18.100C. Quiz 2. Spring 2006.

Name: ____________________ April 25, 2006

Problem 1: ______ /20

Problem 2: ______ /30

Problem 3: ______ /30

Problem 4: ______ /40

Problem 5: ______ /30

Total: ______ /150

Instructions: The exam is closed book, closed notes and calculators are not allowed. You will have 80 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work (and give proofs), unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. Good luck!

Problem 1. (20 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

(a) (10 pts) Show that $f(x)$ is continuous everywhere.
(b) (10 pts) Is $f(x)$ differentiable at 0? Prove your answer.

Problem 2. (30 pts) Consider the function $\alpha(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$. Let f be a bounded real function on $[-1, 1]$.

(a) (15 pts) Prove that f is Riemann-Stieltjes integrable with respect to α if and only if f is continuous at 0.

(b) (15 pts) Assuming that f is continuous at 0, find $\int_{-1}^{1} f \, d\alpha$.

Problem 3. (30 pts) Suppose f is a bounded real function on $[a, b]$. Decide if the following assertions are true or false. Prove your answer or give a counterexample.

(a) (10 pts) If f^2 is Riemann integrable, then f is Riemann integrable.

(b) (10 pts) If f^3 is Riemann integrable, then f is Riemann integrable.

(c) (10 pts) If f is Riemann integrable, the set of discontinuities of f is at most countable.

Problem 4. (40 pts) Let X be a metric space, and M a real number, $0 < M < 1$. Let $f : X \to X$ be a function such that for any $x, y \in X$,

$$d_X(f(x), f(y)) \leq M \cdot d_X(x, y).$$

(a) (10 pts) Prove that f is uniformly continuous.

(b) (15 pts) Assume that X is a complete metric space. Show that there *exists a unique* $x \in X$ such that $f(x) = x$, i.e., f has a unique fixed point. (Hint: define inductively $x_{n+1} = f(x_n)$.)

(c) (8 pts) Exemplify this result using $f(x) = \frac{2+x}{1+x}$ on $[1, \infty)$.

(d) (7 pts) Give an example to show that (b) can be false if we don't assume that X is complete.

Problem 5. (30 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume, in addition, that f is periodic, i.e., there exists $T > 0$ such that $f(x + T) = f(x)$, for all $x \in \mathbb{R}$.

(a) (10 pts) Prove that f is bounded, and that f attains its maximum and minimum on $\mathbb{R}$.

(b) (15 pts) Assume f is diferentiable, and let a be a positive real number. Prove that there exists $t \in \mathbb{R}$ such that

$$f(t + a) - f(t) = af'(t).$$

(c) (5 pts) What is the geometric meaning of the result in (b)?