18.100C. Final. Spring 2006.

NAME: ____________________ May 24, 2006

Check one:

___ Send me my exam score and final grade by email. My e-mail address is

___ Do not send me my exam score and final grade by email.

Problem 1: _____ /50

Problem 2: _____ /60

Problem 3: _____ /35

Problem 4: _____ /50

Problem 5: _____ /50

Problem 6: _____ /55

Total: _____ /300

Instructions: The exam is closed book, closed notes, except for one sheet of A4 paper; calculators are not allowed. You will have 175 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work and give proofs, unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. Good luck!

Problem 1.(50 pts): (10; 15; 10; 15)
Let f be the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

a) Show that f is continuous for all x.

b) Show that f is differentiable for all x, and find the derivative $f'(x)$.

c) Is $f'(x)$ bounded on the interval $(0, 1)$? Justify your answer.

d) Let g be a differentiable function on $(0, 1)$ such that its derivative is bounded on $(0, 1)$. Prove that $g(x)$ is uniformly continuous on $(0, 1)$.

Problem 2. (60 pts): (10; 10; 15; 10; 15)

a) If $n \geq 1$, find an antiderivative for $e^{-nx}\cos(nx)$ using integration by parts. Check your answer by differentiation.

b) Find $\int_1^\infty e^{-nx}\cos(nx)dx$.

c) Consider the series

$$\sum_{n\geq 1}^{\infty} e^{-nx}\cos(nx).$$

Prove that the series converges uniformly on every interval $[a, \infty)$ where $a > 0$.

d) If $f(x)$ denotes the sum of the series in c), show that $f(x)$ is continuous on $(0, \infty)$.

e) Prove that $\left|\int_1^\infty f(x)dx\right| \leq 2$, where $f(x)$ is as defined in parts c) and d).

Problem 3. (35 pts): (15; 20)
a) Define the sequence $\{a_n\}$ by

$$a_{2n} = 4^n, \quad a_{2n+1} = 3^{2n}, \quad n \geq 0.$$

Find the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

b) Determine the radius of convergence of $\sum_{n=1}^{\infty} n z^n$, and find a formula for the sum. (Hint: Start with a well-known formula for $\sum_{n=0}^{\infty} z^n$.) Justify the correctness of your calculations.

Problem 4. (50 pts): (15; 15; 15; 5)

Let E be a nonempty closed subset of a metric space X with metric function d. Define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

a) Prove that $\rho_E(x) = 0$ if and only if $x \in E$.

b) Prove that for all $x \in X, y \in X$,

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y),$$

and therefore $\rho_E : X \to \mathbb{R}$ is uniformly continuous on X.

c) Let K be a compact subset of X, disjoint from E. Prove that there exists $x_0 \in K$ such that $0 < \rho_E(x_0) \leq \rho_E(x)$, for all $x \in K$.

d) If $E \subset \mathbb{R}$ denotes the Cantor set, and $x = \frac{5}{6}$, what is the distance $\rho_E(x)$ equal to?

Problem 5. (50 pts): (15; 20; 15)
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

a) Assume $\int_0^1 f(x)dx = 1$. Show that there exists $c \in (0, 1)$ such that $f(c) = 1$.

b) Now suppose

$$\int_0^1 f(x)x^n dx = \frac{1}{n+1}, \text{ for all } n \geq 0.$$

Prove that $f(x) = 1$ for all $x \in [0, 1]$. (Hint: set $g(x) = f(x) - 1$. You may want to use the Weierstrass theorem.)

c) Prove that if $h : [0, 1] \to \mathbb{R}$ is a continuous nonnegative function, and $\int_0^1 h(x)dx = 0$, then $h(x) = 0$, for all $x \in [0, 1]$.

Problem 6. (55 pts): (10; 15; 15; 15)

Let X be the space of all sequences of real numbers. For any two sequences $\underline{a} = \{a_i\}$ and $\underline{b} = \{b_i\}$, define

$$d(\underline{a}, \underline{b}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|a_i - b_i|}{1 + |a_i - b_i|}.$$

a) Show that d is well defined.

b) Prove that d defines a metric on X.

c) Prove that, with respect to d, X is bounded, but it is not compact. (Hint: construct a sequence $\{\underline{x}_n\}$ in X, such that $d(\underline{x}_n, \underline{x}_m) \geq \frac{1}{2}$ for all $n \neq m$.)

d) Prove that the metric space (X, d) is complete.

(Thank you for taking 100C this semester. Have a great summer break!)