18.100C. Quiz 1. Solutions. Spring 2006.

Problem 1.(40 points) Decide if the following assertions are true or false. Prove your answer (or give a counterexample).

a) Every collection of disjoint intervals in $\mathbb{R}$ is countable.

True. Consider a collection $\{I_\alpha\}$ of disjoint intervals in $\mathbb{R}$. Since every interval contains a rational number, one can assign to each I_α a rational number r_α, such that r_α's are all distinct (because the intervals are disjoint). Since the rational numbers form a countable set, this means that our collection of intervals must have been at most countable.

b) $\mathbb{R}^2$ contains a countable dense subset.

True. Such a set is $\mathbb{Q}^2$, i.e., all points in the plane with rational coordinates. It is countable, being the cartesian product of two copies of $\mathbb{Q}$, and it is dense (because $\mathbb{Q}$ is in $\mathbb{R}$). More precisely, let $z = (x, y)$ be a point in the plane, and let $r > 0$. In the ball $N_r(z)$ one can inscribe some 2-cell $[x-a, x+a] \times [y-b, y+b]$. We can find a rational p in $[x-a, x+a]$, and another one q in $[y-b, y+b]$. Thus, the point $(p, q) \in N_r(z)$.

c) The set of all sequences whose elements are 0 or 1 is countable.

False. This is an example from Rudin. Use Cantor's diagonal process.

d) Every infinite compact subset of $\mathbb{R}$ must be uncountable.

False. A counterexample is the subset $\{0\} \cup \{\frac{1}{n} : n \geq 1\}$ of $\mathbb{R}$. It is closed (the only limit point is 0) and bounded. Being in $\mathbb{R}$, it must be compact.

Problem 2.(20 points) If E is a subset of a metric space X, recall the following notation:

$$E^\circ = \text{ interior of } E, \quad E' = \text{ set of limit points of } E.$$

Suppose that E and F are subsets of X. For each of the following assertions, provide a proof if it is true, or a counterexample if it is false.

a) $(E \cup F)^\circ = E^\circ \cup F^\circ$.

False. A counterexample is $E = (0, 1]$ and $F = [1, 2)$ in $\mathbb{R}$: $(E \cup F)^\circ = (0, 2)$, but $E^\circ \cup F^\circ = (0, 1) \cup (1, 2)$.

b) $(E \cap F)^\circ = E^\circ \cap F^\circ$.

True. If $x \in E^\circ \cap F^\circ$, then there exist $r', r'' > 0$ such that $N_{r'}(x) \subset E$ and $N_{r''}(x) \subset F$. Let r be the minimum of $\{r', r''\}$. Then $N_r(x) \subset E \cap F$, and so $x \in (E \cap F)^\circ$.

For the converse, let $x \in (E \cap F)^\circ$. There is $r > 0$, such that $N_r(x) \subset E \cap F$, so $N_r(x) \subset E$, implying $x \in E^\circ$, and $N_r(x) \subset F$, so $x \in F^\circ$ as well.

c) $(E \cap F)' = E' \cap F'$.

False. A counterexample is $E = (0, 1)$ and $F = (1, 2)$. Then $E \cap F = \emptyset$, so $(E \cap F)' = \emptyset$, but $E' = [0, 1]$, and $F' = [1, 2]$, giving $E' \cap F' = \{1\}$. (Warning: do not confuse limit points with boundary points; E' is not the boundary of E, and it is not the closure of E either in general.)

d) $(E \cup F)' = E' \cup F'$.

True. It should be clear that $E' \cup F' \subset (E \cup F)'$. In the other direction, assume $x \in (E \cup F)'$, but $x \notin E' \cup F'$. This means that there exist $r', r'' > 0$, such that $N_{r'}(x) \cap E \subset \{x\}$, and $N_{r''}(x) \cap F \subset \{x\}$. Let r be the minimum of r', r''; then $N_r \cap (E \cup F) \subset \{x\}$, which would imply that x is not a limit point for $E \cup F$, contradiction.

Problem 3.(30 points) A metric space X is said to be *totally bounded* if for every $r > 0$, there exist finitely many points $x_1, \ldots, x_n$ of X so that

$$X = N_r(x_1) \cup \cdots \cup N_r(x_n).$$

(Recall that $N_r(x) = \{y \in X : \ d(y, x) < r\}$.)

a) Prove that every compact metric space is totally bounded.

For $r > 0$, let $\{G_x\}$ be the open cover consisting of neighborhoods $G_x = N_r(x)$. Since $x \in G_x$ for every x this is a cover of X. As X is compact, there exists a finite subcover; more precisely, there exits $x_1, \ldots x_n$ such that $\cup_{i=1}^n N_r(x) = X$.

One remark: we know that every compact space is complete. This exercise shows that it is also totally bounded. In fact, these are the right notions to characterize compact spaces: in fact, the generalization of the Heine-Borel theorem for metric spaces says that a space is compact if and only if it is complete and totally bounded.

b) Let X be an infinte set. Define the metric function $d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$. (You may assume without proof that d is a metric.) Show that X is bounded, but not totally bounded.

Note that if $x_0 \in X$ is a point, $X = N_2(x_0)$, so X is bounded. It is not totally bounded: take $r = \frac{1}{2}$. Then any neighborhood $N_r(x)$ contains only the point x. Since X is infinite, we cannot cover it with finitely many $N_r(x)'s$.

c) For the space X in b), which subsets of X are compact?

Any finite subset of X is compact (this is true for any metric space). These are all compact subsets of this particular X, because the argument in b) shows that for a subset of X to be totally bounded, it must be finite.

Problem 4.(30 points)

a)(20 points) Let $\{x_n\}$ be a Cauchy sequence in a metric space X. Assume that some subsequence $\{x_{n_k}\}$ converges to a point $x \in X$. Prove that the entire sequence $\{x_n\}$ converges to x.

Let $\epsilon > 0$. Since $\{x_{n_k}\}$ converges to x, there exists $N' > 0$, such that $d(x_{n_k}, x) < \epsilon/2$, for all $n_k > N'$. Since $\{x_n\}$ is Cauchy, there exists $N'' > 0$, such that $d(x_n, x_m) < \epsilon/2$, for all $n, m > N$. In particular, for all $n_k > N$, and $n > N$, $d(x_n, x_{n_k}) < \epsilon/2$. Choose an n_k such that $n_k \geq \max\{N, N'\}$. Then, for all $n > N$,

$$d(x_n, x) < d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

b)(10 points) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that

$$|x_{n+1} - x_n| < \frac{1}{2^n}, \text{ for all } n \geq 1.$$

Prove that $\{x_n\}$ converges.

Let $m \geq n$. Then

(1)

$$|x_m - x_n| < |x_m - x_{m-1}| + |x_{m-1} - x_{m-2} + \cdots + |x_{n+1} - x_n|$$

(2) $$< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} = \frac{1}{2^n}(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}})$$

(3) $$< \frac{1}{2^n}(1 + \frac{1}{2} + \frac{1}{2^2} + \dots)$$

(4) $$= \frac{1}{2^{n-1}}.$$

For every $\epsilon > 0$, we can find N such that $\frac{1}{2^{N-1}} < \epsilon$. Therefore, for all $n, m \geq N$, $|x_m - x_n| < \epsilon$. This shows that $\{x_n\}$ is Cauchy. Since $\{x_n\} \subset \mathbb{R}$, $\{x_n\}$ must be convergent.

Note that just the condition $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, does not imply that the sequence is Cauchy. Look for example at the sequence, $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. We know this sequence is divergent (and being in $\mathbb{R}$), it is not Cauchy. But $|x_{n+1} - x_n| = \frac{1}{n+1} \to 0,\ n \to \infty$.

Problem 5.(30 points) Decide if the following sets are open, closed, compact. (The metric functions are the Euclidean ones.) Justify your answers.

All the examples a)-c) should be pretty clear once we draw pictures for example, so I'll leave the details to you. In the Euclidean spaces, Heine-Borel theorem says that a set is compact if and only if it is closed and bounded.

a) $E_1 = \{(x, y) \in \mathbb{R}^2 :\ y - 2x \leq 1\}$ in $\mathbb{R}^2$.
not open, closed, unbounded, not compact.

b) $E_2 = \{(x, y) \in \mathbb{R}^2 :\ 1 < x^2 + y^2 < 9\}$ in $\mathbb{R}^2$.
open, not closed, bounded, not compact.

c) $E_3 = \{(x, y) \in \mathbb{R}^2 :\ |x| + |y| = 2\}$ in $\mathbb{R}^2$.
not open, closed, bounded, compact.

d) $E_4 = [0, 1] \cap \mathbb{Q}$ in $\mathbb{R}$.
not open (it is not a union of intervals), not closed (the irrationals in $[0, 1]$ are also limit points of E_4), bounded, not compact.