector $\boldsymbol{\Omega}$ is unchanged. On the other hand, motion in the (x, y)-plane perpendicular to the rotation vector can be altered quite substantially, depending upon the phase Ωt. In the case shown in Fig. (7), the change of phase is 2.0 at the end of the trajectory, so that rotation effects are prominent. [23] One important aspect of the trajectory is not changed, however, the (horizontal) radius,

$$\sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2},$$

since the coordinate systems have coincident origins (Fig. 8a)

[23] A well-thrown baseball travels at about 45 m s^{-1}. How much will it be deflected as it travels over a distance of 30 m? Use the

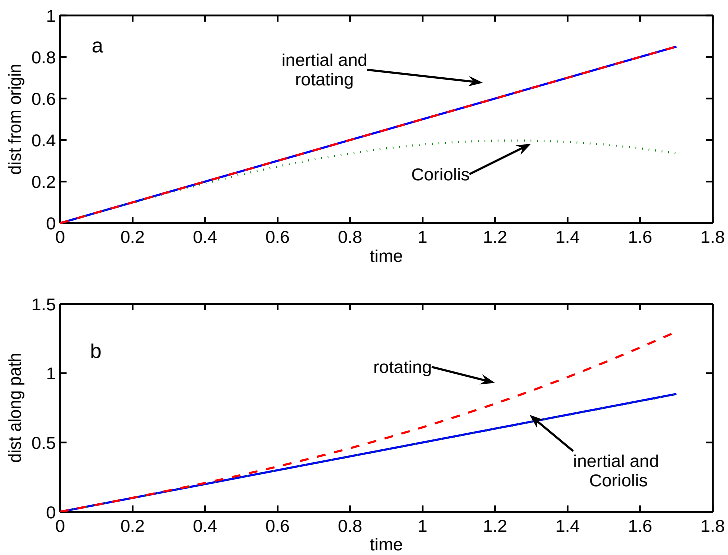

Figure 8: (a) The distance from the origin in the horizontal plane for the trajectories of Fig. (7). The distance from the origin is identical for the inertial and rotating trajectories, and reduced for the Coriolis trajectory (discussed in Section 4). (b) The distance along the path in the horizontal plane for the same trajectories. The slope gives the speed of the parcel. The inertial and Coriolis frame trajectories retain their initial speed and are identical; the rotating frame trajectory accelerates due to the centrifugal force.

How could we compute the trajectory in the rotating frame? The Cartesian component equations in the rotating frame are a bit awkward (the x-component only):

$$\frac{d^2x'}{dt^2} = 2\Omega\frac{dy'}{dt} + \Omega^2\frac{x'^2}{\sqrt{x'^2 + y'^2}}.$$

An elementary problem in the inertial frame transforms into a pair of coupled, nonlinear equations in the rotating frame ($z' = z$). We can always solve these equations numerically, but we are better off in this and many problems involving rotation to use cylindrical polar coordinates where we can take advantage of what we have already learned about the rotating frame solution. We know that

$$r' = r = y_0 + tU_0 \cos\beta,$$

and that the angle in the inertial frame, θ, is constant in time since the motion is purely radial and for the specific case considered, $\theta = \pi/2$. The rotation rates are related by $\omega' = -\Omega$, and thus

$$\theta' = \pi/2 - \Omega t.$$

Both the radius and the angle increase linearly in time, and the horizontal trace of the trajectory as seen from the rotating frame is Archimedes spiral (Fig. 7, lower).

When viewed from the rotating frame, the projectile is obviously deflected to the right, and from the azimuthal component of Eq. (31) we readily attribute this to the Coriolis force,

$$2\omega'\frac{dr'}{dt}M = -2\Omega\frac{dr'}{dt}M,$$

nominal Earth's rotation rate (as we will see in Section 4.2 this is appropriate for the north pole). A long-range artillery shell has an initial speed of about 700 m s^{-1}. Assuming the shell is launched at angle to the ground of 30 degrees, how much will it be deflected over its trajectory (ignoring air resistance)?

since $\omega' = \Omega$. Notice that the horizontal speed and thus the kinetic energy increase with time (Fig. 8, lower). The rate of increase of rotating frame kinetic energy (per unit mass) is

$$\frac{d\mathbf{V}'^2/2}{dt} = \frac{d(U_0^2 + r'^2\Omega^2)/2}{dt} = \frac{dr'}{dt}r'\Omega^2$$

where the term on the right side is the work done by the centrifugal force, $r'\Omega^2$, on the radial velocity, dr'/dt. If the projectile had not returned to the ground, its speed would have increased without limit so long as the radius increased, a profoundly unphysical result of the rotating frame dynamics. [24]

3.4 Summary

This analysis of elementary motions can be summarized with four points:

1) It's our choice. We can make an exact and self-consistent explanation of forced or free motion observed from either an inertial frame or from a rotating frame. If the latter, then in addition to the usual central forces, $\boldsymbol{F}'$ (suitably rotated), there will also arise Coriolis and centrifugal forces. To say it a little differently, there is nothing that occurs in the rotating frame dynamics that can not be understood in terms of inertial frame dynamics and forces $\boldsymbol{F}$. We can use either reference frame — inertial or rotating — that best suits the circumstance.

2) But be consistent. The choice of reference frames is binary, as is the existence or not of the Coriolis and centrifugal forces; either it's inertial, in which case there is no Coriolis or centrifugal force, or it's rotating, and there definitely is (notwithstanding they may be negligibly small). If we choose to (or must) use a rotating frame, then there is no good in calling or thinking of the Coriolis force thereafter as a pseudo force, or a fictitious correction force that seems to question whether the Coriolis force is a full-fledged member of the equation of motion. (And yet, whether we should call it a force or an acceleration is less clear, and considered on closing in Section 6.)

3) Was that an explanation? In the example of azimuthal relative motion on a merry-go-round (Section 3.2.2) the magnitude and direction of the Coriolis force can be thought of as the relative-velocity dependent component of centrifugal force (roughly speaking); in the example of radial relative motion it is equal to the force required to maintain angular momentum balance. These two examples have the feeling of a physical explanation of the Coriolis force but it is probably more appropriate to regard them as a demonstration since they are not general.[25]

4) Gains and losses; global conservation. There is no physical agent or interaction that causes the Coriolis force and so there is no object that is accelerated in the opposite direction by a reaction force. In the same

[24] In the previous example, walking around on a merry-go-round, we indicated that you would be able to feel the Coriolis force directly. Imagine that you are riding along on this projectile (don't try this one at home) — would you be able to feel the Coriolis force?

[25] I began working on the topic of this essay with the hope of finding or developing a *physical* understanding and explanation of the Coriolis force. After a while I concluded that it is futile and even misguided to expect a physical explanation of the Coriolis force since it's origin is mainly kinematic, and moreover, because the Coriolis force has plainly unphysical consequences, the disruption of global momentum conservation. However, other authors, notably Stommel and Moore, [10] indicate that there is a possible (classical) physical understanding of the Coriolis force, and so perhaps this is a matter of semantics only — the meaning of *physical* — rather than hard facts, since we all agree on Eq. (2).

way, there is no source for the work done by the centrifugal force (Section 3.3.2). Global conservation of momentum and energy thus fail to hold when we interpret the Coriolis and centrifugal terms as if they were forces, i.e., if we choose a noninertial, rotating reference frame. Nevertheless, the interpretation of geophysical flow phenomena is usually far simpler when viewed from an Earth-attached, rotating reference frame, as we will see in Section 4.3.

4 Application to the rotating Earth.

The equations of motion appropriate to the atmosphere and ocean differ from that considered up to now in two significant ways. First, it isn't just the reference frame that rotates, but the entire Earth, ocean and atmosphere, aside from the comparatively small (but very important!) relative motion of winds and ocean currents. One consequence of the solid body rotation of the Earth is that the horizontal component of the centrifugal force on a stationary parcel is exactly canceled by a component of the gravitational mass attraction. Thus the centrifugal force does not appear in the rotating frame dynamical equations for the atmosphere and oceans, a welcome simplification (Section 4.1). Second, because the Earth is nearly spherical, the rotation vector is not perpendicular to the plane of horizontal motions except at the poles. This causes the horizontal component of the Coriolis force to vary with latitude (Section 4.2). Lastly, we will compare inertial and rotating frame descriptions of a simple geophysical phenomenon (Section 4.3), and explain why we persist in using the rotating frame equations.

4.1 Cancelation of the centrifugal force

To understand how the centrifugal force is canceled we consider briefly the balance of gravitational mass attraction and centrifugal forces on a rotating Earth. To be sure, the details of this first subsection are a bit beyond the minimum discussion needed for our purpose, but are inherently interesting. A more compact though more abstract way to come to the same result is to consider the definition and measurement of vertical and level in an accelerated fluid environment, Section 4.1.2.

4.1.1 Earth's figure

If Earth was a perfect, homogeneous sphere, the gravitational mass attraction at the surface, $\boldsymbol{g}*$, would be directed towards the center (Fig. 9). Because the Earth is rotating, every parcel on the surface is also subject to a centrifugal force of magnitude $\Omega^2 R \sin\theta$, where R_e is the nominal Earth's radius, and θ is the colatitude ($\pi/2$ - latitude). This centrifugal force has a component parallel to the surface (a shear stress)

$$C_\theta = \Omega^2 R_e \sin\theta \cos\theta \tag{41}$$

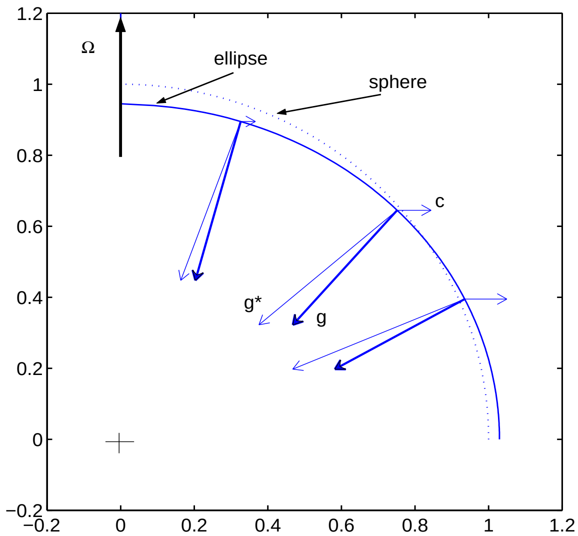

Figure 9: Cross-section through a hemisphere of a gravitating and rotating planet. The gravitational acceleration due to mass attraction is shown as the vector $\boldsymbol{g*}$ that points to the center of a spherical, homogeneous planet. The centrifugal acceleration, $\boldsymbol{C}$, associated with the planet's rotation is directed normal to and away from the rotation axis. The combined gravitational and centrifugal acceleration is shown as the heavier vector, $\boldsymbol{g}$. This vector is in the direction of a plumb line, and defines vertical. A surface that is normal to $\boldsymbol{g}$ similarly defines a level surface, and has the approximate shape of an oblate spheroid (the solid curve). The ellipse of this diagram has a flatness $F = 0.1$ that approximates Saturn; for Earth, $F = 0.0033$.

that is directed towards the equator (except at the equator where it is vertical).[26] C_θ is not large compared to $g*$, $C_\theta/g* \approx 0.002$ at most, but it has been present since the Earth's formation. A fluid can not sustain a shear stress without deforming, and over geological time this holds as well for the Earth's interior and crust. Thus it is highly plausible that the Earth long ago settled into an equilibrium configuration in which this C_θ is exactly balanced by a component of the gravitational (mass) attraction that is parallel to the displaced surface and poleward.

To make what turns out to be a rough estimate of the displaced surface, η, we will assume that the gravitational mass attraction remains that of a sphere and that the meridional slope times the gravitational mass attraction is in balance with the tangential component of the centrifugal force,

$$\frac{g*}{R_e}\frac{d\eta}{d\theta} = \Omega^2 R_e \sin\theta\cos\theta. \tag{42}$$

This may then be integrated with latitude to yield the equilibrium displacement

$$\eta(\theta) = \int_0^\theta \frac{\Omega^2 R_e^2}{g*}\sin\theta\cos\theta d\theta = \frac{\Omega^2 R_e^2}{2g*}\sin\theta^2 + constant. \tag{43}$$

When this displacement is added onto a sphere the result is an oblate (flattened) spheroid, (Fig. 9), which is consistent with the observed shape of the Earth.[27] A convenient measure of flattening is

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[26]Ancient critics of the rotating Earth hypothesis argued that loose objects on a spinning sphere should fly off into space, which clearly does not happen. Even so, given this persistent centrifugal force, why don't we drift towards the equator? Alfred Wegner proposed this as the motive force of Earth's moving continents (see D. McKenzie, 'Seafloor magnetism and drifting continents', in *A Century of Nature*, 131-137. Ed. by L. Garwin and T. Lincoln, The Univ. of Chicago Press, Chicago, Il, 2003.).

[27]The pole-to-equator rise given by Eq. (43), is about 11 km. Precise observations show that Earth's equatorial radius, $R_{eqt} = 6378.2$, is greater than the polar radius, $R_{pol} = 6356.7$ km, by about 21.5 km. This simple model underestimates this displacement because the mass displaced from the pole towards the equator causes a small equatorward mass attraction that is sufficient to compensate for about half of the meridional tilt effect; thus still more mass must be displaced towards the equator in order to achieve a gravitational/rotational equilibrium. The net result is about a factor two greater displacement than Eq. (43) indicates (a very interesting discussion of Earth's shape is available online from http://www.mathpages.com/home/kmath182.htm).

A comprehensive source for physical data on the planets is by C. F. Yoder, 'Astrometric and geodetic data on Earth and the solar system,' Ch. 1, pp 1–32, of *A Handbook of Physical Constants: Global Earth Physics (Vol. 1)*. American Geophysical Union (1995).

$F = (R_{eqt} - R_{pol})/R_{eqt}$, where the subscripts refer to the equatorial and polar radius. Earth's flatness is $F = 0.0033$, which seems quite small, but is nevertheless highly significant in ways beyond that considered here.[28] The flatness of a rotating planet is given roughly by $F \approx \Omega^2 R/g$. If the gravitational acceleration at the surface, g, is written in terms of the planet's mean radius, R, and density, ρ, then $F = \Omega^2/(\frac{4}{3}\pi G\rho)$, where $G = 6.67\times10^{-11}$ m^3 kg^{-1} s^{-2} is the universal gravitational constant. The rotation rate and the density vary a good deal among the planets, and consequently so does F. The gas giant, Saturn, has a rotation rate a little more than twice that of Earth and a very low mean density, about one eighth of Earth's. The result is that Saturn's flatness is large enough, $F \approx 0.10$, that it can be discerned through a good backyard telescope (Fig. 9).

4.1.2 Vertical and level in an accelerating reference frame

Closely related is the notion of 'vertical'. When we measure vertical we do so by means of a plumb bob; the plumb line is, by definition, the direction vertical. Following the discussion above we know that the time-independent, acceleration field of the Earth is made up of two contributions, the first and by far the largest being mass attraction, $\boldsymbol{g}*$, and the second being the centrifugal acceleration associated with the Earth's rotation, $\boldsymbol{C}$, Fig. (9). Just as on the merry-go-round, this centrifugal acceleration adds with the gravitational mass attraction to give the net acceleration, called 'gravity', $\boldsymbol{g} = \boldsymbol{g}* + \boldsymbol{C}$, a vector (field) whose direction and magnitude we can measure with a plumb bob and by observing the period of a simple pendulum. A surface that is normal to the gravitational acceleration vector is said to be a level surface in as much as the acceleration component parallel to that surface is by definition zero. A level surface can be defined by observing the free surface of a water body that is at rest in the rotating frame, since a resting fluid can sustain only normal stresses, i.e., pressure but not shear stress. Thus the measurements of vertical or level that we can readily make necessarily include the centrifugal force due to Earth's rotation summed with gravitational mass attraction. The happy result is that the rotating frame equation of motion applied in an Earth-attached reference frame, Eq. (2), does not include the centrifugal force associated with Earth's rotation (and neither do we tend to roll towards the equator!).

4.1.3 The equation of motion for an Earth-attached frame

The inertial and rotating frame momentum equations are listed again for convenience using velocity in place of the previous time rate change of position,

$$\frac{d\boldsymbol{V}}{dt}M = \boldsymbol{F} + \boldsymbol{g}*M, \quad \text{and,} \tag{44}$$

$$\frac{d\boldsymbol{V}'}{dt}M = -2\boldsymbol{\Omega}\times\boldsymbol{V}'M \; - \; \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{X}'M \; + \; \boldsymbol{F}' + \boldsymbol{g}'_*M, \tag{45}$$

and note that the velocity $\boldsymbol{V}$ of Eq. (44) is the total velocity, $\boldsymbol{V} = \boldsymbol{V_\Omega} + \boldsymbol{V}'$, where $\boldsymbol{V_\Omega} = -\boldsymbol{\Omega} \times \boldsymbol{X}$ is the velocity associated with the solid body rotation of the Earth (Section 2.2). Now we are going to assume the

[28]To note just two: 1) Earth's ellipsoidal shape must be accounted for in highly precise, long range navigation systems, particularly GPS, while shorter range or less precise systems can approximate the Earth as spherical. 2) Because the Earth is not perfectly spherical, the gravitational tug of the Sun, Moon and planets can exert a torque on the Earth and thereby perturb Earth's rotation vector. [18]

result from above that there exists a tangential component of gravitational mass attraction that exactly balances the centrifugal force due to Earth's rotation and that we define vertical in terms of the measurements that we can readily make; thus $\boldsymbol{g} = \boldsymbol{g}* + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{X}$.[32] The equations of motion are then, for the inertial frame,

$$\frac{d\boldsymbol{V}}{dt} M = \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{X} M + \boldsymbol{F} + \boldsymbol{g} M \tag{46}$$

and for the rotating frame,

$$\frac{d\boldsymbol{V}'}{dt} M = -2\boldsymbol{\Omega} \times \boldsymbol{V}' M + \boldsymbol{F}' + \boldsymbol{g} M. \tag{47}$$

Finally we have come to Eq. (2), which we now see is the rotating frame equivalent of Eq. (46) (and we will return to these equations in Section 4.3).[29]

4.2 Coriolis force on motions in a thin, spherical shell

Application to geophysical flows requires one further small elaboration because the rotation vector makes a considerable angle to the vertical except at the poles. An Earth-attached coordinate system is usually envisioned to have $\boldsymbol{e_x}$ in the east direction, $\boldsymbol{e_y}$ in the north direction, and horizontal is defined by a tangent plane to Earth's surface. The vertical direction, $\boldsymbol{e_z}$, is thus radial with respect to the spherical Earth. The rotation vector $\boldsymbol{\Omega}$ thus makes an angle ϕ with respect to the local horizontal x', y' plane, where ϕ is the latitude of the coordinate system and thus

$$\boldsymbol{\Omega} = 2\Omega \cos(\phi)\boldsymbol{e_y} + 2\Omega \sin(\phi)\boldsymbol{e_z}.$$

If $\boldsymbol{V}' = U'\boldsymbol{e_x} + V'\boldsymbol{e_y} + W'\boldsymbol{e_z}$, then the Coriolis force ($/M$) is

$$2\boldsymbol{\Omega} \times \boldsymbol{V}' = (2\Omega \cos(\phi)W' - 2\Omega \sin(\phi)V')\boldsymbol{e_x} + (2\Omega \sin(\phi)U' - 2\Omega \cos(\phi)W')\boldsymbol{e_y} + 2\Omega U' \sin(\phi)\boldsymbol{e_z}. \tag{48}$$

Large scale geophysical flows are very flat in the sense that the horizontal components of wind or current are very much larger than the vertical component, $U' \propto V' \gg W'$, simply because the oceans and the atmosphere are quite thin, having a depth to width ratio of about 0.001. The ocean and atmosphere are stably

[29]This notion of vertical and level turned out to have considerable practical importance beginning on a sweltering September afternoon when the University Housing Office notified you that, because of an unexpectedly heavy influx of freshmen, your old and comfortable dorm room was not going to be available. As a consolation they offered you the use of the merry-go-round (the one in Section 3.3, and still running) at the local, failed amusement park that the University had just gobbled up. You accept. The centrifugal force, amusing at first, was soon a huge annoyance; you had recurring nightmares that you were camping on a mountainside and just about to slide out of bed and over a cliff. To counteract this you decide to build up the floor of the merry-go-round so that the tilt of the floor, combined with the vertical gravitational acceleration, would be just sufficient to balance the centrifugal force, as in Eq. (42). A quick calculation and you find that a parabolic displacement, $\eta \propto r^2$, would be just the thing. A plumb line will then be normal to the new floor at every point, and hence the new floor will be a level surface in the acceleration field of your rotating dorm room. Make sure that we have this right, and specifically, how much does the outside edge ($r = 6$ m) have to be built up to achieve this? Assuming that your bed is 2 m long and flat, can the acceleration field be made zero at both ends at once? (Or is that that why you call it 'the tidal basin'?) How is the calibration of your bathroom scale effected? Visitors are always very impressed with your rotating, parabolic dorm room, and to make sure they have the full experience you send them to the refrigerator for a cold drink. Describe what happens next using Eq. (47). Is their route relevant?

stratified in the vertical, which still further inhibits the vertical component of motion. For these large scale (in the horizontal) flows, the Coriolis terms multiplying W' in the x and y component equations are thus very much smaller than the terms multiplied by U' or V' and as an excellent approximation may be ignored. The Coriolis terms that remain are those having the sine of the latitude, and the important combination

$$f = 2\Omega \sin\phi \tag{49}$$

is dubbed the Coriolis parameter. In the vertical component of the momentum equation the Coriolis term is usually much smaller than the gravitational acceleration, and it too is usually dropped. The result is the thin fluid approximation of the Coriolis force in which only the horizontal force due to horizontal motions is retained,

$$2\boldsymbol{\Omega} \times \boldsymbol{V}' \approx \boldsymbol{f} \times \boldsymbol{V}' = fV'\boldsymbol{e_x} + fU'\boldsymbol{e_y}, \tag{50}$$

and where $\boldsymbol{f}$ is f times the local vertical unit vector. Notice that the Coriolis parameter f varies with the sine of the latitude, having a zero at the equator and maxima at the poles; $f < 0$ for southern latitudes.[30]

For problems that involve parcel displacements, L, that are very small compared to the radius of the Earth, R, a simplification of f is often appropriate. The Coriolis parameter may be expanded in a Taylor series about a central latitude, y_0,

$$f(y) = f(y_0) + (y - y_0)\frac{df}{dy}\,|_{y_0} + HOT \tag{51}$$

and if the second term is demonstrably much smaller than the first term, which follows if $L \ll R_e$, then the second and higher terms may be dropped to leave $f = f(y_0)$, a constant. Under this so-called f-plane approximation [31] the period of inertial motions, $2\pi/f$, is just a little bit less than 12 hrs at the poles, a little less than 24 hrs at 30 N or S, and infinite at the equator. The period of inertial motions is sometimes said to be half of a 'pendulum day', the time required for a Foucault pendulum to precess through 2π radians.[32]

[30]The Coriolis parameter f vanishes at the equator, but the Coriolis force does not, in general. To see this, consider relative velocities that are either eastward or northward, and sketch the resulting Coriolis force $\propto -2\boldsymbol{\Omega} \times \boldsymbol{V}'$ at several latitudes that span pole-to-pole.

[31]The next approximation to Eq. (51) is to retain the first order term, with the symbol β often used to denote the first derivative, viz., $f = f_0 + (y - y_0)\beta$, the beta-plane approximation. A profound change in dynamics follows on this seemingly small change in the Coriolis parameter (see GFD texts[9]).

[32]The effect of Earth's rotation on the motion of a simple (one bob) pendulum, called a Foucault pendulum in this context, is treated in detail in many physics texts, e.g. Marion[6], and need not be repeated here. Here are a few questions, however. Can you calculate the Foucault pendulum motion by rotating the inertial frame solution for a simple pendulum? What approximation(s) have you made in doing this? How does the time required for precession through 360 degrees depend upon latitude? What happens when the pendulum's natural frequency (in the absence of Earth's rotation) equals the Earth's rotation rate? Given the rotated trajectory, can you show that the acceleration of the bob for very short times is consistent with the rotating frame equations of motion?

Foucault pendulums are commonly displayed in science museums, though seldom to large crowds (for a more enthusiastic view see *The Prism and the Pendulum* by R. P. Crease). Much better is to make and observe your very own Foucault pendulum, a simple pendulum having two readily engineered properties. First, the e-folding time of the motion due to frictional dissipation must be long enough, at least 20-30 min, that the precession will become apparent. This can be most easily achieved by using a dense, smooth and symmetric bob having a weight of about half a kilogram or more, and suspended on a fine, smooth monofilament line. It is helpful if the length can be made several meters or more. Second, the pendulum should not interact appreciably with its mounting. This is harder to evaluate, but generally requires a very rigid support, and a bearing that can not exert torque, for example a needle bearing. The rotation effect is proportional to the rotation rate, and so you should plan to bring a simple and rugged pocket pendulum on your merry-go-round ride (Section 3.2). How do your observations (even if qualitative) compare with your solution for a Foucault pendulum? (Hint - consider the initial condition.)

4.3 Why do we insist on the rotating frame equations?

We have emphasized that the rotating frame equation of motion has some inherent awkwardness, viz., the loss of Galilean invariance and global momentum conservation. Why, then, do we insist upon using the rotating frame equations for nearly all of our analyses of geophysical flow? The reasons are several, any one of which would be compelling, but beginning with the fact that the definition and implementation of an inertial frame (outside of the Earth) is simply not a viable option; whatever simplicity we might gain by omitting the Coriolis force would be lost to difficulty with observation. Consider just one aspect of this: the inertial frame velocity, $\boldsymbol{V} = \boldsymbol{V_\Omega} + \boldsymbol{V}'$, is dominated by the solid body rotation $V_\Omega = \Omega R_e \cos(\text{latitude})$, where R_e is earth's nominal radius, 6365 km, and thus $V_\Omega \approx 400$ m s^{-1} near the equator. By comparison, a large wind speed at mid-level of the atmosphere is $V' \approx 50$ m sec^{-1} and a large ocean current is $V' \approx 2$ m sec^{-1}. The very large velocity $\boldsymbol{V_\Omega}$ is accelerated centripetally by a tangential (almost) component of gravitational mass attraction associated with the ellipsoidal shape of the Earth discussed in Section 4.1 that is larger than the Coriolis force in the ratio V_Ω / V' that is O(10) for the atmosphere, or much more for ocean currents. The inertial frame equations have to account for $\boldsymbol{V_\Omega}$ and this very large centripetal force explictly, and yet our interest is almost always the small relative motion of the atmosphere and ocean, $\boldsymbol{V}'$, since it is the relative motion that transports heat and mass over the Earth. In that important regard, the solid body rotation velocity $\boldsymbol{V_\Omega}$ is invisible to us Earth-bound observers, no matter how large it is. To say it a little differently — it is the relative velocity that we measure when observe from Earth's surface, and it is the relative velocity that we seek for most any practical purposes. The Coriolis force follows. [33]

4.3.1 Inertial oscillations from an inertial frame

Given that our goal is the relative velocity, then the rotating frame equation of motion is generally much simpler and more appropriate than is the inertial frame equation of motion. To help make this point we will analyze the free oscillations of Eqs. (46) and (47), i.e., $\boldsymbol{F} = \boldsymbol{F}' = 0$, usually called inertial oscillations, that are interesting in their own right. The domain is presumed to be a small region centered on the pole so that latitude = 90 degrees, and the domain is, in effect, flat. We will consider only horizontal motions, and assume the realistic condition that the motion will be a small perturbation away from the solid body rotation, $\boldsymbol{V}' \ll \boldsymbol{V_\Omega}$. The motion viewed from the inertial frame is thus almost circular and it is appropriate to use the cylindrical coordinate momentum equation, Eq. (30) (dividing out the constant M):

$$\frac{d^2 r}{dt^2} - r\omega^2 = -\Omega^2 r, \tag{52}$$

$$2\omega \frac{dr}{dt} + r\frac{d\omega}{dt} = 0. \tag{53}$$

Notice that when $\omega = \Omega$ and $dr/dt = 0$, the motion is balanced in the sense that $d^2 r/dt^2 = 0$ and r remains constant. We are going to assume an initial condition that is a small radial perturbation away from such a balanced state, $r = R_o$ and $\omega = \Omega$. Since there is no tangential force, Eq. (53) may be integrated,

$$\omega r^2 = \Omega R_o^2 = A, \tag{54}$$

[33] Imagine that Earth's atmosphere is viewed from space, and that we make a map of the absolute velocity, $\boldsymbol{V}' + \boldsymbol{V_\Omega}$. How would this compare (qualitatively) with the weather map of Fig. (1), and how would the pressure surfaces appear in such a map? (Hint, Fig. 9)

which shows that the angular momentum, A, is a conserved quantity. Eq. (54) can then be used to eliminate ω from Eq. (52) to yield an equation for $r(t)$ alone,

$$\frac{d^2r}{dt^2} - \frac{A^2}{r^3} = -\Omega^2 r. \tag{55}$$

To solve this equation it is convenient to move the centripetal acceleration term A^2/r^3 to the right side where it will be summed with the centripetal force,

$$\frac{d^2r}{dt^2} = \frac{A^2}{r^3} - \Omega^2 r, \tag{56}$$

yielding what looks just like an equation of motion. However, it is important to understand that d^2r/dt^2 is *not* the radial component of acceleration in either an inertial or a rotating reference frame (cf. Eqs. 30 and 31) and to acknowledge this explicitly, the right hand side of Eq.(56) will be called a *pseudo* (or false) force. None of this effects the solution *per se*, but only the words we use and the inferences we might then draw. And specifically, if you measured the radial force/ M on the parcel you wouldn't find $A^2/r^3 - \Omega^2 r$, but rather $-\Omega^2 r$, the right hand side of Eq. (55).

Eq. (56) is a well-known, nonlinear oscillator equation and is not difficult to solve. However, because our interest is in the case of small displacements away from the balanced state, $r = R_o$, a simplification is appropriate. To clarify what is meant by small displacement it is helpful to write the radius as

$$r(t) = R_o(1 + \delta(t)).$$

The meaning of 'small displacement' is that δ should be small compared to 1. Substitution into Eq. (56) and rearrangement yields

$$\frac{d^2\delta}{dt^2} = \Omega^2(1+\delta)^{-3} - \Omega^2(1+\delta). \tag{57}$$

When we plot the right side of Eq. (57) it is evident that the net pseudo force is a nearly linear function of δ provided $\delta \leq 0.1$. To exploit this we can expand the nonlinear term of Eq.(57) in Taylor series about $\delta = 0$,

$$\begin{aligned} \frac{d^2\delta}{dt^2} &= \Omega^2(1 - 3\delta + 6\delta^2 + HOT) - \Omega^2(1+\delta) \\ &\approx -4\Omega^2\delta, \end{aligned} \tag{58}$$

where HOT are terms that are higher order in δ. For small displacements the quadratic and higher order terms may be neglected, leaving a simple harmonic oscillator equation, Eq. (58), at a frequency 2Ω.

If the initial condition is a radial impulse that gives a radial velocity V_0, then the initial condition for Eq. (58) is

$$\frac{d\delta}{dt}(t=0) = (V_o/R_o)\cos 2\Omega t.$$

The solution for δ is

$$\delta(t) = (V_0/2\Omega)\sin(2\Omega t)$$

and the radius is then

$$r(t) = R_o(1 + \delta_0 \sin(2\Omega t)), \tag{59}$$

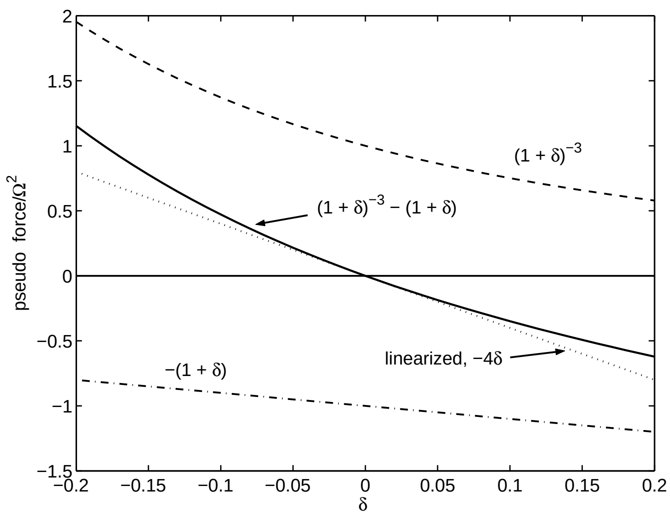

Figure 10: The terms of the right side of Eq. (57), dubbed *pseudo* forces and normalized by Ω^2, shown as a function of δ. Note that the net pseudo force (solid line) is nearly linear in δ when δ is small, roughly $\delta \le 0.1$.

where $\delta_0 = V_0/2\Omega$. The corresponding angular rotation rate can be found by using Eq. (59) together with the angular momentum conservation Eq. (54),

$$\omega(t) = \frac{\Omega}{(1+\delta_0 \sin(2\Omega t))^2} \approx \Omega(1 - 2\delta_0 \sin(2\Omega t)). \tag{60}$$

When graphed, these show that the parcel moves in an ellipsoidal orbit, Fig. (11, left panels), that crosses the (balanced) radius $r = R_o$ four times per complete orbit. The rotating frame turns through 180 degrees just as the parcel returns to $r = R_o$ the second time, after completing a full cycle of the oscillation. When viewed from the rotating frame (Fig. 11, right panels), the parcel appears to be moving in a clockwise-orbiting, circular path, with a frequency $2\,\Omega$.[34]

4.3.2 Inertial oscillations from the rotating frame

It is convenient to expand the rotating frame equation of motion (47) in Cartesian coordinates. Since we have restricted the analysis above to small displacements we can utilize the f-plane approximation that takes f as a constant. Thus the horizontal components U', V' follow

$$\frac{d}{dt}\begin{bmatrix} U' \\ V' \end{bmatrix} = f \begin{bmatrix} -V' \\ U' \end{bmatrix}. \tag{61}$$

Given that the initial condition is an impulse causing a small velocity V_0 in the y-direction then the solution for velocity and displacement is just

$$\begin{bmatrix} U' \\ V' \end{bmatrix} = V_0 \begin{bmatrix} \sin(ft) \\ \cos(ft) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X' \\ Y' \end{bmatrix} = \delta_0 \begin{bmatrix} 1-\cos(ft) \\ \sin(ft) \end{bmatrix}. \tag{62}$$

The velocity of the parcel seen from the rotating frame, V', rotates at a rate of $f = 2\Omega$ in a direction opposite the rotation of the reference frame, Ω.[35] This is exactly the result found in the inertial frame analysis

[34]Which is just *opposite* the sense of rotation Ω. Can you use Eqs. (59) and (60) to explain why?

[35]Two questions for you: 1) How does this compare with the momentum balance and motion described by Eq. (32)? 2) Suppose that the impulse was in the azimuthal direction, what would change?

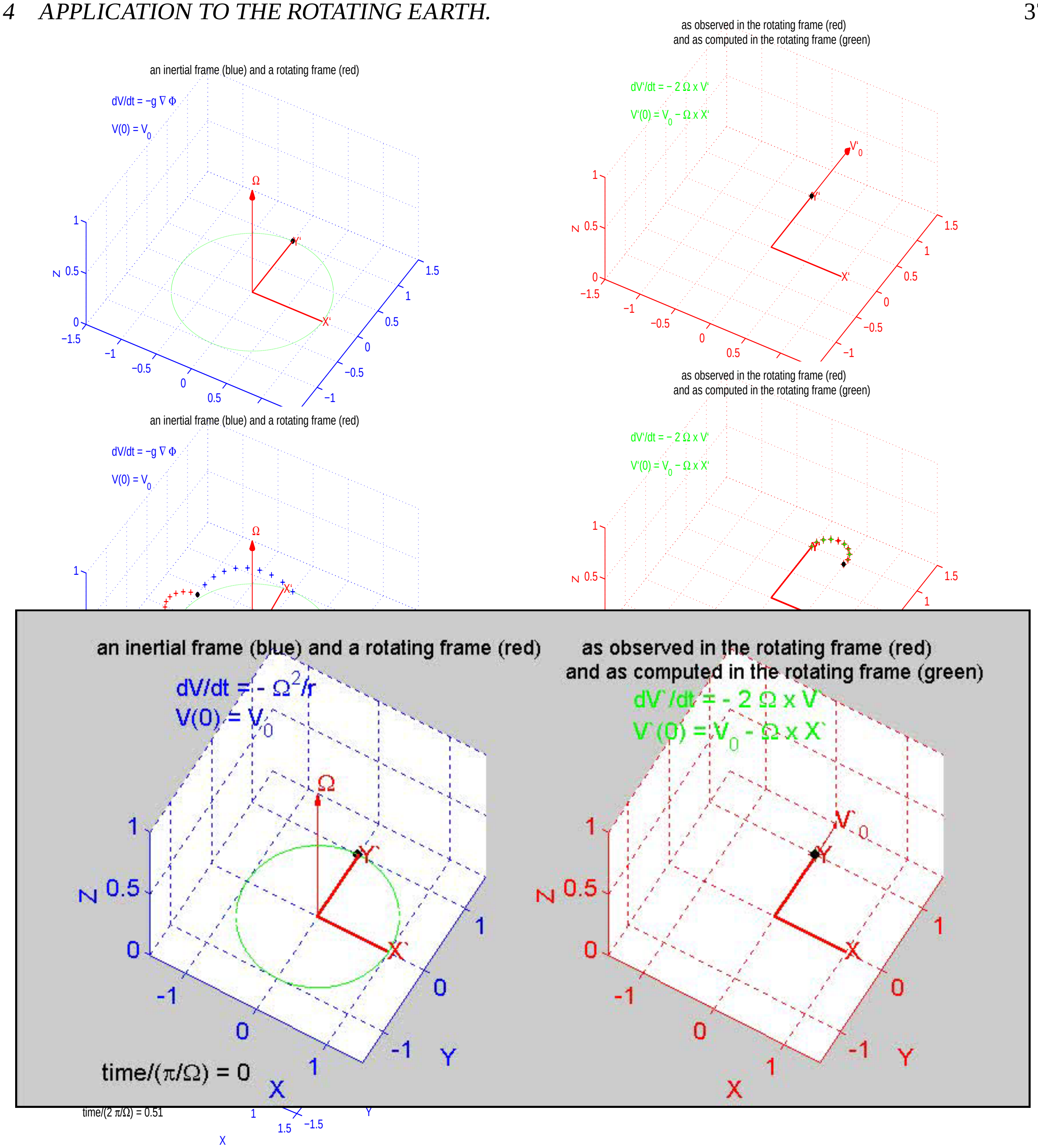

Figure 11: The two-dimensional trajectory of a parcel subject to a centripetal force, $-r\Omega^2$, as if it were on a frictionless parabolic surface. The initial velocity was a solid body rotation in balance with the centripetal force, and a small radial impulse was then superimposed. In this case the ratio $V'/V_\Omega \approx 0.2$, which is far larger than actually occurs in the ocean or atmosphere. The left column shows the resulting ellipsoidal trajectory as seen from an inertial frame, along with the circular trajectory that is seen from a rotating frame (indicated by the rotating, solid unit vectors). The right column shows the trajectory as seen from the rotating frame only, along with the solution computed in the rotating frame (shown as green dots). These lie exactly on top of the 'observed' trajectory and are very difficult to discern if color is not displayed; see the script Coriolis.m[12] that includes this and a number of other cases. Click on the lower half of this figure to start an animation.

but was far simpler to obtain because we did not have to account for the total velocity, $\boldsymbol{V} = \boldsymbol{V_\Omega} + \boldsymbol{V}'$, but the relative velocity only. From the rotating frame perspective, Eq. (47), the rotation of the velocity vector is attributable to deflection by the Coriolis force.[36] This kind of motion, termed an inertial oscillation,[37] is frequently observed in the upper ocean following a sudden shift in the wind speed or direction (Fig. 12).

5 Adjustment to gravity, rotation and friction.

The last problem we consider gives some insight into the establishment of a geostrophic momentum balance, which, as noted in the opening section, is the defining characteristic of large scale flows of the atmosphere and ocean. We are going to stay entirely within an Earth-attached, rotating reference frame, and the Coriolis force will be quite important in some parts of the parameter space (and, dare we say it, considered altogether 'real').

5.1 A dense parcel on a slope

We will model the motion of a single parcel[38] on a rotating Earth (so there is no centrifugal force) and that is subjected to a force that is suddenly turned on at $t = 0$. This force could be a wind stress, a pressure gradient, or the buoyancy force on a relatively dense parcel released onto a sloping sea floor. This latter has the advantage of being similar to the gravitational force due to a tilted Earth's surface (Section 4.1), and so we will take the force to be buoyancy, $b = g\frac{\delta\rho}{\rho_o}$ times the bottom slope $\nabla h = \alpha \boldsymbol{e_y}$, with $\delta\rho$ the density anomaly of the parcel with respect to its surroundings (assumed constant), ρ_o a nominal sea water density, 1030 kg m^{-3}, h the bottom depth and α the slope of the bottom. The depth h is presumed to vary linearly in y only and hence this buoyancy force will appear in the y-component equation only. If the parcel is in contact with a sloping bottom, then it is plausible that the momentum balance should include a frictional term due to bottom

[36] We noted in Section 3.4 that the rotating frame equations of motion does not support global momentum conservation or Galilean invariance. The former can be seen by noting that if all forces except Coriolis were zero, and the initial condition included a velocity, then that velocity would be continually deflected and change direction (as an inertial oscillation) with nothing else showing a reaction force; i.e., global momentum would not be conserved. This evident nonconservation is ignorable in most practical analyses because the Coriolis force is not a spontaneous source of energy. And, when a central force $\boldsymbol{F}$ produces a change of momentum in our parcel, the corresponding reaction force $-\boldsymbol{F}$ generates the complementary change of momentum in the (global) environment that would then undergo a compensating Coriolis deflection. It should be noted that the extent of the global domain within which we can presume exact momentum conservation is not obvious, see, e.g., http://chaos.fullerton.edu/ ∼jimw/nasa-pap/

The Coriolis force is isomorphic to the Lorentz force, $q\boldsymbol{V} \times \boldsymbol{B}$, on a moving, charged particle in a magnetic field $\boldsymbol{B}$. Thus a charged particle moving through a uniform magnetic field will be deflected into a circular orbit with the cyclotron frequency, qB/M, analogous to an inertial oscillation at the frequency f. General Relativity predicts that a rotating (massive) object is accompanied by a 'gravitomagnetic' field analogous to a magnetic field, and that gives rise to a Coriolis-like force on moving objects. The upcoming Gravity Probe B satellite mission aims to test this aspect of General Relativity; see http://einstein.stanford.edu/

[37] The name 'inertial oscillation' is very widely accepted but is not highly descriptive of the dynamics in either the rotating or inertial reference frame. For the rotating frame, 'Coriolis oscillation' might be more appropriate, and for the inertial frame see D. R. Durran, 'Is the Coriolis force really responsible for the inertial oscillation?' *Bull. Am. Met. Soc.*, **74**(11), 2179–2184 (1993).

[38] The applicability of this single-parcel model to the geostrophic adjustment of a real fluid is partial and in fact it omits some of the most interesting aspects. Specifically, if the applied force, $\boldsymbol{F}$, is a pressure gradient associated with a mass anomaly, then $\boldsymbol{F}$ will vary with time as the mass anomaly moves during the adjustment process. Once you understand this single-parcel model you should continue on with a study of geostrophic adjustment in a fluid model, perhaps via the Matlab script geoadjPE.m, also available from the Mathworks File Central, and see Appendix B for an animation from this model.

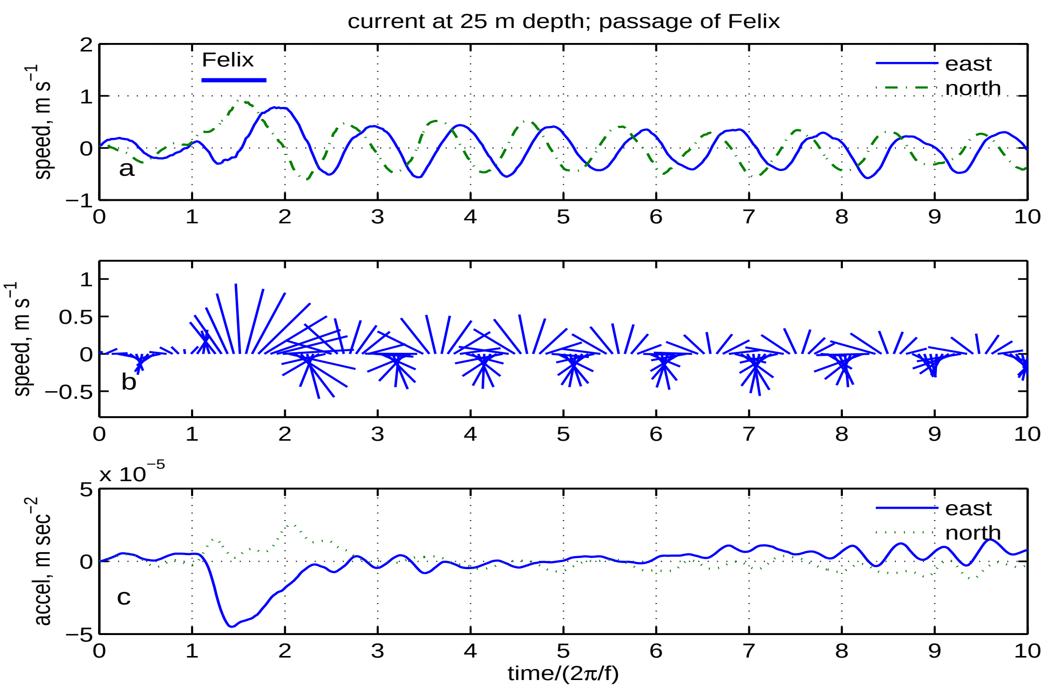

Figure 12: (a and b) Ocean currents at a depth of 25 m, measured by a current meter deployed southwest of Bermuda. The time scale is in inertial periods, which are nearly equal to days at this latitude. Hurricane Felix passed over the current meter mooring at the time noted at upper left, and the strong and rapidly changing wind stress produced energetic, clockwise rotating currents within the upper ocean. To a first approximation these are inertial oscillations. They differ from pure inertial oscillations in that their frequency is usually a few percent higher than f, and their amplitude e-folds over about 5-15 days. These small departures from pure inertial motion are indicative of wave-like dynamics that is not accessible in the single parcel model used here (see the simulation of geostrophic adjustment in Appendix B). (c) Acceleration estimated from the current meter data as $d\boldsymbol{V}'/dt + 2\boldsymbol{\Omega} \times \boldsymbol{V}'$, as if the measurements were made on a specific parcel. (A question for you: what assumption is implicit in this switch from a parcel to a fixed location? Hint: recall the discussion in Sec. 1.1 regarding the single parcel model.) The large acceleration to the west northwest corresponds in time to the passage of Felix. The direction of the estimated acceleration is roughly parallel to the observed winds (not shown here), consistent with being the divergence of wind stress, mainly. Notice also the much smaller oscillations having a period of about 0.5 inertial periods (especially for $t > 8$). These are very likely due to pressure gradients associated with the semidiurnal tide. This is a small part of the data described in detail by Zedler, S.E., T.D. Dickey, S.C. Doney, J.F. Price, X. Yu, and G.L. Mellor, 'Analysis and simulations of the upper ocean's response to Hurricane Felix at the Bermuda Testbed Mooring site: August 13-23, 1995', *J. Geophys. Res.*, **107**, (C12), **25**-1 - **25**-29, (2002), available online at http://www.opl.ucsb.edu/tommy/pubs/SarahFelixJGR.pdf.

drag. The task of estimating an accurate bottom drag for a specific case is beyond the scope here, and we will represent bottom drag by the simplest linear (or Rayleigh) drag law in which the drag is presumed to be proportional to and antiparallel to the velocity difference between the current and the bottom, i.e., bottom drag $= -k(\boldsymbol{V} - \boldsymbol{V_{bot}})$.[39] The ocean bottom is at rest in the rotating frame and hence $\boldsymbol{V_{bot}} = 0$ and omitted from here on. From observations of ocean currents we can infer that a reasonable value of k for a density-driven current on a continental shelf is $k = O(10^{-5})$ sec^{-1}. Thus k is roughly an order of magnitude smaller than a typical mid-latitude value of f. Since k appears in the momentum equations in the same way that f does we can anticipate that rotational effects will be dominant over frictional effects. The equations of motion are then:

$$\begin{aligned}\frac{dU}{dt} &= fV - kU, \\ \frac{dV}{dt} &= -fU - kV + b\alpha,\end{aligned} \tag{63}$$

and we assume initial conditions $U(0) = 0, V(0) = 0$. The depth of the parcel can be computed diagnostically from the y position and the known slope. Notice that we have dropped the superscript prime that had previously been used to indicate the rotating frame variables and we have used the thin fluid approximation for the Coriolis terms. We also use the $f-$plane approximation that f = constant since typical parcel displacements are very small compared to the Earth's radius. The solutions of this linear model are not complex,

$$\begin{aligned}U(t) &= \frac{b\alpha}{k^2 + f^2}\left[f - \exp(-tk)(f\cos(ft) - k\sin(ft))\right], \\ V(t) &= \frac{b\alpha}{k^2 + f^2}\left[k - \exp(-tk)(f\sin(ft) + k\cos(ft))\right],\end{aligned} \tag{64}$$

though they do contain three parameters along with the time, and hence represent a fairly large parameter space. We are not interested in any one solution as much as we are in understanding the qualitative effects of rotation and friction upon the entire family of solutions. How can we display the solution to this end?

One approach that is very widely applicable is to rewrite the governing equations or the solution using nondimensional variables. This will serve to reduce the number of parameters to the fewest possible. To define these nondimensional variables we begin by noting that there are three external parameters in the problem (external in that they do not vary with a dependent variable): the buoyancy and bottom slope, $b\alpha$, which always occur in this combination and so count as one parameter, the Coriolis parameter, f, an inverse time scale, and the bottom friction coefficient, k, also an inverse time scale. To form a nondimensional velocity, $U_* = U/U_{geo}$, we have to make an estimate of the velocity scale as the product of the acceleration and the time scale f^{-1} as $U_{geo} = (b\alpha)/f$ and thus $U_* = U/(b\alpha/f)$ and similarly for the V component. To define a nondimensional time we need an external time scale and choose the inverse of the Coriolis parameter, $t_* = tf$, rather than k^{-1}, since we expect that rotational effects will dominate frictional effects in most cases

[39] A linear drag law of this sort is most appropriate as a model of viscous drag in a laminar boundary layer within which $\tau = \mu\frac{\partial U}{\partial z}$, where μ is the viscosity of the fluid. The boundary layer above a rough ocean bottom is almost always fully turbulent above a very thin, O(10^{-3} m), laminar sublayer that is in contact with the bottom. If the velocity used to estimate drag is measured or computed for a depth that is within the fully turbulent boundary layer, as it is bound to be, then the appropriate drag law can be approximated as independent of the viscosity and is quadratic in the velocity, $\tau \propto \rho U^2$.

of interest. Rewriting the governing equations in terms of these nondimensional variables gives

$$\begin{aligned} \frac{dU_*}{dt_*} &= V_* - EU_*, \\ \frac{dV_*}{dt_*} &= -U_* - EV_* + 1, \end{aligned} \tag{65}$$

and initial conditions $U_*(0) = 0,\ V_*(0) = 0$. The solution to these equations,

$$\begin{aligned} U_*(t_*) &= \frac{1}{1+E^2}\left[1 - \exp(-Et_*)(\cos(t_*) - E\sin(t_*))\right], \\ V_*(t_*) &= \frac{1}{1+E^2}\left[E - \exp(-Et_*)(\sin(t_*) + E\cos(t_*))\right], \end{aligned} \tag{66}$$

$$U_* = \frac{U}{b\alpha/f}, \quad V_* = \frac{V}{b\alpha/f}, \quad t_* = tf \quad \text{and} \quad E = k/f,$$

shows explicitly that the single nondimensional parameter $E = k/f$ serves to define the parameter space of this problem.[40] E, often termed the Ekman number, is the ratio of frictional to rotational forces on the parcel. Thus in place of large friction or large rotation, we have instead large or small E, which implies a standard of comparison. The trajectories computed from Eq. (66) for several values of E are in Fig. (13a).

5.2 Dense parcels on a rotating slope

5.3 Inertial and geostrophic motion

The solution Eq. (66) can be envisioned as the sum of two distinct modes of motion, a time-dependent, oscillatory motion, the now familiar inertial oscillation,

$$\begin{bmatrix} U_* \\ V_* \end{bmatrix} \propto \begin{bmatrix} \cos(t_*) \\ \sin(t_*) \end{bmatrix},$$

and a time mean motion that could also be called a geostrophic current,

$$\begin{bmatrix} U_* \\ V_* \end{bmatrix} \propto \begin{bmatrix} 1 \\ E \end{bmatrix}.$$

[40] A brief introduction to the method and uses of dimensional analysis at about the level of this essay is by J. F. Price, 'Dimensional analysis of models and data sets', *Am. J. Phys.*, **71**(5), 437–447 (2003) and available online from http://www.whoi.edu/science/PO/people/jprice/class/DA.pdf, or from the Mathworks File Exchange, http://www.mathworks.com/matlabcentral/fileexchange/loadCategory.do, under the categories 'Mathematics', 'Linear Algebra', and where the file name is Danalysis. There are usually several plausible ways to accomplish a nondimensionalization. For example, in this problem we could have used $1/k$ to measure (or nondimensionalize) the time. How would this change the solution, Eq. (66)? By simplifying the form of an equation, dimensional analysis can help make clear the comparative sizes of terms; this is effectively what we did in the analysis of Section 4.3.1. With this idea in mind, take another look at Eq. (56) and try measuring (or normalizing) the radius by R_o and the time by Ω^{-1}. How does your result compare with Eq. (56)?

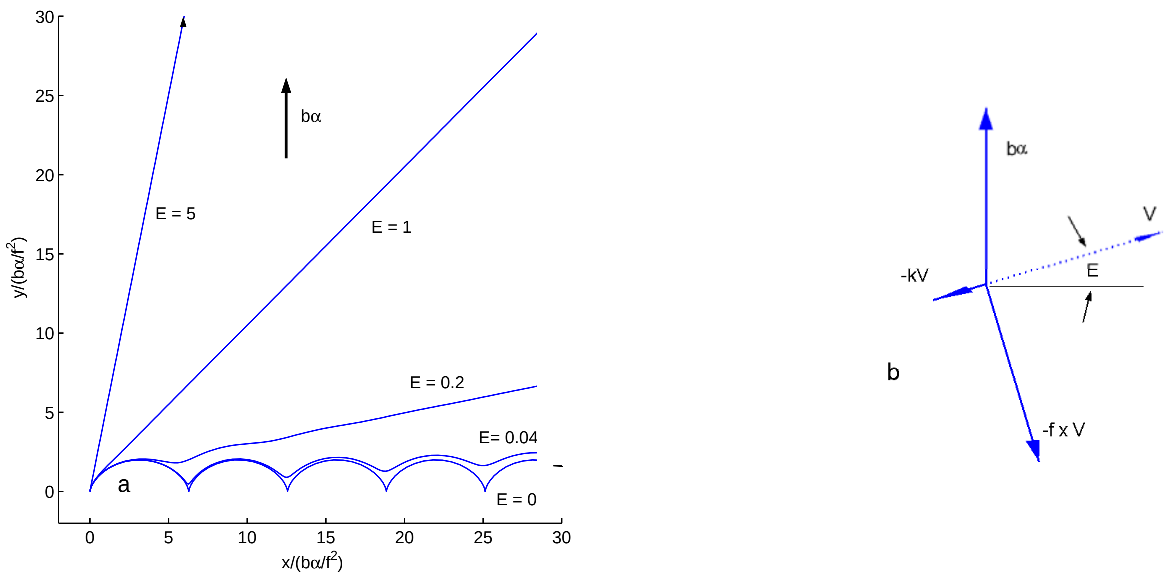

Figure 13: (a) Trajectories of dense parcels released from rest onto a sloping bottom computed by integrating the horizontal velocity, Eq. (66). The buoyancy force is toward positive y and the Ekman number, E, has the value shown next to a trajectory. Notice that for values of E small compared to 1, the long term displacement is nearly at right angles to the imposed force, indicative of geostrophic balance. (b) The force balance (solid arrows) and the time-mean motion (the dashed vector) for the case $E = 0.2$. The angle of the velocity with respect to the isobaths is $E = k/f$, the Ekman number. The Coriolis force ($/M$) is labeled $-\boldsymbol{f} \times \boldsymbol{V}$ where $\boldsymbol{f}$ is f times a vertical unit vector. Simulations of this sort are carried out by the script partslope.m. [12] An animation of several trajectories is in the following figure.

In this linear model the amplitude of either mode [41] is directly proportional to the velocity scale, $U_{geo} = b\alpha/f$. For a dense water parcel on a continental slope (Fig. 15) rough values are $b = g(\delta\rho)/\rho_0 \approx g0.5/1000 = 10^{-2}$ m sec^{-2}, $\alpha = 1.3\times10^{-2}$, and f at 62 degrees latitude = 1.3 $\times10^{-4}$ sec^{-1}, and thus a typical geostrophic density current has a speed $U_{geo} \approx 0.5$ m sec^{-1}, Fig. (15).

The inertial oscillations found in this solution are the consequence of starting from a state of rest and the imposition of what amounts to an impulsively started external force (the buoyancy force). This is not an especially realistic model of a density current released onto a continental slope, but it does serve as a first approximation of the rapidly changing wind stress exerted upon the upper ocean during the passage of a storm (Fig. 12). A pure inertial oscillation, that is, an exact two term balance between the time rate of change and the Coriolis force (Eq. 61), would not evolve with time in the sense that the amplitude and frequency would remain constant. The (near) inertial oscillations of Figs. (12) and (13) decrease with time; the latter decay with time as exp $(-Et_*)$ on account of bottom drag.[42]

[41] This usage of 'mode' may be a little unusual; the intent is to identify a possible two term balance within the equation of motion, i.e., between a pressure gradient and the Coriolis force, for example. By this we are not asserting that such a balance necessarily holds strictly even within our very simple models but rather that modes are a kind of vocabulary useful for describing the complex balances that actually do occur. When the equations or physical phenomenon are nonlinear, as they usually are, then the actual solution or flow will not be a sum over the modes computed individually. Beyond those already noted, what other modes could be present in Eqs. (66)?

[42] A couple of questions for you: 1) Can you devise an initial condition for this problem that would eliminate the inertial oscillations? Could you make the inertial oscillations larger than those shown in Fig. (13a)? To test your hypothesis you might try experimenting with the Matlab script partslope.m 2) Draw the vector force balance for inertial oscillations (include the acceleration) with and without

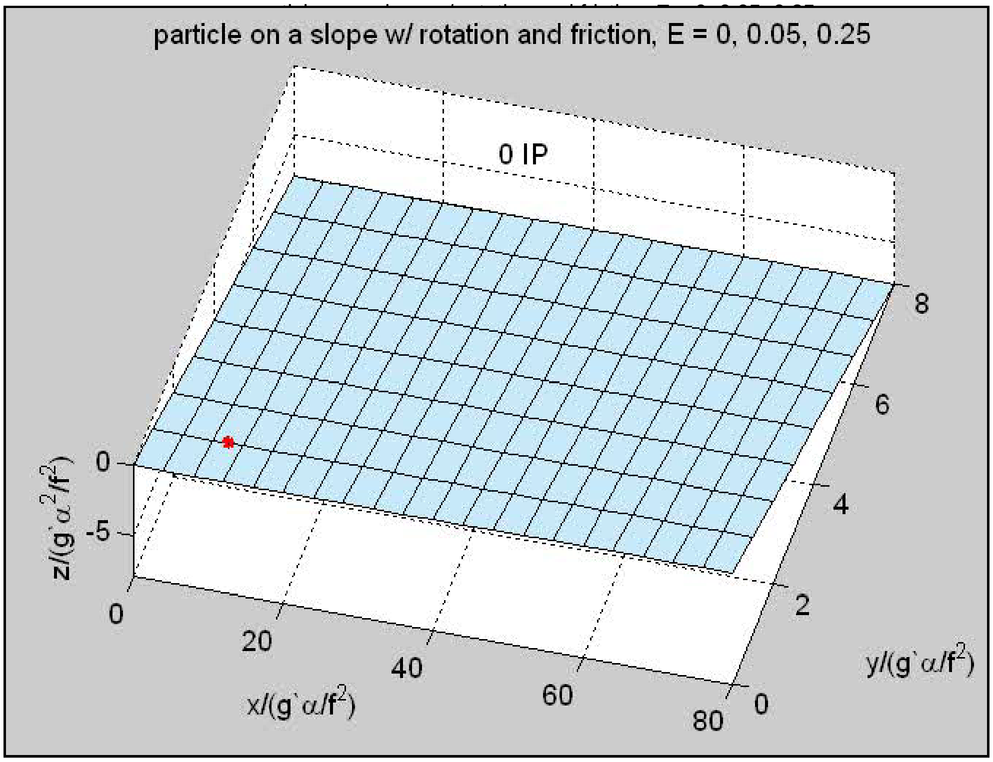

Figure 14: A simulation of the motion of dense parcels released onto a slope, computed by partslope.m. Northern hemisphere is assumed, and the Ekman number is 0., 0.05 and 0.25 for the red, green and blue trajectories, respectively. The elapsed time in units of inertial periods, $2\pi/f$, is at upper left.

5.4 Energy budget

The energy budget for the parcel makes an interesting diagnostic. To find the energy budget (per unit mass) we simply multiply the x-component momentum equation by U_* and the y-component equation by V_* and add:

$$\frac{d(U_*^2 + V_*^2)/2}{dt_*} = V_* - E(U_*^2 + V_*^2). \tag{67}$$

The Coriolis terms add to zero, and the rate of work by the buoyancy force is V_* in these nondimensional units. It can be helpful to integrate with time to see energy changes;

$$\begin{aligned} (U_*^2 + V_*^2)/2 &= \int_o^t V_* dt_* - \int_o^t E(U_*^2 + V_*^2) dt_* \\ KE &= BW - FW \end{aligned} \tag{68}$$

bottom drag as in Fig. (13b). Explain how the amplitude can decay while the frequency is unaltered.

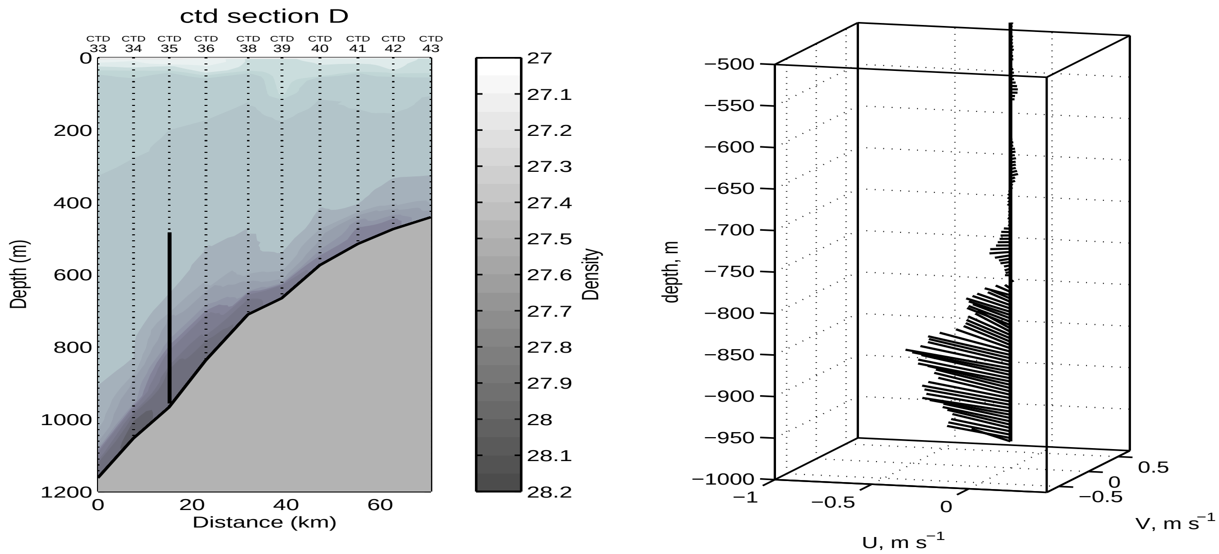

Figure 15: Observations of density and currents along the southern flank of the Scotland-Iceland Ridge, about 90 km west of the Faroe Islands. The dense water found along the bottom is an outflow from the Norwegian-Greenland Sea that has come through the narrow Faroe Bank Channel (about 15 km width, at latitude 62 North) and that will eventually settle into the deep North Atlantic. The units of density are kg m $^{-3}$, and 1000 has been subtracted away. Currents were measured at the thick vertical line shown on the density section. The density section is aligned normal to the isobaths and the current appeared to be flowing roughly along the isobaths, but the direction was temporally varying. What is clearer is that the core of the dense water has descended roughly 200 m between this site and the narrow Faroe Bank Channel, about 90 km upstream from this site. A question: which trajectory of Fig. (13) is analogous to this current? Said a little differently, what is the approximate Ekman number of this current?

where KE is the kinetic energy, BW is the work by the buoyancy force (proportional to the change of the potential energy),

$$BW = \int_o^t V_* dt_* = \delta z \frac{b}{U_{geo}^2} = \delta z \frac{f^2}{b\alpha^2},$$

where $\delta z = \int_o^t V\alpha dt$ is the change in the depth of the parcel (dimensional units), and FW is the work by the bottom friction (Fig. 16). The Coriolis force *per se* drops out of the energy budget since it is normal to the current and does no work.

Nevertheless, rotation has a profound effect on the energy budget overall in as much as the cross-isobath component of the mean motion (which carries the parcel to greater bottom depth and thus releases potential energy) is directly proportional to the Ekman number, from Eq. (66),

$$\frac{V_*}{U_*} = E,$$

(Fig. 13b) and thus inversely proportional to f for a given frictional coefficient, k. Whether friction or rotation is dominant, and thus whether circulations are rapidly dissipated or long-lived, depends solely upon

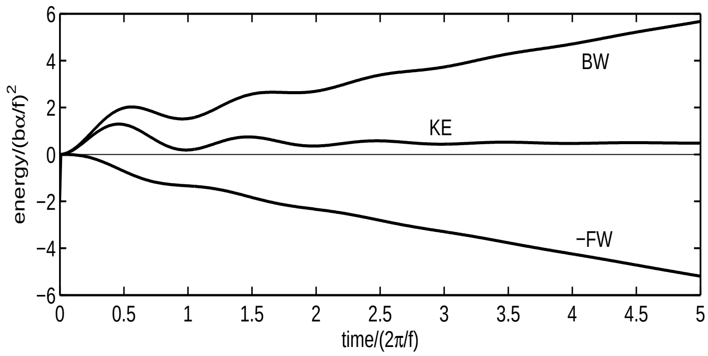

Figure 16: The energy budget for the trajectory of Fig. (13) having $E = 0.2$. These data are plotted in a nondimensional form in which the energy or work is normalized by the square of the velocity scale, $U_{geo} = b\alpha/f$ and time is nondimensionalized by the inertial period, $2\pi/f$. After inertial oscillations are damped away, the energy balance settles into a balance between work by the buoyancy force, BW, and work against bottom (frictional) drag, FW.

the Ekman number in this highly simplified system.[43] In the limit $E \rightarrow 0$ the time-averaged motion becomes perpendicular to the buoyancy force and the parcel coasts along isobaths with no energy exchanges and no temporal evolution, the energy budget consequence of geostrophic motion. A crucial, qualitative effect of rotation is that it makes possible a steady balance between the external force and the Coriolis force, where in the absence of rotation the only possible steady balance is between the external force and friction. Thus rotation admits the comparatively long-lived (geostrophic) circulations that make up the most important winds and ocean currents outside of tropical regions (e.g., Fig. 1).[44]

6 Summary and closing remarks.

To close we will return to the first of the topics/questions noted in Section 1.2 — the origin of the term $2\boldsymbol{\Omega} \times \boldsymbol{V}'M$, and whether it is appropriate to think of this term as the Coriolis 'force'. Before we respond directly to this we should recognize that if we had the ability to compute trajectories in an inertial frame, we could then transform those trajectories into the rotating frame and would never have to consider the Coriolis force (an example of this procedure was in Section 3.4). Appealing as that is, inertial frame solutions are almost never attempted for oceanic and atmospheric flows, which in practice are much more readily analyzed

[43]Two questions for you: (1) Can you show that the time-averaged solution of the single parcel model is the solution of the time-averaged model equations? Suppose the model equations were not linear, then what? (2) How would geostrophic adjustment look if viewed from an inertial frame, as in Fig. (11a)? Consider that there is an initial, balanced solid body rotation, and then impose a small radial or azimuthal force. Compare your (qualitative) result with the solution computed by the script Coriolis-forced.m.

[44]An exact geostrophic balance thus implies exactly steady motion. The atmosphere and ocean evolve continually, and so geostrophic balance must be an approximation to the momentum balance of large-scale, extra-tropical motions, albeit a very good one. To understand how or why atmospheric and oceanic circulations evolve will evidently require an understanding of the small *departures* from geostrophic balance of these motions, a task that is well beyond the scope here. We may have noticed that geostrophy does not appear to hold even approximately in near-equatorial regions (Fig. 1), and the single parcel model seems to indicate as much; for a given k, E will become very large as the latitude goes to zero and hence geostrophy would not hold at low latitudes because of large frictional effects. The conclusion is correct, but the reason is not, and positively misleading. Long before being damped by friction, an equatorial mass and pressure anomaly will disperse into large scale gravity waves that may propagate for thousands of kilometers along the equator. The rapid and far reaching gravity wave response of equatorial regions is a key element of the El Nino phenomenon that could not have been anticipated from the single parcel model considered here. It isn't exactly that the single parcel model is wrong, so much as it is irrelevant (because it is incomplete) when applied to near-equatorial regions. This kind of failure of an otherwise useful model is a fairly common occurrence in fluid mechanics. (An animation of geostrophic adjustment that gives a suggestion of this gravity wave response is in Appendix B).

from an Earth-attached, rotating reference frame (Section 4.3). Once we decide to use a rotating frame, the centrifugal and Coriolis forces are exact consequences of transforming the equation of motion (Section 2.4); there is nothing *ad hoc* or discretionary about their appearance in the rotating frame equation of motion. [6] In the special case of an Earth-attached and essentially fluid reference frame that is in gravitational-rotational equilibrium, the centrifugal force is exactly canceled by a small component of gravitational mass attraction (Section 4.1).

The Coriolis and centrifugal forces are inertial forces that arise from the rotational acceleration of a reference frame rather than from an interaction between physical objects. This has significance on two levels. (1) Since there is no physical origin for these forces, neither should we expect a physical explanation of these forces. The sole, reliable explanation of these terms is their origin in the transformation law for acceleration combined with our practice to observe and analyze the acceleration as seen in the rotating frame (Section 2.5). (2) Because the centrifugal and Coriolis are not central forces they contribute peculiar, unphysical behavior to the rotating frame dynamics. Recall the elementary trajectory of Section 3.4; when observed from a rotating frame the parcel veered to the right as expected from the deflecting Coriolis force. However, there was no object in the environment that revealed a corresponding reaction force. Similarly, the parcel speed and kinetic energy increased with time due to work by the centrifugal force, and yet there was no source for the energy. The rotating frame equation of motion thus does not support global conservation of momentum and neither does it preserve invariance to Galilean transformations. In practical analysis these are not serious flaws, and are more than compensated by the simplicity of the rotating frame equations compared to their inertial frame counterpart (Section 4.3).

What we call the Coriolis term, e.g., whether an acceleration or a force, *is* a matter of choice, of course, though our usage should reflect our understanding and guide our subsequent interpretation along useful lines. The former is sensible insofar as the Coriolis term arises from the transformation of acceleration, and too because it is independent of the mass of the parcel. However, when we use an Earth-attached, rotating reference frame we seek to analyze (and necessarily observe) the acceleration seen in the rotating frame, $d^2\boldsymbol{X}'/dt^2$, and not the rotated acceleration, $(d^2\boldsymbol{X}/dt^2)'$, of which the Coriolis term is a part (Section 2.5). To understand the contribution of the Coriolis term to the acceleration that we observe, it is then natural to regard the Coriolis term as the Coriolis *force*, for example in the interpretation of pressure and wind fields (Fig. 1) or of the motion of dense parcels along a slope (Fig. 13b). In the end, this seems the most important consideration. But we have also emphasized throughout that the Coriolis force is something apart from a central force, such as a pressure gradient or frictional force. If it is helpful to acknowledge this, then we could say instead Coriolis *inertial force* which gives at least a hint at the origin, in the rotational motion of Earth itself.

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7 Appendix A: Circular motion and polar coordinates

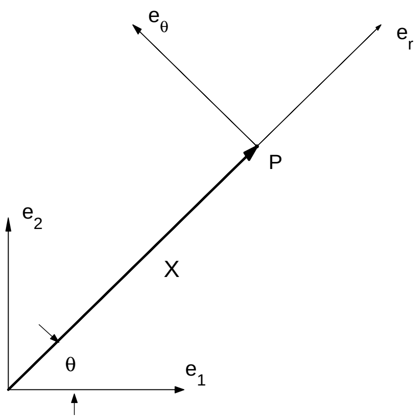

Figure 17: The unit vectors $\boldsymbol{e_1}, \boldsymbol{e_2}$ define a stationary reference frame. The unit vectors for a polar coordinate system, $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$, are defined at the position of a given parcel, P. These unit vectors are time-dependent since the angle θ is time-dependent.

Rotational phenomena are often analyzed most efficiently with cylindrical polar coordinates, reviewed here briefly. The vertical coordinate is exactly the z or x_3 of Cartesian coordinates, and we need consider only the horizontal (two-dimensional) position, which can be specified by a distance from the origin, r, and the angle, θ between the radius vector and (arbitrarily) the x_1 axis (Fig. 17). The corresponding unit vectors are given in terms of the time-independent Cartesian unit vectors that define the stationary frame by

$$\boldsymbol{e_r} = cos(\theta)\boldsymbol{e_1} + sin(\theta)\boldsymbol{e_2} \text{ and, } \boldsymbol{e_\theta} = -sin(\theta)\boldsymbol{e_1} + cos(\theta)\boldsymbol{e_2}. \tag{69}$$

The position vector in this system is

$$\boldsymbol{X} = r\boldsymbol{e_r},$$

and hence the velocity is

$$\frac{d\boldsymbol{X}}{dt}M = \frac{dr}{dt}\boldsymbol{e_r}M + r\frac{d\boldsymbol{e_r}}{dt}M = \frac{dr}{dt}M\boldsymbol{e_r} + r\omega M\boldsymbol{e_\theta}, \tag{70}$$

where we have taken account of the time-dependence of $\boldsymbol{e_r}$ and $\omega = d\theta/dt$. Continuing, the equation of motion is

$$\frac{d^2\boldsymbol{X}}{dt^2}M = \left(\frac{d^2r}{dt^2} - r\omega^2\right)M\boldsymbol{e_r} + \left(2\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right)M\boldsymbol{e_\theta} \tag{71}$$

$$= F_r\boldsymbol{e_r} + F_\theta\boldsymbol{e_\theta}. \tag{72}$$

Notice that there are acceleration terms, $r\omega^2$ and $2\omega\frac{dr}{dt}$, that look just like the centrifugal and Coriolis forces (and are sometimes deemed to be such, even by otherwise very careful authors, e.g., Boas, [13] p. 399), though this equation holds in an inertial frame where centrifugal and Coriolis forces (or accelerations) *do not arise*.

To find the rotating frame equation of motion is straightforward; the radius is identical in the rotating frame, $r' = r$, since the origins are assumed coincident. The unit vectors are identical since they are defined at the location of the parcel, $\boldsymbol{e}'_{\boldsymbol{r}} = \boldsymbol{e}_{\boldsymbol{r}}$ and $\boldsymbol{e}'_{\boldsymbol{\theta}} = \boldsymbol{e}_{\boldsymbol{\theta}}$; the force components are thus also identical. The only thing different is that the angular velocity ω is decomposed into a time mean and a relative angular velocity, $\omega = \omega' + \Omega$. Substituting this into the inertial frame equation of motion Eq. (72), and rearrangement to move terms containing Ω to the right hand side yields the formidable-looking rotating frame equation of motion for polar coordinates,

$$\begin{aligned} \frac{d^2\boldsymbol{X}'}{dt^2}M &= \left(\frac{d^2r'}{dt^2} - r'\omega'^2\right)M\boldsymbol{e}'_{\boldsymbol{r}} + \left(2\omega'\frac{dr'}{dt} + r'\frac{d\omega'}{dt}\right)M\boldsymbol{e}'_{\boldsymbol{\theta}} \\ &= \left(r'\Omega^2 M + 2\Omega\omega' r' M + F'_r\right)\boldsymbol{e}'_{\boldsymbol{r}} + \left(-2\Omega\frac{dr'}{dt}M + F'_\theta\right)\boldsymbol{e}'_{\boldsymbol{\theta}}. \end{aligned} \tag{73}$$

Notice that there are genuine centrifugal and Coriolis force terms on the righthand side of Eq. (31) and that we have derived these terms for the third time now; for Cartesian coordinates, Eq. (25), for vectors, Eq. (26), and here for cylindrical polar coordinates. You should be sure to verify this, as it is perhaps the most direct derivation of the Coriolis force, and most easily shows how or where the factor of 2 arises.

8 Appendix B: Adjustment to gravity and rotation in a single fluid layer

This appendix uses a model of a single layer fluid to illustrate some aspects of gesotrophic adjustment. The model is highly idealized in that it varies in the X or east-west direction only, but is nevertheless a significant step closer to a realistic model of geostrophic flow than is the single parcel model (it is also somewhat beyond the scope of this essay). The initial condition is taken to be a ridge in the height of a dense (blue) layer that is at rest and released at time t = 0. Dimensional variables are chosen with the oceanic main thermocline in mind; the ridge has a half-width of L =200 km and a density anomaly $\delta\rho = 2$ kg m^3. The nominal layer thickness is $H = 500$ m and the ridge has an initial height $\eta_o = 100$ m. The main (contact) force on fluid parcels in the dense layer is the gradient of the hydrostatic pressure, $P = g\delta\rho\eta$, and thus an elevated dense layer is a region of high pressure. Some qualitative features of these simulations are relevant to the comparison of equatorial and high latitude circulations noted in the discussion of Fig. (1).

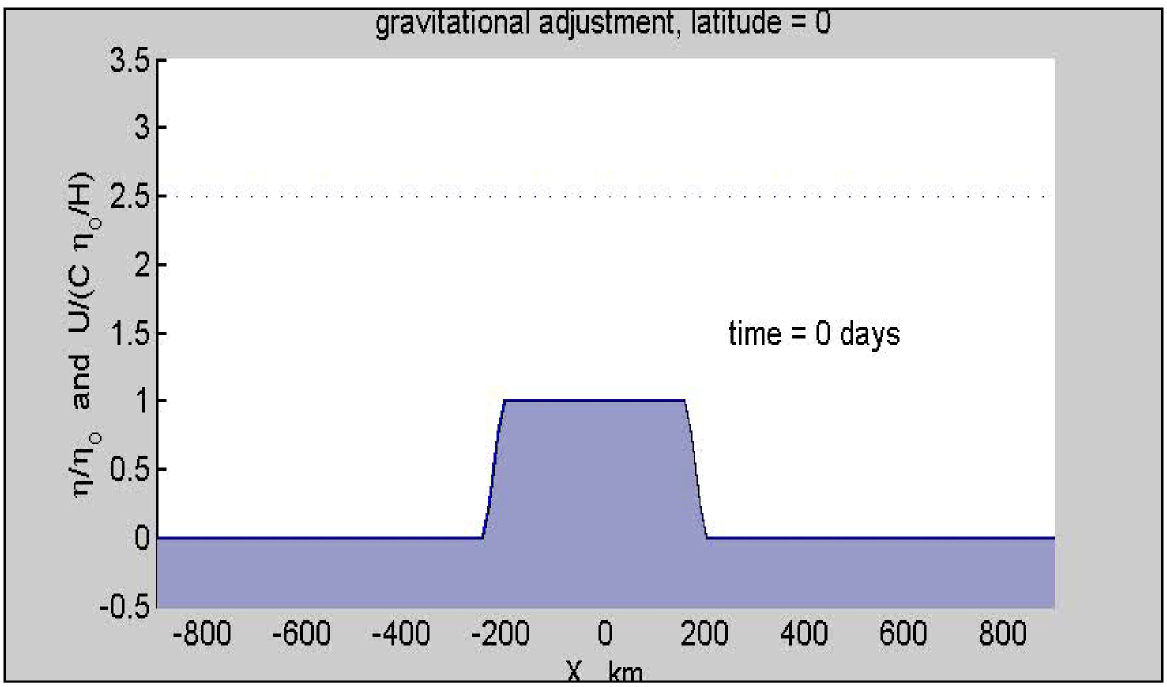

Figure 18: Adjustment to gravity only in a single fluid layer, simulated by the primitive equation model, geoadjPE.m.[12] The present case is set at latitude = 0 so that f = 0 and rotational effects vanish. In the context of this idealized model, this case is equatorial (the equivalent problem with rotation is shown next). After the ridge is released, it splits into two equal pieces of amplitude $\eta_o/2$ that move outward at the long gravity wave speed $C = \sqrt{(g\delta\rho/\rho_o)H} = 3.1$ m s^{-1}, qualitatively like the D'Alembert solution for the initial value problem of the elementary wave equation. The current, which is shown by the array of lines plotted above the ridge, is in the X-direction only (left-to-right in this figure), and is scaled with $C\eta_o/H = 0.6$ m s^{-1}. In this case the gravity wave dispersion is entirely dominant, and there is no steady response whatever.

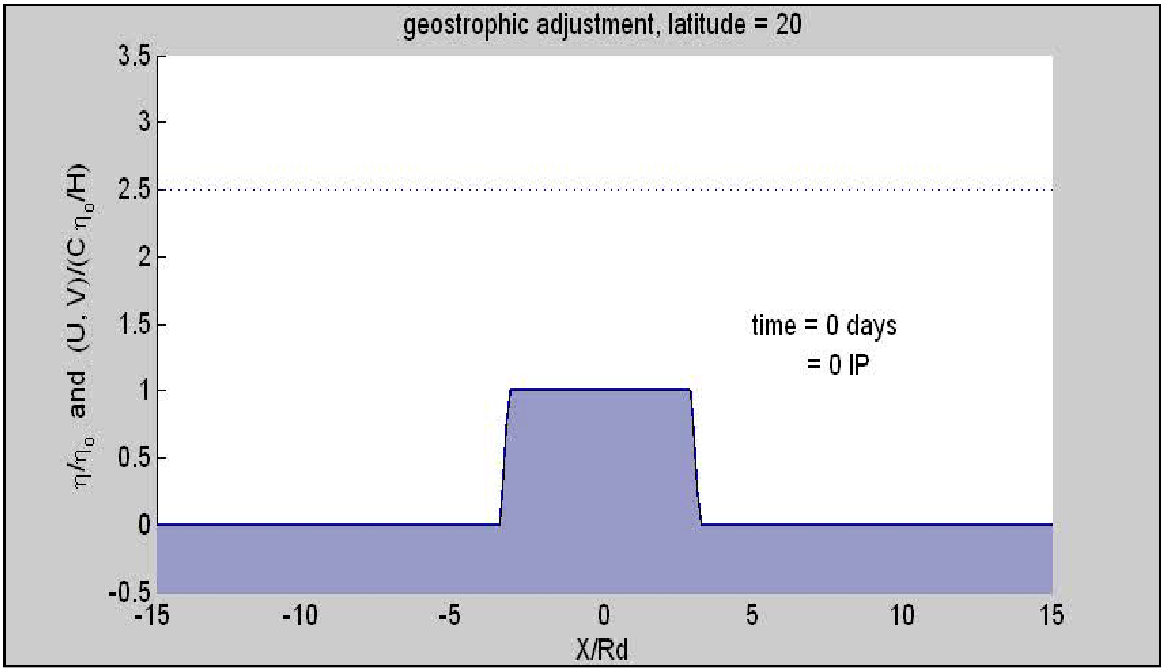

Figure 19: Geostrophic adjustment to gravity and rotation in a single fluid layer, simulated by the primitive equation model, geoadjPE.m. This case is identical to the previous example except that the latitude = 20 N. When this ridge is released, the resulting gravity waves have a somewhat smaller amplitude than in the previous (non-rotating) example and their associated current rotates clockwise in time, very much like an inertial oscillation (cf. Fig. 12). The phase and group speed of the fastest moving waves is about the same as in the previous example. The key result is that most of this rather wide ridge remains after the pressure and velocity fields have settled into an approximate geostrophic balance, $-fV \approx -(g\delta\rho/\rho_o)\partial\eta/\partial x$, where V is the velocity along the ridge (normal to the page). To begin to understand why rotational effects are important in this case we can compare the time required for dispersion by gravity waves, roughly $T_d = L/C$, and the time required for rotational effects to become apparent, roughly $T_r \approx 1/f$. The ratio of these time scales gives a nondimensional parameter $T_d/T_r = Lf/C = L/Rd$, where $C/f = R_d$ is the called the radius of deformation, an important length scale in rotating, stratified flows (see the GFD texts [9] for much more on this). In the oceanic thermocline at this latitude, $R_d \approx 60$ km; the comparable scale of the atmosphere is much larger, $R_d \approx 1000$ km, because the atmosphere is much more strongly stratified. The condition $L/R_d \gg 1$ indicates a wide ridge (in this case $L \approx 3.3R_d$), most of which will survive geostrophic adjustment, while a narrow ridge, $L/Rd \ll 1$, would be largely dispersed by gravity waves before coming into geostrophic balance. The previous 'equatorial' example could be regarded as the limit $L/R_d \to 0$ in which rotational effects were negligible (literally zero in that case) compared to gravity wave dispersion.