18.100C. Quiz 2. Solutions.

Problem 1.

(a)
We know that the function $\frac{1}{x}$ is continuous away from 0, and so the function $1+e^{1/x}$ is continuous except at 0. Note that $1+e^{1/x}$ is strictly positive on this range, so that $\frac{x}{1+e^{1/x}}$ is continuous on all of $\mathbb{R}$ except possibly at 0. (4 points)
To check continuity at 0, we take limits:

$$\lim_{x\to 0+} \frac{x}{1+e^{1/x}} = 0$$

$$\lim_{x\to 0-} \frac{x}{1+e^{1/x}} = 0$$

and $f(0) = 0$, so this function is continuous. (6 points)

(b)
The function $f(x)$ is not differentiable at 0. We check the definition:
We have:

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{1}{1+e^{1/x}} = 0$$

(5 points)

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{1}{1+e^{1/x}} = 1$$

(5 points)
So $f(x)$ is not differentiable at 0.

Problem 2.

(a) We must show that f is Riemann-Stieltjes integrable with respect to α iff f is continuous at 0.
First assume that f is integrable. Then for every $\epsilon > 0$, there is some partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Of course, this sum $U - L$ can only decrease if we refine P, so if necessary we may further refine P to include 0 and still have that $U - L < \epsilon$. Now the (new) partition P has $P = \{-1 = x_1, x_2, \ldots, x_n = 1\}$ and $x_k = 0$ for some k, so that $x_{k-1} < 0$ and $x_{k+1} > 0$. As usual, define M_j, m_j as the maximum and minimum (respectively) of f on the interval $[x_j, x_{j+1}]$. Then:

$$\begin{aligned} U(P, f, \alpha) &= M_{k-1}(\alpha(0) - \alpha(x_{k-1})) + M_k(\alpha(x_{k+1}) - \alpha(0)) \\ &= M_{k-1}/2 + M_k/2 \end{aligned}$$

and

$$L(P, f, \alpha) = m_{k-1}/2 + m_k/2$$

Since $U - L < \epsilon$, we have

$$(M_{k-1} - m_{k-1})/2 + (M_k - m_k)/2 < \epsilon$$

and both of these quantities are positive.
Now, choose a δ such that $[-\delta, \delta] \subset [x_{k-1}, x_{k+1}]$. Then, for any $y > 0$ such that $y - 0 < \delta$, we have $|f(y) - f(0)| \leq M_k - m_k < 2\epsilon$. If we choose $-\delta < y < 0$ and repeat the computation with M_{k-1} and m_{k-1}, we again find that $|f(y) - f(0)| < 2\epsilon$. So, f is continuous at 0. (9 points)

Conversely, suppose that f is continuous at 0. For $0 < \zeta < 1$, Define the partition P_ζ to be $P_\zeta = \{-1, -\zeta, 0, \zeta, 1\}$. Now, choose δ such that $|y - 0| < \delta$ implies that $|f(y) - f(0)| < \epsilon$ as guaranteed by continuity. Then $P_\delta = \{-1, -\delta, 0, \delta, 1\}$. Let M_3, m_3 denote the maximum and minimum (respectively) of $f(y)$ on $[0, \delta]$, and similarly for M_2, m_2 on $[-\delta, 0]$. In particular, this means that $M_3 - m_3 < 2\epsilon$ and $M_2 - m_2 < 2\epsilon$. Then:

$$\begin{aligned} U(P_\delta, f, \alpha) - L(P_\delta, f, \alpha) &= (M_2/2 + M_3/2) - (m_2/2 + m_3/2) \\ &< 2\epsilon \end{aligned}$$

Since we can find such a P_δ for any ϵ, f is integrable. (6 points)

(b)
We show $\int_{-1}^{1} f \, d\alpha = f(0)$. To show this, we show that U and L both get arbitrarily close to $f(0)$ for appropriate partitions P.

By assumpution f is continuous at 0. For any ϵ, we can pick a δ and P_δ as above. Then $|M_2 - f(0)| < \epsilon$ and $|M_3 - f(0)| < \epsilon$.

$$\begin{aligned} U(P_\delta, f, \alpha) - f(0) &= (M_2/2 + M_3/2) - f(0) \\ &= (M_2 - f(0))/2 + (M_3 - f(0))/2 \\ &< \epsilon \end{aligned}$$

and similarly $f(0) - L(P_\delta, f, \alpha) < \epsilon$. Since such a P_δ can be found for any ϵ, we must have that $\int_{-1}^{1} f \, d\alpha = f(0)$. (15 points)

Problem 3.

(a) This is false.
One counterexample is as follows: we work with the interval $[0, 1]$ and define

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Then $f(x)^2 = 1$ is integrable, but f is not; it is easy to check that for any partition $U(P, f) = 1$ and $L(P, f) = -1$. (10 points)

(b) This is true.
On any interval $[a, b]$ the function $g(x) = x^{1/3}$ is continuous. Since $f(x)^3$ is integrable, so is $g(f(x)^3) = f(x)$. (Note that the same argument fails for the previous case, as the function x^2 does not have an inverse.) (10 points)

(c) This is false.
We know that if f is bounded on $[a, b]$, then f is integrable iff the set of discontinuities has measure 0. So, the question reduces to: is every set of measure 0 countable? The answer is no. One example is the Cantor set, which is uncountable, but (as can be checked) has measure $\lim_{n\to\infty}(2/3)^n = 0$. So, for example, the function f defined on $[0, 1]$ as

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is in the Cantor set} \\ 0 & \text{if } x \text{ is not in the Cantor set} \end{cases}$$

is integrable. (10 points)

Problem 4.
(a) Let $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{M}$. For every $x, y \in X$, such that $d_X(x, y) < \delta$, we have

$$d_X(f(x), f(y)) \leq M \cdot d_X(x, y) < M \cdot \frac{\epsilon}{M} = \epsilon.$$

This verifies the definition of uniform continuity.

(b) Let $x_0 \in X$ be arbitrary, and define inductively $x_{n+1} = f(x_n)$. If we show that the sequence $\{x_n\}$ converges, let x be its limit. Then by taking limit in $x_{n+1} = f(x_n)$, since f is continuous, we find that $f(x) = x$, so x is a fixed point. We will actually show that $\{x_n\}$ is Cauchy, which implies convergent, since the space X is Cauchy.
Using the Lipschitz condition on f, we have that $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq Md(x_n, x_{n-1}$. Proceeding by induction, we find that $d(x_{n+1}, x_n) \leq M^n d(x_1, x_0)$. Now, if $m \geq n$, by the triangle inequality,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \leq (M^{m-1} + \cdots + M^n)d(x_1, x_0) \leq \frac{M^n}{1-M} d(x_1, x_0).$$

As $n \to \infty$, the right hand side tends to 0, which implies that $\{x_n\}$ is Cauchy.

To verify the uniqueness of the fixed point, assume there are two fixed points, x and x'. Then $d(x, x') = d(f(x), f(x')) \leq Md(x, x')$. Since $0 < M < 1$, necessarily $d(x, x') = 0$.

(c) Solving the equation $f(x) = x$ on $[1, \infty)$, we find that the unique fixed point of f on this interval is $x = \sqrt{2}$. This is an example of the above result, since $f(x) = \frac{2+x}{1+x}$ satisfies the hypothesis in this problem. Indeed, $|f(x) - f(y)| = \frac{1}{(1+x)(1+y)}|x - y| \leq \frac{1}{4}|x - y|$, since $x, y \geq 1$.

(d) The example in (c), when restricted to $X = \mathbb{Q}$ works.

Problem 5.
(a) Since f is periodic, with period T, $f(\mathbb{R}) = f([0, T])$. Since $[0, T]$ is compact in $\mathbb{R}$, and f is continuous, by a theorem from the textbook, we know that $f([0, T])$ is bounded, and that f attains its maximum and minimum.

(b) Note that what we want to prove is stronger than the Mean Value Theorem. Consider the function $g(x) = f(x + a) - f(x) - af'(x)$. Let x_0 and x_1 be points where f attains its maximum, respectively minimum. Since these are automatically interior points ($\mathbb{R}$ is open in itself), $f'(x_0) = 0 = f'(x_1)$. But then $g(x_0) \leq 0$, and $g(x_1) \geq 0$. By the Intermediate Value Property of g, there must exists t between x_0 and x_1, such that $g(t) = 0$.

(c) In terms of the graph of f, part (b) says that for every $a > 0$, there exists a point t such that the tangent line at t to the graph of f intersects the graph again a units to the right of t.